Bulletin of the Iranian Mathematical Society Vol. 36 *No.* 1 *(2010), pp 149-166.*

ON CHAINS OF CLASSICAL PRIME SUBMODULES AND DIMENSION THEORY OF MODULES[†]

M. BEHBOODI* AND S. H. SHOJAEE

Communicated by Siamak Yassemi

Dedicated to O.A.S. Karamzadeh on the occasion of his 65th birthday

Abstract. We introduce and study the notion of "classical prime dimension" of modules as a new generalization of the notion of "classical Krull dimension" of commutative rings to modules over arbitrary rings.

1. Introduction

The classical Krull dimension of a ring R, denoted by $\dim(R)$, was originally defined to be the supremum of the lengths of all chains of prime ideals in R. Then, in order to distinguish among rings with infinite classical Krull dimension, Krause [12] introduced a refinement of the definition allowing infinite ordinal values (see also [9]). The importance of the classical Krull dimension is that it has provided an invariant with certain good features and with the property that it distinguishes between a prime ring R and a prime factor R/P. In particular, classical Krull dimension provides a basis for proofs via transfinite induction.

MSC(2000): Primary: 16P60; Secondary: 16D80.
Keywords: Classical prime submodule, classical Krull dimension, classical prime dimension.
Received: 2 January 2009, Accepted: 5 April 2009.
† The research of the first author was in part supported by a grant from IPM (No. 87160026).
∗ Corresponding author

Throughout, all rings are associative rings with identity, and all modules are unital left modules. The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets. If N is a submodule (resp. proper submodule) of M, we write $N \leq M$ (resp. $N < M$). We denote the left annihilator of a factor module M/N of M by $(N : M)$. We call M faithful if $(0 : M) = 0$. Also, we denote the set of all prime (two-sided) ideals of R by $\mathrm{Spec}(R)$.

A proper submodule P of M is called a *classical prime submodule* of M if, for all ideals $\mathcal{A}, \mathcal{B} \subseteq R$ and every submodule $N \leq M$, $\mathcal{AB}N \subseteq P$, then either $\mathcal{A}N \subseteq P$ or $\mathcal{B}N \subseteq P$. This notion of classical prime submodule has been extensively studied by the first author in [2,3] (see also, [4,6], in which the notion of "weakly prime submodule" is investigated). Also, a proper submodule P of M is called a *semiprime submodule* of M if, for every ideal $\mathcal{A} \subseteq R$ and every submodule $N \leq M$, $\mathcal{A}^2N \subseteq P$, then $\mathcal{A}N \subseteq P$. An R-module M is called a classical prime (resp. semiprime) module if $(0) < M$ is a classical prime (resp. semiprime) submodule. It is clear that for a submodule $P < M$, M/P is classical prime (resp., semiprime) if and only if P is a classical prime (resp. semiprime) submodule of M. One can easily see that a two-sided ideal I of any ring R is a prime (resp. semiprime) ideal if and only if it is a classical prime (resp. semiprime) submodule of $M = R$. Therefore, in case $M = R$, where R is any commutative ring, classical prime (resp. semiprime) submodules coincide with prime (resp. semiprime) ideals.

Let M be an R-module and let N_1 and N_2 be submodules of M. Then, we say that N_1 is strongly properly contained in N_2, and we write $N_1 \subset_s N_2$, if $N_1 \subset N_2$ and $(N_1 : M) \subset (N_2 : M)$. Also, we say that N_1 is strongly contained in N_2, and we write $N_1 \subseteq_s N_2$ if $N_1 \subset_s N_2$ or $N_1 = N_2$. Clearly, $\subseteq_s$ is an order relation, called *strong containment*, on the set of all submodules of M. In particular, the chain $N_1 \subseteq_s N_2 \subseteq_s N_3 \subseteq_s \cdots$ of submodules of M is called a *strong ascending chain of submodules*. Also, an R-module M is said to satisfy *virtually ascending chain condition* on submodules (or *virtually acc*) if for every strong chain $N_1 \subseteq_s N_2 \subseteq_s N_3 \subseteq_s \cdots$ of submodules of M, there is an integer n such that $N_i = N_n$, for all $i \geq n$ (see [1] for more details on virtual chain conditions of modules).

We recall that a proper submodule P of M is called a *prime submodule* of M if, for every ideal $\mathcal{A} \subseteq R$ and every submodule $N \leq M$, $\mathcal{A}N \subseteq P$, then either $N \subseteq P$ or $\mathcal{A}M \subseteq P$. This notion of prime submodule was first introduced and systematically studied in [8] and recently it has received a good deal of attention from several authors; see for example, [1,5,13-15]. We denote the set of all prime submodules of M by Spec$({}_RM)$.

There is already a generalization of the classical Krull dimension of rings to modules via prime submodules. In fact, the notion of classical Krull dimension of a left R-module M, denoted by cl.k.dim(M), was introduced in [1], as supremum of the lengths of all strong chains of prime submodules of M (see [1, Section 3] for definition of cl.k.dim(M) and more details).

A submodule P of M is called virtually maximal classical prime if P is a classical prime submodule of M and there is no prime submodule Q of M such that $P \subset_s Q$ (see Definition 3.1 for various maximality of submodules). For example, every proper submodule of a homogeneous semisimple module is virtually maximal classical prime. Also, $(0) < \mathbb{Q}$ as $\mathbb{Z}$-submodule is virtually maximal classical prime.

Here, we study the *classical prime dimension* of a module, defined to be the length of the longest strong chain of classical prime submodules. In fact, we denote the set of all classical prime submodules of M by cl.Spec$({}_RM)$, and then we define, by transfinite induction, sets X_α of classical prime submodules of M. To start, let X_{-1} be the empty set. Next, consider an ordinal $\alpha \geq 0$; if X_β has been defined for all ordinals $\beta < \alpha$, then let X_α be the set of those classical prime submodules P in M such that all classical prime submodules strongly properly containing P belong to $\bigcup_{\beta<\alpha} X_\beta$. In particular, X_0 is the set of virtually maximal classical prime submodules of M. One obtains an ascending chain $X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_\alpha \cdots$ of subsets of cl.Spec$({}_RM)$. If some X_γ contains all classical prime submodules of M, then we say that *cl.p.dim*(M) exists, and we set *cl.p.dim*(M)-the *classical prime dimension* of M to be equal to the smallest such γ. We write "*cl.p.dim*$(M) = \gamma$" as an abbreviation for the statement that *cl.p.dim*(M) exists and equals γ (we note that cl.p.dim(M) may be -1; see Section 4).

In Section 2, we show that cl.p.dim(M) exists if and only if M has virtual *acc* on classical prime submodules. Also, if R is a ring for which dim(R) exists, then for each R-module M, cl.p.dim(M) exists and cl.k.dim$(M) \leq$ cl.p.dim$(M) \leq$ dim(R). In particular, if M is a free R-module or R is commutative and M is a faithful finitely generated R-module, then cl.p.dim(M) exists if and only if cl.k.dim(M) exists, if and only if dim(R) exists and, cl.p.dim(M) =cl.k.dim(M) = dim(R), in case one of them exists. In Section 3, we study modules of classical prime dimension 0. In particular, it is shown that over any commutative ring, all co-semisimple modules, as well as all Artinian modules with a classical prime submodule, lie in the class of modules of classical prime dimension zero. In Section 4, we study modules of classical prime dimension -1. Finally, in Section 5 we characterize left bounded prime left Goldie rings (or PI-rings) over which, classical prime dimension and the classical Krull dimension of any module coincide.

2. **Some properties of classical prime dimension**

We begin this section with the following proposition which shows that the classical prime dimension of a ring R as an R-module coincides with its usual classical Krull dimension of R.

Proposition 2.1. *For any ring R, the following statements are equivalent:*

(1) *$dim(R)$ exists.*
(2) *$cl.k.dim({}_RR)$ exists.*
(3) *$cl.p.dim({}_RR)$ exists.*

Moreover, if one of the three exists, then

$$dim(R) = cl.k.dim({}_RR) = cl.p.dim({}_RR).$$

Proof. (1) $\Leftrightarrow$ (2). This follows from [1, Proposition 2.3].
(1) $\Leftrightarrow$ (3). Define the sets X_γ of classical prime left ideals as in the definition of classical prime dimension. It is clear that $\mathrm{Spec}(R) \subseteq \mathrm{cl.Spec}({}_RR)$. If P is a classical prime left ideal of R, then $\mathcal{P} = (P : R)$ is a prime (two-sided) ideal of R such that $\mathcal{P} \subseteq P$ and $\mathcal{P} \not\subset_s P$. It follows that every minimal classical prime left ideal of R is a minimal prime (two-sided) ideal of R. Therefore, if for each ordinal γ, we define $\bar{X}_\gamma := \{\mathcal{P} \in X_\gamma \mid \ \mathcal{P} \textit{ is an ideal of } R\}$, then $Spec(R) = \bar{X}_\gamma$ if and only if $\bar{X}_\gamma$ contains all minimal prime (left) ideals of R, if and only if $cl.Spec({}_RR) = X_\gamma$. Thus, $\mathrm{cl.p.dim}({}_RR)$ exists if and only if $\mathrm{dim}(R)$ exists and $\mathrm{cl.p.dim}({}_RR)=\mathrm{dim}(R)$. □

Let M be an R-module. In [1, Theorem 3.11], it was shown that $\mathrm{cl.k.dim}(M)$ exists if and only if M satisfies virtual *acc* on prime submodules. Here, by the same method we generalize this fact for classical prime dimension of modules.

Lemma 2.2. *Let M be an R-module such that cl.p.dim(M) exists. Then, for any submodule N of M, cl.p.dim(M/N) exists and is not larger than cl.p.dim(M).*

Proof. Let cl.p.dim(M) exist and $N \subseteq P$ be submodules of M. Clearly, P/N is a classical prime submodule of M/N if and only if P is a classical prime submodule of M. Thus, cl.p.dim(M/N) exists and is not larger than cl.p.dim(M). □

Lemma 2.3. *Let M be an R-module for which cl.p.dim(M) exists. If N and K are classical prime submodules of M such that $N \subset_s K$, then*

$$cl.p.dim(M/K) < cl.p.dim(M/N).$$

Proof. Immediate from Lemma 2.2. □

Theorem 2.4. *Let M be an an R-module. Then, cl.p.dim(M) exists if and only if M satisfies virtually acc on classical prime submodules.*

Proof. (⇒). Let cl.p.dim$(M) = \gamma$, where γ is an ordinal number. If $P_1 \subset_s P_2 \subset_s P_3 \subset_s \cdots$ is a strong assenting chain of classical prime submodules of M, then by lemmas 2.2 and 2.3, we have

$$\cdots < cl.p.dim(M/P_3) < cl.p.dim(M/P_2) < cl.p.dim(M/P_1) \leq \gamma,$$

which is impossible. Therefore, M has virtually *acc* on classical prime submodules.
(⇐). Define the sets X_γ of classical prime submodules as in the definition of classical prime dimension. Since there is a bound for the cardinalities of these sets (e.g., $2^{card(M)}$), the transfinite chain $X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots$ cannot be properly increasing forever. Hence, there exists an ordinal γ such that $X_\gamma = X_{\gamma+1}$. If cl.p.dim(M) dose not exist, then X_γ dose not contain all the classical prime submodules of M. Using the virtual *acc* on classical prime submodules, there is a classical prime submodule P of M virtually maximal with respect to the property $P \notin X_\gamma$. Hence, all classical prime submodules strongly properly containing P lie in X_γ. But then $P \in X_{\gamma+1} = X_\gamma$, which is a contradiction. □

Corollary 2.5. *Let M be an R-module for which cl.p.dim(M) exists. Then, cl.k.dim(M) also exists and cl.k.dim$(M) \leq$ cl.p.dim(M).*

Proof. Immediate from Theorem 2.4, [1, Theorem 3.11] and the fact that every prime submodule of M is a classical prime submodule. □

The following example shows that, in general, the inequality in Corollary 2.5 is not an equality.

Example 2.6. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_p \oplus \mathbb{Q}$, where p is a prime number. One can easily see that the zero submodule of M is a classical prime submodule, but it is not a prime submodule. Moreover, cl.Spec$({}_RM) = \{(0), \mathbb{Z}_p \oplus (0), (0) \oplus \mathbb{Q}\}$ and Spec$({}_RM) = \{\mathbb{Z}_p \oplus (0), (0) \oplus \mathbb{Q}\}$. It follows that cl.p.dim$(M) = 1$ and cl.k.dim$(M) = 0$.

Theorem 2.7. *Let R be a ring for which dim(R) exists. Then for each R-module M, cl.p.dim(M) exists and*

$$cl.k.dim(M) \leq cl.p.dim(M) \leq dim(R).$$

Proof. Let dim(R) exist. Then, by [9, Exercise 14.A(b)], R satisfies the *acc* on prime ideals. It follows that for each R-module M, cl.Spec$({}_RM)$ satisfies the virtually *acc*. Thus, by Theorem 2.4, cl.p.dim(M) exists. Now, we define the sets X_γ of classical prime submodules as in the definition of classical prime dimension. Also, we define,

$$Y_\gamma := \{\mathcal{P} \in Spec(R) \mid \mathcal{P} = (P : M),\ for\ some\ P \in X_\gamma\}.$$

It is clear that if $X_\alpha \subset X_\beta$, then $Y_\alpha \subset Y_\beta$. It follows that cl.p.dim(M) exist and cl.p.dim$(M) \leq$ dim(R). Also, by Corollary 2.5, cl.k.dim$(M) \leq$ cl.p.dim(M). $\square$

Let R be any ring. In [1, Proposition 3.8], it was shown that for each free R-module F, cl.k.dim(F) exists if and only if dim(R) exists and also, if one of them exists, then cl.k.dim(F) =dim(R). Also, in [7, Theorem 2.3], it was shown that for each faithful finitely generated module M over a commutative ring R, cl.k.dim(M) exists if and only if dim(R) exists and also, if one of them exists, then cl.k.dim(M) =dim(R). Thus, by using these facts and Theorem 2.7, we have the following interesting result.

Proposition 2.8. *Let R be a ring. If M is a free R-module or R is commutative and M is a faithful finitely generated R-module, then the following statements are equivalent:*

(1) *dim(R) exists.*
(2) *cl.k.dim(M) exists.*
(3) *cl.p.dim(M) exists.*

Moreover, if one of the three exists, then $dim(R) = cl.k.dim(M) = cl.p.dim(M)$.

Corollary 2.9. *Let R be a commutative domain, and let M be a nonzero finitely generated projective R-module. Then, the following statements are equivalent:*

(1) $dim(R)$ *exists.*
(2) $cl.k.dim(M)$ *exists.*
(3) $cl.p.dim(M)$ *exists.*

Moreover, if one of the three exists, then $dim(R) = cl.k.dim(M) = cl.p.dim(M)$.

Proof. There is a free R-module F and R-module K such that $F \cong K \oplus M$. Since R is a prime ring, then F is also a prime module. Thus, $Ann(M) = Ann(F) = 0$ and so M is a faithful R-module. Now, apply Proposition 2.8. □

3. On modules of classical prime dimension 0

Here, we introduce various maximality conditions on submodules of a module M which, for $M = R$ and R commutative, are equivalent to notion of maximal ideal in R. Then, we apply these conditions to study modules of classical prime dimension 0. In particular, we will show that over any commutative ring, all co-semisimple modules as well as all Artinian modules with a classical prime submodule lie in the class of modules with classical prime dimension zero.

Definition 3.1. Let R be a ring and M be an R-module. A submodule P of M is called:

− *virtually maximal* if the factor module M/P is a homogeneous semisimple module.
− *maximal prime* if P is a prime submodule of M and there is no prime submodule Q of M such that $P \subset Q$.
− *virtually maximal prime* if P is a prime submodule of M and there is no prime submodule Q of M such that $P \subset_s Q$.
− *maximal classical prime* if P is a classical prime submodule of M and there is no classical prime submodule Q of M such that $P \subset Q$.

− *virtually maximal classical prime* if P is a classical prime submodule of M and there is no classical prime submodule Q of M such that $P \subset_s Q$.
− *maximal semiprime* if P is a semiprime submodule of M and there is no semiprime submodule Q of M such that $P \subset Q$.
− *virtually maximal semiprime* if P is a semiprime submodule of M and there is no semiprime submodule Q of M such that $P \subset_s Q$.

Using the above definitions we have the following evident proposition.

Proposition 3.2. *Let M be an R-module. Then,*

(1) *$cl.p.dim(M) = 0$ if and only if $cl.Spec({}_RM) \neq \emptyset$ and every classical prime submodule of M is a virtually maximal classical prime submodule.*
(2) *$cl.k.dim(M) = 0$ if and only if $Spec({}_RM) \neq \emptyset$ and every prime submodule of M is a virtually maximal prime submodule.*

The following result shows that for any module M the notion of maximal semiprime submodule and maximal classical prime submodule coincide.

Proposition 3.3. *Let M be an R-module. Then, every maximal semiprime submodule of M is a maximal classical prime submodule of M.*

Proof. Let P be a maximal semiprime submodule of M. Let $\overline{M} := M/P$. Then, the zero submodule of $\overline{M}$ is the only semiprime submodule of $\overline{M}$. We claim that it is the only prime (classical prime) submodule of $\overline{M}$. It suffices to show that $\overline{M}$ is a prime module. To see this, let $rRm = 0$, $0 \neq m \in \overline{M}$, $r \in R$ and $r\overline{M} \neq 0$. Thus, if $N := \{m \in \overline{M} \ : \ rRm = 0\}$, then $0 \subset N \subset \overline{M}$. We claim that N is a semiprime submodule of $\overline{M}$. Let $aRa(Rm) \subseteq N$, $a \in R$, $m \in \overline{M}$; i.e., $rRaRa(Rm) = 0$. Then, $(rRa)R(rRa)(Rm) = 0$. Since $\overline{M}$ is semiprime, then $rRa(Rm) = 0$, and so $aRm \subseteq N$, which means that N is a semiprime submodule of $\overline{M}$, which is a contradiction. □

A prime ring R is called left bounded if for each regular element c in R there exists an ideal A of R and a regular element d such that $Rd \subseteq A \subseteq Rc$. A general ring R is called left fully bounded if every prime homomorphic image of R is left bounded. A ring R is called a left FBN-ring if R is left fully bounded and left Noetherian. It is well known that if R is a PI-ring (ring with polynomial identity) and $\mathcal{P}$ is a prime ideal of R, then the ring $R/\mathcal{P}$ is (left and right) bounded and (left and right) Goldie [16, 13.6.6].

Lemma 3.4. *Let R be a PI-ring (or an FBN-ring) and let M be an R-module in which every proper submodule is contained in a maximal submodule. Then, for each proper submodule P of M, the following statements are equivalent:*

(1) *P is a virtually maximal submodule.*

(2) *P is a virtually maximal prime submodule.*

(3) *P is a virtually maximal classical prime submodule.*

(4) *P is a virtually maximal semiprime submodule.*

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4) is clear.
(4)$\Rightarrow$(1). Assume that P is a virtually maximal semiprime submodule of M. Then, there exists a maximal submodule Q of M such that $P \subseteq Q$. It follows that $(P : M) = (Q : M) = \mathcal{P}$ and $\overline{M} = M/Q$ is a simple $R/\mathcal{P}$-module. Since R is a PI-ring (or an FBN-ring), then the ring $R/\mathcal{P}$ is a left bounded, left Goldie ring. Now, by [9, Proposition 9.7] we have that $R/\mathcal{P}$ embeds as a left R-module in a finite direct sum of copies of $\overline{M}$. Thus, $R/\mathcal{P}$ is a left Artinian ring, and hence $R/\mathcal{P}$ is simple Artinian. Therefore, the left $R/\mathcal{P}$-module M/P is a direct sum of isomorphic simple modules. It follows that M/P is a homogeneous semisimple R-module; i.e., P is a virtually maximal submodule of M. $\square$

Clearly, in any commutative rings R every proper ideal is contained in a maximal ideal, and $\dim(R) = 0$ if and only if every prime ideal of R is maximal. Next, we generalize this fact to modules over a PI-ring (or an FBN-ring).

Theorem 3.5. *Let R be a PI-ring (or an FBN-ring), and let M be an R-module in which every proper submodule is contained in a maximal submodule. Then, the following statements are equivalent:*

(1) *$cl.p.dim(M) = 0$.*

(2) *$cl.k.dim(M) = 0$.*

(3) *every prime submodule of M is a virtually maximal submodule.*

(4) *every classical prime submodule of M is a virtually maximal submodule.*

(5) *every semiprime submodule of M is a virtually maximal submodule.*

Proof. Immediate from Proposition 3.2 and Lemma 3.4. $\square$

We recall that if U and M are $R-$modules, then following Azumaya U is called $M-$injective if for any submodule N of M, each homomorphism $N \longrightarrow U$ can be extended to $M \longrightarrow U$, and an R-module M is called co-semisimple if every simple module is $M-$injective (see, for example, [5,17,18], for several characterization). Every semisimple module is of course co-semisimple.

Corollary 3.6. *Let M be a co-semisimple module over a commutative ring R. Then, cl.p.dim$(M) = 0$ and every classical prime submodule of M is virtually maximal.*

Proof. Let R be a commutative ring and M be a co-semisimple module. By [18, 23.1], every proper submodule of M is an intersection of maximal submodules of M, and hence every proper submodule is contained in a maximal submodule. On the other hand, by [1, Proposition 3.1], cl.k.dim$(M) = 0$ and every prime submodule of M is virtually maximal. Now, apply Theorem 3.5. $\square$

Corollary 3.7. *Let M be a semisimple module over a PI-ring (or an FBN-ring) R. Then, cl.p.dim$(M) = 0$ and every classical prime submodule of M is virtually maximal.*

Proof. Let R be a PI-ring (or an FBN-ring) and M be a semisimple module. Clearly, every proper submodule is contained in a maximal submodule. On the other hand, by [1, Proposition 3.1], cl.k.dim$(M) = 0$ and every prime submodule of M is virtually maximal. Now, apply Theorem 3.5. $\square$

Also, in [1, Corollary 1.6], it is shown that every prime submodule of an Artinian module M over a PI-ring (or an FBN-ring) is virtually maximal, and hence if Spec$({}_RM) \neq \emptyset$, then cl.k.dim$(M) = 0$. In the following theorem we show that this fact is also true for classical prime submodules of Artinian modules over commutative rings.

Theorem 3.8. *Let R be a commutative ring, and let M be an Artinian R-module. Then, every classical prime submodule of M is virtually maximal. Consequently, if cl.Spec$(_RM) \neq \emptyset$, then cl.p.dim$(M) = 0$.*

Proof. By [1, Corollary 1.6], it suffices to show that if P is a classical prime submodule of M, then P is a prime submodule. Let P be a classical prime submodule of M. Since M is Artinian, then $\overline{M} := M/P$ is an Artinian classical prime R-module. Thus, $Ann(\overline{M}) = \bigcap_{0\neq m\in\overline{M}} Ann(m)$ and by [2, Proposition 1.1], $\{Ann(m) \mid 0 \neq m \in \overline{M}\}$ is a chain of prime ideal of R. Clearly, for each $0 \neq m \in \overline{M}$, Rm is also an Artinian classical prime R-module. Since $Rm \cong R/Ann(m)$ and R is commutative, then Rm is an Artinian prime module. Now, by [5, Corollary 1.9], Rm is a homogenous semisimple R-module; i.e., $Ann(m) = \mathcal{P}$ is a maximal ideal. It follows that $\{Ann(m) \mid 0 \neq m \in \overline{M}\}$ is singleton. Thus, $Ann(\overline{M}) = Ann(m)$ for each $0 \neq m \in \overline{M}$; i.e, $\overline{M}$ is a prime R-module. Therefore, P is a prime submodule of M. $\square$

4. On modules of classical prime dimension -1

Unlike rings with unity, not every R-module contains a prime (classical prime) submodule; for example, any torsion divisible module over a commutative domain dose not contain a (classical) prime submodule (see [6] and [14]). Therefore, an R-module M does not contain a prime submodule (resp. classical prime submodule) if and only if *cl.k.dim*$(M) = -1$ (resp. cl.p.dim$(M) = -1$). Here, we investigate modules of classical prime dimension -1.

Let M be an R-module. Since Spec$(_RM) \subseteq$cl.Spec$(_RM)$, we infer that if cl.p.dim$(M) = -1$, then cl.k.dim$(M) = -1$. However, we have not found any R-module M, where cl.k.dim$(M) = -1$ and cl.p.dim$(M) \neq -1$. The lack of such counterexamples together with the fact that cl.k.dim$(M) = -1$ if and only if cl.p.dim$(M) = -1$, for modules over a large class of rings (we will shortly present these rings), motivates the following conjecture.

Conjecture 4.1. *An R-module M has a classical prime submodule if and only if it has a prime submodule (i.e., cl.k.dim$(M) = -1$ if and only if cl.p.dim$(M) = -1$).*

The following lemma is crucial for our investigation.

Lemma 4.2. *Let R be a ring with dcc on prime ideals and M be an R-module. Then, M has a prime submodule if and only if it has a classical prime submodule.*

Proof. Let M have no prime submodules, and let K be a classical prime submodule of M. For creation of a contradiction, it suffices to show that M/K has a prime submodule. Since, by [2, Proposition 1.1], $(K : M)$ is a prime ideal, without loss of generality we may assume that M is a faithful classical prime R-module and R is a prime ring. Again, by [2, Proposition 1.1], the set $T := \{Ann(Rm) : 0 \neq m \in M\}$ is a chain of prime ideals of R and $Ann(M) = \bigcap_{0 \neq m \in M} Ann(Rm)$. Since R is a ring with the *dcc* on prime ideals, we infer that T has *dcc*; i.e., there exists $0 \neq m \in M$ such that $Ann(Rm) = Ann(M) = 0$. If T is a singleton, then for any $0 \neq m \in M$, $Ann(Rm) = Ann(M) = 0$; i.e., M is a prime module and we are through. Thus, we may assume that T contains a nonzero element. It follows that $N = \{m \in M : 0 \neq Ann(Rm) \in T\}$ is a proper nonzero submodule of M, and there is a nonzero prime ideal $\mathcal{P}$ in T such that it is a minimal element among the nonzero elements of T (since T has *dcc*). Clearly, $\mathcal{P} = Ann(N)$ and we claim that N is a prime submodule of M. To see this, let $\mathcal{A}Rm \subseteq N$, where $m \in M$ and $\mathcal{A}$ is an ideal of R. We must show that either $m \in N$ or $\mathcal{A}M \subseteq N$. Thus, we may assume that $\mathcal{A} \neq 0$; i.e., $\mathcal{P}\mathcal{A}(Rm) = 0$. Since R is a prime ring, we infer that $\mathcal{P}\mathcal{A} \neq 0$; i.e., $Ann(Rm) \neq 0$, which means that $m \in N$ and the proof is complete. □

Corollary 4.3. *Let R be a ring with $dim(R) < \infty$, and let M be an R-module. Then, $cl.k.dim(M) = -1$ if and only if $cl.p.dim(M) = -1$.*

Proof. Since $dim(R) < \infty$, we infer that R has both *acc* and *dcc* on prime ideals. Now, by Lemma 4.2, the proof is complete. □

It is well known that any commutative Noetherian ring satisfies *dcc* on prime ideas. This is also true for any left Noetherian PI-ring (see, for example, [16, 13.7.15]). Thus, we have the following result.

Corollary 4.4. *Let R be a left Noetherian PI-ring, and let M be an R-module. Then, $cl.k.dim(M) = -1$ if and only if $cl.p.dim(M) = -1$.*

Next, we give more information about modules of classical prime dimension -1.

Proposition 4.5. *For modules over any ring R, the following properties hold:*

(i) *All direct sums of modules of classical prime dimension* -1 *have classical prime dimension* -1.

(ii) *All direct summands of modules of classical prime dimension* -1 *have classical prime dimension* -1.

(iii) *All factor modules of modules of classical prime dimension* -1 *have classical prime dimension* -1.

(iv) *If N is a submodule of M and cl.p.dim*(N) =*cl.p.dim*$(M/N) = -1$, *then cl.p.dim*$(M) = -1$.

(v) *The statements* (i) – (iv) *are also true when we replace "classical prime dimension" with "classical Krull dimension"* (*see also* [14, *Proposition*1.7]).

Proof. Straight forward. □

Remark 4.6. A submodule of a module of classical prime dimension (resp. classical Krull dimension) -1 does not need to be a module of classical prime dimension (resp. classical Krull dimension) -1. To see this, consider the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$, where p is a prime number. One can easily see that $\mathbb{Z}_{p^\infty}$ has no classical prime submodule; i.e., cl.p.dim$(\mathbb{Z}_{p^\infty}) =$ cl.k.dim$(\mathbb{Z}_{p^\infty}) = -1$. But, every proper submodule of $\mathbb{Z}_{p^\infty}$ has a prime (maximal) submodule. In fact, any finitely generated submodule of any module of classical prime dimension (classical Krull dimension) -1 has a prime (maximal) submodule. Also, a direct product of modules of classical prime dimension (resp. classical Krull dimension) -1 does not need to be a module of classical prime dimension (resp. classical Krull dimension) -1 (see, [14, Proposition 1.8]).

Let R be a ring and $\mathcal{P}$ be a maximal ideal of R. One can easily see that if M is an R-module such that $\mathcal{P}M \neq M$, then $\mathcal{P}M$ is a (classical) prime submodule of M. Thus, we have the following evident lemma.

Lemma 4.7. *For any R-module M, the following statements are equivalent:*

(1) *$cl.p.dim(M) = -1$.*
(2) *for every $P \in Spec(R)$, either $PM = M$ or $cl.p.dim(M/PM) = -1$ as R/P-module.*

Moreover, (1) $\Leftrightarrow$ (2) *when we replace "classical prime dimension" with "classical Krull dimension".*

Clearly, over a simple ring R, the zero module is the only R-module of classical prime dimension (classical Krull dimension) -1. In the following proposition, we give a characterization for modules of classical prime dimension (classical Krull dimension) -1 over zero-dimensional non-simple rings.

Proposition 4.8. *Let R be a ring with $dim(R) = 0$ and M be an R-module. Then, the following statement are equivalent:*

(1) *$cl.k.dim(M) = -1$.*
(2) *$cl.p.dim(M) = -1$.*
(3) *for every $\mathcal{P} \in Spec(R)$, $\mathcal{P}M = M$.*

Proof. (1) $\Leftrightarrow$ (2). It follows from Corollary 4.3.
(1) $\Rightarrow$ (3). Let $\mathcal{P} \in Spec(R)$ (i.e., $\mathcal{P}$ is a maximal ideal of R). If $\mathcal{P}M \neq M$, then $\mathcal{P}M$ is a (classical) prime submodule of M, which is a contradiction.
(3) $\Rightarrow$ (1). It follows from Lemma 4.7. □

5. Left bounded prime left Goldie rings over which, classical prime dimension and classical Krull dimension of any module coincide

Let R be a ring with $\dim(R) = 0$. Then, for each R-module M, the classical prime dimension of M and the classical Krull dimension of M coincide. In fact, if $\dim(R) = 0$, then by Theorem 2.7 and Proposition 4.8, for any R-module M, cl.k.dim(M) =cl.p.dim$(M) = -1$ or 0. Here, we characterize left bounded prime left Goldie rings (or PI rings), in which classical prime dimension and the classical Krull dimension of any module coincide.

Let M be an R-module. A proper submodule P of M is called *strongly prime* if:

(i) $\mathcal{P} = (P : M)$ is a prime ideal of R and the ring $R/\mathcal{P}$ is a left Goldie ring, and
(ii) M/P is a torsion-free left $(R/\mathcal{P})$-module (see [15], for more results on strongly prime submodules).

We need the following lemma given in [15].

Lemma 5.1. *([15, Lemma 2.6]). Let R be a ring and $\mathcal{P}$ be a prime ideal such that the ring $R/\mathcal{P}$ is left bounded, left Goldie. Let M be an R-module. Then, the following statements are equivalent for a submodule P of M:*

(1) *P is a prime submodule of M such that $\mathcal{P} = (P : M)$.*
(2) *P is a strongly prime submodule of M such that $\mathcal{P} = (P : M)$.*

Lemma 5.2. *Let R be a prime left Goldie ring, and let Q be the left Goldie quotient ring of R. If R is left bounded, then the zero R-submodule of Q is the only prime submodule of Q.*

Proof. It is clear that the zero submodule of Q is a prime submodule. Let P be a nonzero prime submodule of Q with $(P : Q) = \mathcal{P}$. If $\mathcal{P} \neq 0$, then $\mathcal{P}$ contains a regular element of R. Since Q is divisible, then $\mathcal{P}Q = Q \subseteq P$, which is a contradiction. But, if $\mathcal{P} = 0$, then we claim that $P = 0$, for if not, then by Lemma 5.1, P is a nonzero strongly prime submodule of Q with $(P : Q) = 0$; i.e., Q/P is a torsion-free left R-module. Since $P \neq 0$ and R is a prime ring, then $P \leq_e Q$ (see also [9, Exercise 5A and Proposition 5.6(a)]). Now, by [9, Proposition 7.8(c)], Q/P is a torsion, which is a contradiction. □

Theorem 5.3. *Let R be a left bounded, prime left Goldie ring. Then, the following statements are equivalent:*

(1) *the classical prime dimension and the classical Krull dimension of any R-module coincide.*
(2) *R is simple Artinian.*
(3) *$dim(R) = 0$.*

Proof. (1) ⇒ (2). Let $M = N \oplus Q$, where N is a simple R-module and Q is the left Goldie quotient ring of R; i.e., $Q = E({}_RR)$, the injective hull of ${}_RR$. Clearly, N and Q are prime submodules of M (since $M/N \cong Q$

and $M/Q \cong N$). We claim that $Ann(N) = 0$. Let $Ann(N) \neq 0$. Then, the zero submodule of M is not a prime submodule of M, for otherwise, $Ann(M) = Ann(N) = Ann(Q) = \mathcal{P}$ is a prime ideal of R. Since R is a prime ring and $\mathcal{P} \neq 0$, then $\mathcal{P}$ contains a regular element of R (see [9, Proposition 7.3]) . Since Q is divisible, then $\mathcal{P}Q = Q$, which is a contradiction. Now, assume P is a nonzero prime submodule of M with $(P : M) = \mathcal{P}$. If $\mathcal{P} \neq 0$, then $\mathcal{P}$ contains a regular element of R. Since Q is divisible, then $\mathcal{P}Q = Q$; i.e., $Q \subseteq \mathcal{P}M \subseteq P$, and it follows that $P = Q$ (since P is a proper submodule of M and N is a simple R-module). But, if $\mathcal{P} = 0$, then we claim that $P = N$. First, we show that $N \subseteq P$, for if not, then $N \cap P = 0$; i.e., $P \subseteq Q$ is a nonzero prime submodule of Q, which a contradiction (see Lemma 5.2). Next, we show that $P \cap Q = 0$, for if not, then $0 \neq P \cap Q$ is a prime submodule of Q, which is a contradiction. Therefore, cl.k.dim$(M) = 0$ and so by our hypothesis, cl.p.dim$(M) = 0$. Now, one can easily see that for each $m \in M$, $Ann(Rm)$ is a prime ideal of R (either $Ann(Rm) = 0$ or $Ann(Rm) = Ann(N)$). Thus, by [2, Proposition 1.1], the zero submodule is a classical prime submodule of M. It follows that (0) and Q are two classical prime submodule of M such that $(0) \subset_s Q$, and so cl.p.dim$(M) \geq 1$, which is a contradiction. Thus, $Ann(N) = 0$ and so N is a simple faithful R-module. Now, by [9, Proposition 9.7], ${}_RR$ embeds in some finite direct sum of copies of N. Thus, R is simple Artinian.
(2) $\Rightarrow$ (3) $\Rightarrow$ (1). The proof is clear. □

Corollary 5.4. *Let R be a PI-ring (or an FBN-ring). Then, the following statements are equivalent:*

(1) *the classical prime dimension and the classical Krull dimension of any R-module coincide.*

(2) *for every prime ideal $\mathcal{P}$ of R, the ring $R/\mathcal{P}$ is simple Artinian.*

(3) *$dim(R) = 0$.*

Proof. Let R be ring, over which, the classical prime dimension and the classical Krull dimension of any R-module coincide. Clearly, for every ideal I of R, the ring R/I has also this property. Now, if R is a PI-ring (or an FBN-ring), then for each prime ideal $\mathcal{P}$ of R, the ring $R/\mathcal{P}$ is a left bounded, prime left Goldie ring. Now, apply Theorem 5.3. □

A ring R is called a Max-ring (or a left Max-ring) if every nonzero left R-module has a maximal submodule (see, for example, [10]). In [1, Theorem 5.6], there are several characterizations for PI-rings whose nonzero modules have zero classical Krull dimension. Thus, by [1, Theorem 5.6] and Theorem 2.7, we have the following proposition.

Proposition 5.5. *Let R be a PI-ring. Then, the following statements are equivalent:*

(1) *each nonzero R-module has zero classical prime dimension.*

(2) *each nonzero R-module has zero classical Krull dimension.*

(3) *R is a Max-ring.*

References

[1] M. Behboodi, A generalization of the classical Krull dimension for modules, *J. Algebra.* **305** (2006) 1128-1148.

[2] M. Behboodi, A generalization of Baer's lower nilradical for modules, *J. Algebra and its Appl.* **6** (2) (2007) 337-353.

[3] M. Behboodi, Classical prime submodules, *Ph.D. Thesis, Chamran University, Ahvaz, Iran* (2004).

[4] M. Behboodi, On weakly prime radical of modules and semi-compatible modules, *Acta Math. Hungar.* **113** (3) (2006) 239-250.

[5] M. Behboodi, O. A. S. Karamzadeh and H. Koohy, Modules whose certain submodules are prime, *Vietnam J. Math.* **32** (3) (2004) 303-317.

[6] M. Behboodi and H. Koohy, Weakly prime modules, *Vietnam J. Math.* **32** (2) (2004) 185-195.

[7] M. Behboodi and S.H. Shojaee, On the classical Krull dimension of modules, *Int. J. Algebra.* **3** (6) (2009) 287-296.

[8] J. Dauns, Prime modules, *J. Reine Angew. Math.* **298** (1978) 156-181.

[9] K. R. Goodearl and R. B. Warfield, An introduction to non-commutative Noetherian rings (Second Edition), London Math. Soc. Student Texts 16, Cambridge University Press, Cambridge, 2004.

[10] R. Hamsher, Commutative Noetherian rings over which every modules has a maximal sub-module, *Proc. Ammer. Math. Soc.* **17** (1966) 1471-1472.

[11] O. A. S. Karamzadeh, On the classical Krull dimension of rings, *Fund. Math.* **117** (1983) 103-108.

[12] G. Krause, On fully left bounded left Noetherian rings, *J. Algebra.* **23** (1972) 88-99.

[13] H. Man and P. F. Smith, On chains of prime submodules, *Israel Journal of Mathematics* **127** (2002) 131-155.

[14] R. L. McCasland, M. E. Moore and P. F. Smith, On the spectrum of a module over a commutative ring, *Comm. Algebra* **25** (1997) 79-103.
[15] R. L. McCasland and P. F. Smith, Prime submodules of Noetherian modules, *Rocky Mountain J. Math.* **23** (1993) 1041-1062.
[16] J. C. McConnell and J. C. Robson, Noncommutative Noetherian rings, *Wily, Chichester* (1987).
[17] R. Wisbauer, Co-semisimple modules and nonassociative V-rings, *Comm. Algebra* **5** (1977) 1193-1209.
[18] R. Wisbauer, Foundations of Modules and Rings Theory, *Gordon and Breach, Reading* (1991).

Mahmood Behboodi
Department of Mathematical Sciences, Isfahan University of Technology, P.O. Box 84156-83111, Isfahan, Iran
and
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran
Email: mbehbood@cc.iut.ac.ir

Seyed Hossein Shojaee
Department of Mathematical Sciences, Isfahan University of Technology, P.O. Box 84156-83111, Isfahan, Iran
Email: hshojaee@math.iut.ac.ir