

RELATIVE COMPACTNESS AND PRODUCT STABLE QUOTIENT MAPS

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ABSTRACT. It is well known that, for Hausdorff spaces, a quotient map $f : X \rightarrow Y$ is product stable, in the sense that the product map $f \times g$ is a quotient map, for every quotient map g , if and only if Y is locally compact and f is biquotient. Here, we present a generalization of those quotient maps which are product stable.

1. Introduction

Recall that, for arbitrary subsets A and B of a topological space X , we say that A is compact relative to B , written $A \ll B$, if for every open cover of B , there exist finitely many elements in the cover that cover A . The extension of the notion of relative compactness to arbitrary subsets generalizes the familiar concept of compact subset. Indeed, a subset A is compact if and only if $A \ll A$; see [4].

The notion of relative compactness, restricted to open sets, plays an important role in the study of core compact spaces. Indeed, the definition of core compactness says that X is core compact if, for every $x \in X$ and every open neighborhood V of x , there exists an open neighborhood U of x with $U \ll V$. For Hausdorff spaces, core compactness

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coincides with local compactness; see [3]. There exist many characterizations of the core compact spaces. A useful characterizations is related to the quotient maps, that is, a space X is core compact if and only if $q \times i_X : A \times X \rightarrow B \times X$ is a quotient map for every quotient map $q : A \rightarrow B$, where i_X is the identity map; see [4].

Here, we present the notion of f -relative compactness and we study a few basic properties of continuous maps which reflect relative compactness to f -relative compactness. Also, we investigate some properties of these maps related to the proper, biquotient and product stable quotient maps.

Lemma 1.1. [3] *If X is a core compact space and U, V are open subsets of X with $U \ll V$, then there exists an open subset W of X , such that $U \ll W \ll V$.*

Lemma 1.2. [4] *Let X and Y be topological spaces.*

(a) *If $f : X \rightarrow Y$ is a continuous map and $A \ll B$ in X , then $f(A) \ll f(B)$ in Y .*

(b) *If $A' \ll A$ in X and $B' \ll B$ in Y , then $A' \times B' \ll A \times B$ in $X \times Y$.*

Theorem 1.3. [1] *A quotient map $f : X \rightarrow Y$ is product stable if and only if Y is a core compact space and the product map $f \times i_Z$ is quotient for every space Z , where i_Z is the identity map.*

2. f -relative compactness

In this section, we give the notion of f -relative compactness and some properties of it. Also, we present a necessary and sufficient condition for f -relative compactness.

Definition 2.1. Let $f : X \rightarrow Y$ be a continuous map. For arbitrary subsets A and B of Y and X , respectively, we say that A is f -relatively compact in B , written $A \ll_f B$, if for every open covering $(U_i)_{i \in I}$ of B , there is some finite subset F of I such that $A \subseteq \bigcup_{i \in F} f(U_i)$.

The following lemma is an immediate consequence of the definition of the relation \ll_f .

Lemma 2.2. *Let $f : X \rightarrow Y$ be a continuous map.*

(a) *For open subsets U and V of Y and X , respectively, $U \ll_f V$ implies $U \subseteq f(V)$.*

(b) *$A' \subseteq A \ll_f B \subseteq B'$ implies $A' \ll_f B'$.*

(c) *$A \ll_f B$ and $A' \ll_f B$ together imply $A \cup A' \ll_f B$.*

Lemma 2.3. *Let $f : X \rightarrow Y$ be a continuous map.*

(a) *If A and B are subsets of Y and X , respectively, such that $A \ll_f B$, then $A \ll f(B)$.*

(b) *If A and B are subsets of X such that $A \ll B$, then $f(A) \ll_f B$.*

Proof. (a): Let $(U_i)_{i \in I}$ be an open cover of $f(B)$. Then, $B \subseteq \bigcup_{i \in I} f^{-1}(U_i)$,

and hence there exists a finite subset F of I such that $A \subseteq \bigcup_{i \in F} f f^{-1}(U_i)$.

Therefore, $A \subseteq \bigcup_{i \in F} U_i$, as desired. The proof of (b) is similar to (a). \square

The following example shows that if $f : X \rightarrow Y$ is a continuous map and $A \ll B$ in Y , then the relation $A \ll_f f^{-1}(B)$ may be false.

Example 2.4. Let $X = \{a, b, c\}$ and $f : (X, \tau) \rightarrow (X, \tau')$ be the identity map such that $\tau' = \{\emptyset, X, \{a\}\}$, and τ be the discrete topology on X . Then, $\{b\} \ll \{c\}$ in (X, τ') , but the relation $\{b\} \ll_f f^{-1}(\{c\})$ is false.

Lemma 2.5. *Let $f : X \rightarrow Y$ be a continuous map and A, B be subsets of Y and X , respectively. Then, the following conditions are equivalent:*

(a) *$A \ll_f B$.*

(b) *For every space Z , if $z \in Z$ and W is an open subset of $Z \times X$ such that $\{z\} \times B \subseteq W$, then $V \times A \subseteq (i_z \times f)(W)$, for some neighborhood V of z .*

Proof. Similar to the proof of the Theorem 6.2 in [2]. \square

3. Product stable quotient maps

A quotient map $f : X \rightarrow Y$ is called product stable if the product map $f \times g : X \times Z \rightarrow Y \times W$ is a quotient for every quotient map

$g : Z \rightarrow W$. In [8], Michael showed that for locally compact Hausdorff spaces, product stable quotient maps were precisely biquotient maps, where a continuous surjective map $f : X \rightarrow Y$ is called biquotient, if for each $y \in Y$ and each open covering $(U_i)_{i \in I}$ of $f^{-1}(y)$, there is some finite subset F of I , such that $y \in \text{int}(\bigcup_{i \in F} f(U_i))$. In [1], Day and Kelly showed that biquotient maps were exactly universal quotient maps, or pullback stable quotient maps, that is, quotient maps whose pullback along any map is still a quotient.

Here, we present some properties of continuous maps which reflect relative compactness to f -relative compactness. In particular, we show that (main result) such maps are product stable quotient maps, for core compact spaces, and are precisely the biquotient maps, for Hausdorff spaces. The importance of the main theorem is in its usefulness to be able to identify a quotient map as a product stable map, for non-Hausdorff spaces.

For the proof of the main theorem, we use several concepts from main stream general topology. In particular, the characterization of the exponentiable spaces and the exponential topology has a long history, which is discussed in detail by Isbell [6] and goes back to at least 1945 with the work of Fox [5]. The first general solution is implicit in the work of Day and Kelly [1], who characterized the space X for which the identity map $i_X : X \rightarrow X$ is product stable if and only if X is a core compact space. By virtue of the Adjoint Functor Theorem [7], such spaces coincide with the exponentiable spaces; see [3] and [6] for details. Day and Kelly's characterization amounts to the fact that the open sets of X (denoted by OX) form a continuous lattice in the sense of Scott [9], but continuous lattices were introduced independent of the work of Day and Kelly.

Definition 3.1. Let $f : X \rightarrow Y$ be a continuous map.

(a) We say that f reflects relative compactness between arbitrary sets, if for arbitrary subsets A and B of Y such that $A \ll B$, then $f^{-1}(A) \ll f^{-1}(B)$.

(b) We say that f reflects relative compactness to f -relative compactness, if for arbitrary subsets A and B of Y such that $A \ll B$, then $A \ll_f f^{-1}(B)$.

Example 3.2. Let $f : X \rightarrow Y$ be a continuous surjective map and $A \ll B$ in Y . If $f^{-1}(A) \ll f^{-1}(B)$, then by Lemma 2.3, $ff^{-1}(A) \ll_f f^{-1}(B)$, and hence $A \ll_f f^{-1}(B)$. Therefore, if f reflects relative compactness, then f reflects relative compactness to f -relative compactness. Every proper map reflects relative compactness between arbitrary sets; see [2]. Therefore, proper maps reflect relative compactness to f -relative compactness.

Proposition 3.3. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous maps.*

(a) *If g is surjective and reflects relative compactness and f reflects relative compactness to f -relative compactness, then gf reflects relative compactness to gf -relative compactness.*

(b) *If gf reflects relative compactness to gf -relative compactness and g is injective, then f reflects relative compactness to f -relative compactness.*

(c) *If gf reflects relative compactness to gf -relative compactness and f is surjective, then g reflects relative compactness to g -relative compactness.*

Proof. (a): Let $A \ll B$ in Z . Then, $g^{-1}(A) \ll g^{-1}(B)$ in Y , and so $g^{-1}(A) \ll_f f^{-1}g^{-1}(B)$. Now, let $(U_i)_{i \in I}$ be an open cover of $f^{-1}g^{-1}(B)$. Then, there is some finite subset F of I such that $g^{-1}(A) \subseteq \bigcup_{i \in F} f(U_i)$,

and hence $A = gg^{-1}(A) \subseteq \bigcup_{i \in F} gf(U_i)$. Thus, $A \ll_{gf} f^{-1}g^{-1}(B)$.

(b): Let $A \ll B$ in Y . By Lemma 1.2, $g(A) \ll g(B)$ in Z . So, $g(A) \ll_{gf} f^{-1}g^{-1}g(B)$, and hence $g(A) \ll_{gf} f^{-1}(B)$. Now, let $(U_i)_{i \in I}$ be an open cover of $f^{-1}(B)$. Then, there is some finite subset F of I such that $g(A) \subseteq \bigcup_{i \in F} gf(U_i)$. Since g is injective, then $A \subseteq \bigcup_{i \in F} f(U_i)$, and thus $A \ll_f f^{-1}(B)$.

(c): Similar to the proof of (b). □

Proposition 3.4. *Let $f : X \rightarrow Y$ and $g : Z \rightarrow W$ be continuous maps.*

(a) *If $f \times g$ reflects relative compactness to $f \times g$ -relative compactness, then f reflects relative compactness to f -relative compactness.*

(b) *If f and g reflect relative compactness to f -relative compactness and g -relative compactness, respectively, $A \times B \ll C \times D$ in $Y \times W$, then $A \times B \ll_{f \times g} (f \times g)^{-1}(C \times D)$.*

Proof. (a): Let $A \ll B$ in Y and w be an arbitrary point in W . Then, by Lemma 1.2, $A \times \{w\} \ll B \times \{w\}$ in $Y \times W$, and therefore $A \times \{w\} \ll_{f \times g} f^{-1}(B) \times g^{-1}\{w\}$, and hence $A \ll_f f^{-1}(B)$.

(b): Let $A \times B \ll C \times D$ in $Y \times W$. Then, by Lemma 1.2, $A = \pi(A \times B) \ll C = \pi(C \times D)$, where $\pi : Y \times W \rightarrow Y$ is the projection map. So, $A \ll_f f^{-1}(C)$, and similarly, we have that $B \ll_g g^{-1}(D)$. Thus, similar to the proof of Lemma 1.2, we have that $A \times B \ll_{f \times g} f^{-1}(C) \times g^{-1}(D) = (f \times g)^{-1}(C \times D)$. \square

Lemma 3.5. *Let $f : X \rightarrow Y$ be a biquotient map. Then, f reflects relative compactness to f -relative compactness between arbitrary sets.*

Proof. Let $A \ll B$ in Y and $(U_i)_{i \in I}$ be an open cover of $f^{-1}(B)$. Then, for every point y in B , there is a finite subset F_y of I and an open neighborhood U_y of y such that $y \in U_y \subseteq \bigcup_{i \in F_y} f(U_i)$. Therefore, there is a finite subset S of B such that $A \subseteq \bigcup_{y \in S} f(U_y)$, and hence

$$A \subseteq \bigcup_{y \in S} \bigcup_{i \in F_y} f(U_i), \text{ as desired.} \quad \square$$

Remark 3.6. Let $f : X \rightarrow Y$ be a continuous map such that Y be a Hausdorff space. Then, f is a biquotient map if and only if for each $y \in Y$ and each open covering $(U_i)_{i \in I}$ of X , there is some finite subset F of I such that $y \in \text{int}(\bigcup_{i \in F} f(U_i))$; see proposition 2.1 in [8].

Lemma 3.7. *Let $f : X \rightarrow Y$ be a continuous surjective map such that reflects relative compactness to f -relative compactness between open sets, and Y be a locally compact Hausdorff space. Then, f is a biquotient map.*

Proof. Let $y \in Y$ and $(U_i)_{i \in I}$ be an open cover of X . Since Y is a core compact space, then there exists an open neighborhood U of y such that $U \ll Y$. By assumption, $U \ll_f f^{-1}(Y) = X$, and thus there is a finite subset F of I such that $y \in U \subseteq \bigcup_{i \in F} f(U_i)$, and hence by Remark 3.6, the result follows. \square

Proposition 3.8. *Let $f : X \rightarrow Y$ be a continuous surjective map such that Y be a locally compact Hausdorff space. Then, f is a biquotient map if and only if f reflects relative compactness to f -relative compactness between open sets.*

Proof. The result follows by Lemmas 3.5 and 3.7.

Remark 3.9. Let X and Y be topological spaces and let $C(X, Y)$ denote the set of continuous maps from X to Y . Given any continuous map $f : A \times X \rightarrow Y$, one has a function $\bar{f} : A \rightarrow C(X, Y)$ defined by $\bar{f}(a)(x) = f(a, x)$, called the exponential transpose of f . A topology on $C(X, Y)$ is said to be exponential, if continuity of a function $f : A \times X \rightarrow Y$ is equivalent to that of its transpose $\bar{f} : A \rightarrow C(X, Y)$. A space X is said to be exponentiable, if for every space Y there is an exponential topology on $C(X, Y)$. A space is exponentiable if and only if it is core compact. Moreover, if X is a core compact space and Y is any space, then the exponential topology on $C(X, Y)$ is generated by the sets $\{f \in C(X, Y) \mid U \ll f^{-1}(V)\}$, where U and V range over open sets of X and Y , respectively; see Theorem 5.3 in [3].

Theorem 3.10. *(Main result) Let $q : X \rightarrow Y$ be a quotient map which reflects relative compactness to q -relative compactness between open sets, and let Y be a core compact space. Then, q is a product stable quotient map.*

Proof. By Theorem 1.3, it is enough to show that $i_A \times q$ is a quotient map, for every space A . Let $g : A \times Y \rightarrow Z$ be an arbitrary function such that $h = g \circ (i_A \times q) : A \times X \rightarrow A \times Y \rightarrow Z$ is a continuous map. we show that g is a continuous map. Since Y is a core compact space, then by Remark 3.9, $C(Y, Z)$ has an exponential topology which is generated by the sets $N(U, V) = \{f \in C(Y, Z) \mid U \ll f^{-1}(V)\}$, where U and V range over open sets of Y and Z , respectively. Let $C(X, Z)$ be endowed with a topology which is generated by the sets $N'(U, V) = \{f \circ q \mid f \in C(Y, Z), U \ll f^{-1}(V)\}$. Then, the function $\hat{q} : C(Y, Z) \rightarrow C(X, Z)$, defined by $\hat{q}(f) = f \circ q$, is continuous and injective. Assume that \bar{g} and \bar{h} are the exponential transposes of g and h , respectively. Then, we have $\hat{q} \circ \bar{g}(a)(x) = (\bar{g}(a) \circ q)(x) = g(a, q(x)) = g \circ (i \times q)(a, x) = h(a, x) = \bar{h}(a)(x)$. Therefore, $\hat{q} \circ \bar{g} = \bar{h}$.

Now, we show that \bar{h} is a continuous map. It is enough to show that $\bar{h}^{-1}(N'(U, V))$ is an open subset of A , where U and V range over open sets of X and Z , respectively. Let $W = h^{-1}(V)$.

First, we show that $\bar{h}^{-1}(N'(U, V)) = \bar{W}^{-1}(O)$, where $O = \{V' \in OX \mid U \ll_q V'\}$ and the function $\bar{W} : A \rightarrow OX$ is defined by $\bar{W}(a) = \{x \in X \mid (a, x) \in W\}$, and OX is the lattice of open sets of X . Suppose that $a \in \bar{h}^{-1}(N'(U, V))$. Then, $\bar{h}(a) = f \circ q$, for some $f \in C(Y, Z)$, such that $U \ll f^{-1}(V)$. By assumption, $U \ll_q q^{-1}f^{-1}(V)$. But, we have $x \in q^{-1}f^{-1}(V) \Leftrightarrow f \circ q(x) \in V \Leftrightarrow \bar{h}(a)(x) \in V \Leftrightarrow h(a, x) \in V \Leftrightarrow (a, x) \in h^{-1}(V) \Leftrightarrow x \in \bar{W}(a)$. Thus, $\bar{W}(a) = q^{-1}f^{-1}(V)$, and hence $a \in \bar{W}^{-1}(O)$. Conversely, let $a \in \bar{W}^{-1}(O)$ and $f = g \circ \alpha$, where the map $\alpha : Y \rightarrow A \times Y$ is defined by $\alpha(y) = (a, y)$. Then, $f \circ q = \bar{h}(a)$, and since q is a quotient map, f is continuous. So $\bar{W}(a) = q^{-1}f^{-1}(V)$, and thus we have $q^{-1}f^{-1}(V) \in O \Rightarrow U \ll_q q^{-1}f^{-1}(V) \Rightarrow U \ll f^{-1}(V) \Rightarrow f \circ q \in N'(U, V) \Rightarrow \bar{h}(a) \in N'(U, V) \Rightarrow a \in \bar{h}^{-1}(N'(U, V))$.

Next, we show that $\bar{W}^{-1}(O)$ is an open subset of A . Let $a \in \bar{W}^{-1}(O)$. Similar to the above argument, we have $\bar{W}(a) = q^{-1}f^{-1}(V)$, for some $f \in C(Y, Z)$ and $U \ll f^{-1}(V)$. By Lemma 1.1, there exists an open subset V' of Y such that $U \ll V' \ll f^{-1}(V)$, and hence $U \ll_q q^{-1}(V')$ and $V' \ll_q \bar{W}(a)$. Now, for every point x in $\bar{W}(a)$, there exist open neighborhoods U_x and V_x of a and x , respectively, such that $(a, x) \in U_x \times V_x \subseteq W$. Thus, there is a finite set F such that $V' \subseteq \bigcup_{x \in F} q(V_x)$.

Let $U_0 = \bigcap_{x \in F} U_x$ and $V_0 = \bigcup_{x \in F} V_x$. Then, we show that $U_0 \subseteq \bar{W}^{-1}(O)$.

Let $b \in U_0$. It is enough to show that $V_0 \subseteq \bar{W}(b)$. Suppose that $x \in V_0$. Then, $x \in V_{x'}$, for some $x' \in F$. Therefore, $(b, x) \in U_{x'} \times V_{x'} \subseteq W$, and hence $x \in \bar{W}(b)$. But similar to the above argument, $\bar{W}(b) = q^{-1}f^{-1}(V)$, for some $f \in C(Y, Z)$. Thus, we have $U \ll_q q^{-1}(V') \subseteq q^{-1}q(V_0) \subseteq q^{-1}q(\bar{W}(b)) = \bar{W}(b)$, and hence $b \in \bar{W}^{-1}(O)$. Therefore, $a \in U_0 \subseteq \bar{W}^{-1}(O)$, which shows that $\bar{W}^{-1}(O)$ is an open subset of A .

Finally, since \hat{q} is an injective map, $\hat{q} \circ \bar{g} = \bar{h}$ and $N'(U, V) = \hat{q}(N(U, V))$, then we have $\bar{h}^{-1}(N'(U, V)) = \bar{g}^{-1}\hat{q}^{-1}(\hat{q}(N(U, V))) = \bar{g}^{-1}(N(U, V))$, which shows that \bar{g} is a continuous map, and hence g is a continuous map because Y is a core compact space, as desired. \square

By Theorem 1.3, the core compactness is a necessary condition for the product stability of the quotient maps.

Question. Does the converse of Theorem 3.10 hold?

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