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ON THE RATIONAL RECURSIVE SEQUENCE $X_{N+1} = \gamma X_{N-K} + (AX_N + BX_{N-K}) / (CX_N - DX_{N-K})$

E.M.E. ZAYED AND M.A. EL-MONEAM*

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Abstract. Our main objective is to study some qualitative behavior of the solutions of the difference equation

$$x_{n+1} = \gamma x_{n-k} + (ax_n + bx_{n-k}) / (cx_n - dx_{n-k}) \ , \quad n = 0, 1, 2, ...,$$

where the initial conditions $x_{-k},..., x_{-1}, x_0$ are arbitrary positive real numbers and the coefficients γ, a, b, c and d are positive constants, while k is a positive integer number.

1. Introduction

Our goal is to investigate some qualitative behavior of the solutions of the difference equation,

$$x_{n+1} = \gamma x_{n-k} + \frac{ax_n + bx_{n-k}}{cx_n - dx_{n-k}}, \qquad n = 0, 1, 2, ..., \tag{1.1}$$

where the initial conditions $x_{-k},..., x_{-1}, x_0$ are arbitrary positive real numbers and the coefficients γ, a, b, c and d are positive constants, while k is a positive integer number. The case where any of α, a, c, d is allowed to be zero gives different special cases of the equation (1) which are studied by many authors (see for example, [3, 6, 8, 12, 17, 29]). For

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$*$ Corresponding author

related work, see also [1, 2, 4, 5, 7, 9-11, 13-16, 18-28, 30–40]. The study of these equations is challenging and rewarding and still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own rights. Furthermore, the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations. Note that Eq. (1.1) can be considered as a generalization of that obtained in [8, 33].

Definition 1.1. A difference equation of order $(k+1)$ is of the form

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \qquad n = 0, 1, 2, ..., \tag{1.2}$$

where F is a continuous function which maps some set J^{k+1} into J and J is a set of real numbers. An equilibrium point $\widetilde{x}$ of this equation is a point that satisfies the condition $\widetilde{x} = F(\widetilde{x}, \widetilde{x}, ..., \widetilde{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \widetilde{x}$, for all $n \geq -k$, is a solution of that equation.

Definition 1.2. Let $\widetilde{x} \in (0, \infty)$ be an equilibrium point of the difference equation (1.2). Then, we have:
(i) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-k}, ..., x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \widetilde{x}| + ... + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \delta$, then $|x_n - \widetilde{x}| < \varepsilon$, for all $n \geq -k$.

(ii) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-k}, ..., x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \widetilde{x}| + ... + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \gamma$, then

$$\lim_{n\to\infty} x_n = \widetilde{x}.$$

(iii) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called a global attractor if $x_{-k}, ..., x_{-1}, x_0 \in (0, \infty)$, then

$$\lim_{n\to\infty} x_n = \widetilde{x}.$$

(iv) An equilibrium point $\widetilde{x}$ of the equation (1.2) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\widetilde{x}$, of the difference equation (2) is called unstable if it is not locally stable.

Definition 1.3. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$, for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

The linearized equation of the difference equation (1.2) about the equilibrium point $\widetilde{x}$ is the linear difference equation,

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\widetilde{x}, \widetilde{x}, ..., \widetilde{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.3}$$

Now, assume that the characteristic equation associated with (1.3) is

$$p(\lambda) = p_0 \lambda^k + p_1 \lambda^{k-1} + ... + p_{k-1}\lambda + p_k = 0, \tag{1.4}$$

where,

$$p_i = \partial F(\widetilde{x}, \widetilde{x}, ..., \widetilde{x}) / \partial x_{n-i}.$$

Theorem 1.1. *(See [18]). Assume that $p_i \in R$, $i = 1, 2, ...$, and $k \in \{0, 1, 2, ...\}$. Then,*

$$\sum_{i=1}^{k} |p_i| < 1 \tag{1.5}$$

is a sufficient condition for the asymptotic stability of the difference equation,

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, 2, ... \tag{1.6}$$

Theorem 1.2. *(The linearized stability theorem; see[15,18,19]).*
Suppose F is a continuously differentiable function defined on an open neighbourhood of the equilibrium $\widetilde{x}$. Then, the following statements are true.
(i) If all roots of the characteristic equation (1.4) of the linearized equation (1.3) have absolute value less than one, then the equilibrium point $\widetilde{x}$ is locally asymptotically stable.
(ii) If at least one root of Eq.(1.4) has absolute value greater than one, then the equilibrium point $\widetilde{x}$ is unstable.

The following Theorem will be useful for the proof of our main results.

Theorem 1.3. $(See[15, p.18])$. *Let* $F : [a, b]^{k+1} \longrightarrow [a, b]$ *be a continuous function, where* k *is a positive integer, and* $[a, b]$ *is an interval of real numbers and consider the difference equation (1.2). Suppose that* F *satisfies the following conditions:*
(i) For every integer i *with* $1 \leq i \leq k+1$, *the function* $F(z_1, z_2, ..., z_{k+1})$ *is weakly monotonic in* z_i, *for fixed* $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.
(ii) If (m, M) *is a solution of the system*

$$m = F(m_1, m_2, ..., m_{k+1}) \quad \text{and} \quad M = F(M_1, M_2, ..., M_{k+1}), \tag{1.7}$$

then $m = M$, *where for each* $i = 1, 2, ..., k+1$, *we set*

$$m_i = \begin{cases} m & if \;\; F is\ nondecreasing \;\; in \;\; z_i \\ \\ M & if \;\; F is\ nonincreasing \;\; in \;\; z_i \end{cases}$$

and

$$M_i = \begin{cases} M & if \;\; F is\ nondecreasing \;\; in \;\; z_i \\ \\ m & if \;\; F is\ nonincreasing \;\; in \;\; z_i. \end{cases}$$

Then, there exists exactly one equilibrium point $\widetilde{x}$ *of the difference equation (1.2), and every solution of (1.2) converges to* $\widetilde{x}$.

2. Periodic solutions

Theorem 2.1. *If* k *is an even positive integer and* $c \neq d$, *then Eq. (1.1) has no positive solution of prime period two.*

Proof. Assume that there exists a distinctive positive solution

$$..., P, Q, P, Q, ...$$

of prime period two of Eq. (1.1). If k is even, then $x_n = x_{n-k}$. It follows from Eq. (1.1) that

$$P = \gamma Q + \frac{aQ + bQ}{cQ - dQ} \quad \text{and} \quad Q = \gamma P + \frac{aP + bP}{cP - dP},$$

provided that $c \neq d$. Hence, we have $(P - Q)(\gamma + 1) = 0$. Thus, $P = Q$, which is a contradiction. The proof of Theorem 2.1 is now complete. $\square$

Theorem 2.2. *If k is an odd positive integer, $\gamma > 1,$ and $b > a$, then the difference equation (1.1) has no positive solution of prime period two.*

Proof. Assume that there exists a distinctive positive solution

$$..., P, Q, P, Q, ...$$

of prime period two of Eq. (1.1). If k is odd, then $x_{n+1} = x_{n-k}$. It follows from the difference equation (1.1) that

$$P = \gamma P + \frac{aQ + bP}{cQ - dP} \quad \text{and} \quad Q = \gamma Q + \frac{aP + bQ}{cP - dQ}.$$

Consequently, we obtain:

$$cPQ - dP^2 = \gamma cPQ - \gamma dP^2 + aQ + bP, \tag{2.1}$$

and

$$cPQ - dQ^2 = \gamma cPQ - \gamma dQ^2 + aP + bQ. \tag{2.2}$$

By subtracting, we deduce:

$$P + Q = \frac{b - a}{d\,(\gamma - 1)}, \tag{2.3}$$

while by adding we obtain:

$$PQ = -\ \frac{a\,(b - a)}{d\,(c + d)\,(\gamma - 1)^2}. \tag{2.4}$$

Since $\gamma > 1$ and $b > a$, then PQ is negative. But P and Q are both positive, and we have a contradiction. Therefore, the proof of Theorem 2.2 is complete. □

Theorem 2.3. *If k is an odd positive integer, then the necessary and sufficient condition for the difference equation (1.1) to have a positive prime period two solution is that the inequality*

$$(c + d)\,(a - b) > 4ad \tag{2.5}$$

is valid, provided that $a > b$ *and* $0 < \gamma < 1$.

Proof. First, suppose that there exists a positive prime period two solution

$$..., P, Q, P, Q, ...$$

of the difference equation (1.1). If k is odd, then $x_{n+1} = x_{n-k}$. We shall prove that the condition (2.5) holds. It follows from the difference equation (1.1) that

$$P = \gamma P + \frac{aQ + bP}{cQ - dP} \quad \text{and} \quad Q = \gamma Q + \frac{aP + bQ}{cP - dQ}.$$

Consequently, we have,

$$cPQ - dP^2 = \gamma cPQ - \gamma dP^2 + aQ + bP, \tag{2.6}$$

and

$$cPQ - dQ^2 = \gamma cPQ - \gamma dQ^2 + aP + bQ. \tag{2.7}$$

By subtracting (2.6) from (2.7), we deduce:

$$P + Q = \frac{a - b}{d(1 - \gamma)}. \tag{2.8}$$

while, by adding (2.6) and (2.7), we have

$$PQ = \frac{a(a - b)}{d(c + d)(1 - \gamma)^2}, \tag{2.9}$$

where $a > b$ and $0 < \gamma < 1$. Assume that P and Q are two positive distinct real roots of the quadratic equation,

$$t^2 - (P + Q)t + PQ = 0. \tag{2.10}$$

Thus, we deduce:

$$\left(\frac{a - b}{d(1 - \gamma)}\right)^2 > \frac{4a(a - b)}{d(c + d)(1 - \gamma)^2}. \tag{2.11}$$

From (2.11), we obtain the condition (2.5). Thus, the necessary condition is satisfied. Conversely, suppose that the condition (2.5) is valid. Then, we deduce immediately from (2.5) that the inequality (2.11) holds.

Consequently, there exist two positive distinct real numbers P and Q such that

$$P = \frac{(a-b)+\beta}{2d\,(1-\gamma)} \quad \text{and} \quad Q = \frac{(a-b)-\beta}{2d\,(1-\gamma)}\,, \tag{2.12}$$

where $\beta = \sqrt{(a-b)^2 - 4ad\,(a-b)\,/\,(c+d)}$. Thus, P and Q represent two positive distinct real roots of the quadratic equation (2.10). Now, we prove that P and Q form a positive prime period two solution of the difference equation (1.1). To this end, we assume that

$$x_{-k} = P, \quad x_{-k+1} = Q, ..., \quad x_{-1} = P, \quad \text{and} \quad x_0 = Q.$$

We wish to show that

$$x_1 = P \quad \text{and} \quad x_2 = Q.$$

To this end, we deduce from the difference equation (1.1) that

$$x_1 = \gamma x_{-k} + \frac{ax_0 + bx_{-k}}{cx_0 - dx_{-k}} = \gamma P + \frac{aQ+bP}{cQ-dP}. \tag{2.13}$$

Thus, we deduce from (2.12) and (2.13) that

$$\begin{aligned} x_1 - P &= \frac{aQ+bP}{cQ-dP} - (1-\gamma)\,P \\ &= \frac{aQ+bP-c\,(1-\gamma)\,PQ + d\,(1-\gamma)\,P^2}{cQ-dP} = \\ &\frac{a\left[\frac{(a-b)-\beta}{2d(1-\gamma)}\right] + b\left[\frac{(a-b)+\beta}{2d(1-\gamma)}\right] - c\,(1-\gamma)\left[\frac{a(a-b)}{d(c+d)(1-\gamma)^2}\right] + d\,(1-\gamma)\left[\frac{(a-b)+\beta}{2d(1-\gamma)}\right]^2}{c\left[\frac{(a-b)-\beta}{2d(1-\gamma)}\right] - d\left[\frac{(a-b)+\beta}{2d(1-\gamma)}\right]}. \end{aligned} \tag{2.14}$$

Multiplying the denominator and numerator of (2.14) by $4d^2(1-\gamma)^2$, we get:

$$(2.15)\quad x_1 - P =$$

$$\frac{4d^2(1-\gamma)^2\left(a\left[\frac{(a-b)-\beta}{2d(1-\gamma)}\right]+b\left[\frac{(a-b)+\beta}{2d(1-\gamma)}\right]-c(1-\gamma)\left[\frac{a(a-b)}{d(c+d)(1-\gamma)^2}\right]\right)}{4d^2(1-\gamma)^2\left(c\left[\frac{(a-b)-\beta}{2d(1-\gamma)}\right]-d\left[\frac{(a-b)+\beta}{2d(1-\gamma)}\right]\right)}$$

$$+\frac{4d^2(1-\gamma)^2\left(d(1-\gamma)\left[\frac{(a-b)+\beta}{2d(1-\gamma)}\right]^2\right)}{4d^2(1-\gamma)^2\left(c\left[\frac{(a-b)-\beta}{2d(1-\gamma)}\right]-d\left[\frac{(a-b)+\beta}{2d(1-\gamma)}\right]\right)}$$

$$=\frac{2ad(1-\gamma)[(a-b)-\beta]+2bd(1-\gamma)[(a-b)+\beta]-4cd(1-\gamma)\left[\frac{a(a-b)}{(c+d)}\right]}{2cd(1-\gamma)[(a-b)-\beta]-2d^2(1-\gamma)[(a-b)+\beta]}$$

$$+\frac{d(1-\gamma)[(a-b)+\beta]^2}{2cd(1-\gamma)[(a-b)-\beta]-2d^2(1-\gamma)[(a-b)+\beta]}$$

$$=\frac{2a[(a-b)-\beta]+2b[(a-b)+\beta]-4c\left[\frac{a(a-b)}{(c+d)}\right]+[(a-b)+\beta]^2}{2[c[(a-b)-\beta]-d[(a-b)+\beta]]}$$

$$=\frac{2(a-b)(a+b)-2(a-b)\beta-4c\left[\frac{a(a-b)}{(c+d)}\right]+(a-b)^2+2(a-b)\beta+\beta^2}{2[c[(a-b)-\beta]-d[(a-b)+\beta]]}$$

$$=\frac{2\left(a^2-b^2\right)-4c\left[\frac{a(a-b)}{(c+d)}\right]+(a-b)^2+\beta^2}{2[c[(a-b)-\beta]-d[(a-b)+\beta]]}$$

$$=\frac{2\left(a^2-b^2\right)-4c\left[\frac{a(a-b)}{(c+d)}\right]+(a-b)^2+(a-b)^2-\frac{4ad(a-b)}{c+d}}{2[c[(a-b)-\beta]-d[(a-b)+\beta]]}=0.$$

Thus, $x_1 = P$. Similarly, we can show,

$$x_2=\gamma x_{1-k}+\frac{ax_1+bx_{1-k}}{cx_1-dx_{1-k}}=\gamma Q+\frac{aP+bQ}{cP-dQ}=Q.$$

Using the mathematical induction, we have,

$$x_n = P \quad \text{and} \quad x_{n+1} = Q, \quad \text{for all } n \ge -k.$$

Thus, the difference equation (1.1) has a positive prime period two solution,

$$..., P, Q, P, Q, ...$$

Hence, the proof is now complete. □

3. Local stability of the equilibrium point

Here, we study the local stability character of the solutions of the difference equation (1.1). The equilibrium points of the difference equation (1.1) are given by the relation

$$\widetilde{x} = \gamma\widetilde{x} + \frac{a\widetilde{x} + b\widetilde{x}}{c\widetilde{x} - d\widetilde{x}}. \tag{3.1}$$

If $(1-\gamma)(c-d) > 0$, then the only positive equilibrium point $\widetilde{x}$ of the difference equation (1.1) is given by

$$\widetilde{x} = \frac{a+b}{(1-\gamma)(c-d)}, \tag{3.2}$$

where $0 < \gamma < 1$ and $c > d$. Let $F : (0,\infty)^{k+1} \longrightarrow (0,\infty)$ be a continuous function defined by

$$F(u_0, u_1) = \gamma u_1 + \frac{au_0 + bu_1}{cu_0 - du_1}, \tag{3.3}$$

provided that $cu_0 \neq du_1$. Therefore,

$$\frac{\partial F(u_0,u_1)}{\partial u_0} = -\frac{(ad+bc)\,u_1}{(cu_0 - du_1)^2} \quad \text{and} \quad \frac{\partial F(u_0,u_1)}{\partial u_1} = \gamma + \frac{(ad+bc)\,u_0}{(cu_0 - du_1)^2}. \tag{3.4}$$

Then, we see that

$$\frac{\partial F(\widetilde{x},\widetilde{x})}{\partial u_0} = -\frac{(1-\gamma)(ad+bc)}{(a+b)(c-d)} = \rho_0$$

and

$$\frac{\partial F(\widetilde{x},\widetilde{x})}{\partial u_1} = \gamma + \frac{(1-\gamma)(ad+bc)}{(a+b)(c-d)} = \rho_1.$$

Then, the linearized equation of the difference equation (1.1) about $\widetilde{x}$ is:

$$y_{n+1} - \rho_0\, y_n - \rho_1\, y_{n-k} = 0. \tag{3.5}$$

Theorem 3.1. *Assume that* $0 < \gamma < 1$, $c > d$ *and*

$$2(1-\gamma)(ad+bc) + \gamma(a+b)(c-d) < (a+b)(c-d). \tag{3.6}$$

Then, the equilibrium point $\widetilde{x}$ *of the difference equation* (1.1) *is locally asymptotically stable.*

Proof. From (3.5), we get:

$$\begin{aligned}
|\rho_0| + |\rho_1| &= \left| -\frac{(1-\gamma)(ad+bc)}{(a+b)(c-d)} \right| + \left| \gamma + \frac{(1-\gamma)(ad+bc)}{(a+b)(c-d)} \right| \\
&= \frac{(1-\gamma)(ad+bc)}{(a+b)(c-d)} + \gamma + \frac{(1-\gamma)(ad+bc)}{(a+b)(c-d)} \\
&= \frac{2(1-\gamma)(ad+bc) + \gamma(a+b)(c-d)}{(a+b)(c-d)} \\
&= \frac{2(1-\gamma)(ad+bc) + \gamma(a+b)(c-d)}{(a+b)(c-d)}. \qquad (3.7)
\end{aligned}$$

From (3.6) and (3.7), we deduce that

$$|\rho_0| + |\rho_1| < 1. \qquad (3.8)$$

It is followed by Theorem 1.1 that Eq. (1.1) is locally asymptotically stable. Thus, the proof of Theorem 3.1 is now complete. □

4. Global attractor of the equilibrium point

Here, we investigate the global attractivety character of the solutions of the difference equation (1.1) .

Theorem 4.1. *The equilibrium point $\widetilde{x}$ of the difference equation* (1.1) *is a global attractor if* $0 < \gamma < 1$.

Proof. By using (3.4), we can see that the function $F(u_0, u_1)$ which is defined by (3.3) is decreasing in u_0 and increasing in u_1. Suppose that (m, M) is a solution of the system

$$m = F(M, m) \quad \text{and} \quad M = F(m, M). \qquad (4.1)$$

Then, we get:

$$\begin{aligned}
m &= F(M, m) = \gamma m + \frac{aM + bm}{cM - dm}, \\
M &= F(m, M) = \gamma M + \frac{am + bM}{cm - dM},
\end{aligned}$$

and we have,

$$\frac{aM + bm}{m(cM - dm)} = (1-\gamma) \quad \text{and} \quad \frac{am + bM}{M(cm - dM)} = (1-\gamma). \qquad (4.2)$$

From (4.2) we deduce that $M = m$. It follows by Theorem 1.3 that $\widetilde{x}$ is a global attractor of the difference equation (1.1). Thus, the proof of Theorem 4.1 is now complete. $\square$

References

[1] M.T. Aboutaleb, M.A. El-Sayed and A. E. Hamza, Stability of the recursive sequence $x_{n+1} = (\alpha - \beta x_n)/(\gamma + x_{n-1})$, J. Math. Anal. Appl. **261** (2001) 126–133.

[2] R. Agarwal, Difference Equations and Inequalities. Theory, Methods and Applications, Marcel Dekker Inc., New York, 1992.

[3] A.M. Amleh, E.A. Grove, G.Ladas and D.A. Georgiou, On the recursive sequence $x_{n+1} = \alpha + (x_{n-1}/x_n)$, J. Math. Anal. Appl. **233** (1999) 790–798.

[4] C.W. Clark, A delayed recruitment model of population dynamics with an application to baleen whale populations, J. Math. Biol. **3** (1976) 381-391.

[5] R. Devault, W. Kosmala, G. Ladas and S.W. Schultz, Global behavior of $y_{n+1} = (p + y_{n-k})/(qy_n + y_{n-k})$, Nonlinear Analysis, **47** (2001) 4743–4751.

[6] R. Devault, G. Ladas and S.W. Schultz, On the recursive sequence $x_{n+1} = \alpha + (x_n/x_{n-1})$, Proc. Amer. Math. Soc. **126**(11) (1998) 3257–3261.

[7] R. Devault and S.W. Schultz, On the dynamics of $x_{n+1} = (\beta x_n + \gamma x_{n-1})/(Bx_n + Dx_{n-2})$, Comm. Appl. Nonlinear Analysis **12** (2005) 35-40.

[8] E.M. Elabbasy, H.El-Metwally and E.M. Elsayed, On the difference equation $x_{n+1} = ax_n - bx_n/(cx_n - dx_{n-1})$, Advances in Difference Equations, Volume 2006, Article ID 82579, pages 1-10, doi: 10.1155/2006/82579.

[9] H. El-Metwally, E.A. Grove and G. Ladas, A global convergence result with applications to periodic solutions, J. Math. Anal. Appl. **245** (2000) 161–170.

[10] H. El-Metwally, G. Ladas, E.A. Grove and H.D. Voulov, On the global attractivity and the periodic character of some difference equations, J. Difference Equations and Appl. **7** (2001) 837–850.

[11] H.A. El-Morshedy, New explicit global asymptotic stability criteria for higher order difference equations, J. Math. Anal. Appl. **336** (2007), 262-276.

[12] H.M. EL- Owaidy, A.M. Ahmed and M.S. Mousa, On asymptotic behavior of the difference equation $x_{n+1} = \alpha + (x_{n-1}^p/x_n^p)$, J. Appl. Math. & Computing **12** (2003) 31-37.

[13] H.M. EL- Owaidy, A.M. Ahmed and Z. Elsady, Global attractivity of the recursive sequence $x_{n+1} = (\alpha - \beta x_{n-k})/(\gamma + x_n)$, J. Appl. Math. & Computing **16** (2004) 243-249.

[14] C.H. Gibbons, M.R.S. Kulenovic and G. Ladas, On the recursive sequence $x_{n+1} = (\alpha + \beta x_{n-1})/(\gamma + x_n)$, Math. Sci. Res. Hot-Line **4** (2) (2000) 1–11.

[15] E.A. Grove and G. Ladas, Periodicities in nonlinear difference equations, Vol. 4, Chapman & Hall / CRC, 2005.

[16] G. Karakostas, Convergence of a difference equation via the full limiting sequences method, Diff. Equations and Dynamical. System **1** (1993) 289–294.

[17] G. Karakostas and S. Stevic$'$, On the recursive sequences $x_{n+1} = A + f(x_n, ..., x_{n-k+1})/x_{n-1}$, Comm. Appl. Nonlinear Analysis **11** (2004) 87–100.

[18] V.L. Kocic and G. Ladas, Global behavior of nonlinear difference equations of higher order with applications, Kluwer Academic Publishers, Dordrecht, 1993.

[19] M.R.S. Kulenovic and G. Ladas, Dynamics of second order rational difference equations with open problems and conjectures, Chapman & Hall / CRC, Florida, 2001.

[20] M.R.S. Kulenovic, G. Ladas and W.S. Sizer, On the recursive sequence $x_{n+1} = (\alpha x_n + \beta x_{n-1})/(\gamma x_n + \delta x_{n-1})$, Math. Sci. Res. Hot-Line **2**(5) (1998) 1–16.

[21] S.A. Kuruklis, The asymptotic stability of $x_{n+1} - ax_n + bx_{n-k} = 0$, J. Math. Anal. Appl. **188** (1994) 719–731.

[22] G. Ladas, C.H. Gibbons, M.R.S. Kulenovic and H.D. Voulov, On the trichotomy character of $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + x_n)$, J. Difference Equations and Appl. **8** (2002) 75–92.

[23] G. Ladas, C.H. Gibbons and M.R.S. Kulenovic, On the dynamics of $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + Bx_n)$, Proceeding of the Fifth International Conference on Difference Equations and Applications, Temuco, Chile, Jan. 3-7, 2000, Taylor and Francis, London, (2002) 141–158.

[24] G. Ladas, E. Camouzis and H.D. Voulov, On the dynamic of $x_{n+1} = (\alpha + \gamma x_{n-1} + \delta x_{n-2})/(A + x_{n-2})$, J. Difference Equations and Appl. **9** (2003) 731–738.

[25] G. Ladas, On the rational recursive sequence $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + Bx_n + Cx_{n-1})$, J. Difference Equations and Appl. **1** (1995) 317–321.

[26] W. T. Li and H.R. Sun, Global attractivity in a rational recursive sequence, Dynamical Systems. Appl. **11** (2002) 339–346.

[27] S. Stevic′, On the recursive sequences $x_{n+1} = x_{n-1}/g(x_n)$, Taiwanese J. Math. **6** (2002) 405–414.

[28] S. Stevic′, On the recursive sequences $x_{n+1} = g(x_n, x_{n-1})/(A + x_n)$, Appl. Math. Letter **15** (2002) 305–308.

[29] S. Stevic′, On the recursive sequences $x_{n+1} = \alpha + (x_{n-1}^p/x_n^p)$, J. Appl. Math.& Computing **18** (2005) 229–234.

[30] E.M.E. Zayed and M.A. El-Moneam, On the rational recursive sequence $x_{n+1} = (D + \alpha x_n + \beta x_{n-1} + \gamma x_{n-2})/(Ax_n + Bx_{n-1} + Cx_{n-2})$, Comm. Appl. Nonlinear Analysis **12** (2005) 15–28.

[31] E.M.E. Zayed and M.A. El-Moneam, On the rational recursive sequence $x_{n+1} = (\alpha x_n + \beta x_{n-1} + \gamma x_{n-2} + \delta x_{n-3})/(Ax_n + Bx_{n-1} + Cx_{n-2} + Dx_{n-3})$, J. Appl. Math. & Computing **22** (2006) 247-262.

[32] E.M.E. Zayed and M.A. El-Moneam, On the rational recursive sequence $x_{n+1} = \left(A + \sum_{i=0}^{k} \alpha_i x_{n-i}\right) / \left(B + \sum_{i=0}^{k} \beta_i x_{n-i}\right)$, Int. J. Math. & Math. Sci, Volume 2007, Article ID23618, 12 pages, doi: 10.1155/2007/23618.

[33] E.M.E. Zayed and M.A. El-Moneam, On the rational recursive sequence $x_{n+1} = ax_n - bx_n / (cx_n - dx_{n-k})$, Comm. Appl. Nonlinear Analysis 15(2008) 47-57.

[34] E.M.E. Zayed and M.A. El-Moneam, On the rational recursive sequence $x_{n+1} = \left(A + \sum_{i=0}^{k} \alpha_i x_{n-i}\right) / \sum_{i=0}^{k} \beta_i x_{n-i}$, Mathematica Bohemica 133(2008) 225-239.

[35] E.M.E. Zayed and M.A. El-Moneam, On the rational recursive sequence $x_{n+1} = (ax_n + bx_{n-k}) / (cx_n - dx_{n-k})$, Comm. Appl. Nonlinear Analysis 15(2008) 67-76.

[36] E.M.E. Zayed and M.A. El-Moneam, On the rational recursive sequence $x_{n+1} = (\alpha + \beta x_{n-k}) / (\gamma - x_n)$, J. Appl. Math. & Computing, 10.1007/s12190-008-0205-6, in press.
[37] E.M.E. Zayed and M.A. El-Moneam, On the global attractivity of two nonlinear difference equations, J. Math. Sci., to appear.
[38] E.M.E. Zayed and M.A. El-Moneam, On the rational recursive two sequences $x_{n+1} = ax_{n-k} + bx_{n-k} / (cx_n + \delta dx_{n-k})$, Acta Math. Vietnamica, to appear.
[39] E.M.E. Zayed and M.A. El-Moneam, On the rational recursive sequence $x_{n+1} = Ax_n + (\beta x_n + \gamma x_{n-k}) / (Cx_n + Dx_{n-k})$, Comm. Appl. Nonlinear Analysis **16** (2009) 91–106.
[40] E.M.E. Zayed and M.A. El-Moneam, On the rational recursive sequence $x_{n+1} = Ax_n + Bx_{n-k} + (\beta x_n + \gamma x_{n-k}) / (Cx_n + Dx_{n-k})$, Acta Appl. Math. doi 10.1007/s10440-009-9545-y, in press.

E. M. E. Zayed
Mathematics Department, Faculty of Science, Taif University, El-Taif, EL-Hawiyah, P.O. Box 888, Kingdom of Saudi Arabia
Email: emezayed@hotmail.com

M. A. El-Moneam
Mathematics Department, Faculty of Science and Arts, Jazan University, Farasan, Jazan, Kingdom of Saudi Arabia
Email: mabdelmeneam2004@yahoo.com