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SECOND HOCHSCHILD COHOMOLOGY OF CONVOLUTION ALGEBRAS

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ABSTRACT. We show that the second-order Hochschild cohomology groups of measure algebra $M(G)$ and group algebra $L^1(G)$ with coefficients in $\mathbb{C}_\varphi$ are Banach spaces where G is a locally compact group and φ is the augmentation character.

1. Introduction

We study the structure of Hochschild cohomology groups of convolution algebras with coefficients in $\mathbb{C}$. We first recall some basic results and introduce our notations.
Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule, so that $\mathcal{X}$ is an $\mathcal{A}$-bimodule and $\mathcal{X}$ is a Banach space for a norm $\|\cdot\|$ such that

$$\|a\cdot x\| \leq \|a\| \, \|x\| \, , \qquad \|x\cdot a\| \leq \|a\| \, \|x\| \qquad (a\in\mathcal{A},\, x\in\mathcal{X}).$$

A Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is symmetric if $a\cdot x = x\cdot a$, when $a\in\mathcal{A}$ and $x\in\mathcal{X}$. The character space on the Banach algebra $\mathcal{A}$ is denoted by $\Phi_\mathcal{A}$. For all $\varphi\in\Phi_\mathcal{A}\cup\{0\}$, the set of complex numbers $\mathbb{C}$ with the following module actions,

$$a\cdot z = z\cdot a = \varphi(a)z \qquad (a\in\mathcal{A}, z\in\mathbb{C}),$$

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becomes a symmetric Banach $\mathcal{A}$-bimodule, which is denoted by $\mathbb{C}_\varphi$.

Let G be a locally compact group and let $\mathcal{X}$ be a Banach G-bimodule, so that $\mathcal{X}$ is a G-bimodule, $\mathcal{X}$ is a Banach space for a norm $\|\cdot\|$ and there exists $K \geq 0$ such that

$$\|g \cdot x\| \leq K \|a\| \|x\|, \qquad \|x \cdot g\| \leq K \|g\| \|x\| \qquad (a \in \mathcal{A},\, x \in \mathcal{X}).$$

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule in the usual way. An n-cochain is a bounded n-linear map T from $\mathcal{A}$ to $\mathcal{X}$, which we denote by $T \in \mathcal{C}^n(\mathcal{A}, \mathcal{X})$. The map $\delta^n : \mathcal{C}^n(\mathcal{A}, \mathcal{X}) \longrightarrow \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{X})$ is defined by

$$\begin{aligned}(\delta^n T)(a_1, \ldots, a_{n+1}) &= a_1 \cdot T(a_2, \ldots, a_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i T(a_1, \ldots, a_i a_{i+1}, \ldots a_{n+1}) \\ &\quad + (-1)^{n+1} T(a_1, \ldots, a_n) \cdot a_{n+1}.\end{aligned}$$

The n-cochain T is an n-cocycle if $\delta^n T = 0$ and it is an n-coboundary if $T = \delta^{n-1} S$ for some $S \in \mathcal{C}^{n-1}(\mathcal{A}, \mathcal{X})$. The linear space of all n-cocycles is denoted by $\mathcal{Z}^n(\mathcal{A}, \mathcal{X})$, and the linear space of all n-coboundaries is denoted by $\mathcal{B}^n(\mathcal{A}, \mathcal{X})$. We also recall that $\mathcal{B}^n(\mathcal{A}, \mathcal{X})$ is included in $\mathcal{Z}^n(\mathcal{A}, \mathcal{X})$ and that the n-th Hochschild cohomology group $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ is defined by the quotient,

$$\mathcal{H}^n(\mathcal{A}, \mathcal{X}) = \frac{\mathcal{Z}^n(\mathcal{A}, \mathcal{X})}{\mathcal{B}^n(\mathcal{A}, \mathcal{X})}.$$

The space $\mathcal{Z}^n(\mathcal{A}, \mathcal{X})$ is a Banach space, but in general $\mathcal{B}^n(\mathcal{A}, \mathcal{X})$ is not closed. We regard $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ as a complete seminormed space with respect to the quotient seminorm. This seminorm is a norm if and only if $\mathcal{B}^n(\mathcal{A}, \mathcal{X})$ is a closed subspace of $\mathcal{C}^n(\mathcal{A}, \mathcal{X})$, which means that $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ is a Banach space.
Before giving our main results, we explain the general idea for showing that a cohomology group is a Banach space. Let $\delta : \mathcal{C}^n(\mathcal{A}, \mathcal{X}) \to \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{X})$ be the boundary map. Then, $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ is a Banach space if and only if the range of δ is closed, which is the case if and only if δ is open onto its range, that is, there exist a constant K such that if $\psi = \delta(\phi)$ is so that $\|\psi\| < 1$, then there exist $\phi_1 \in \mathcal{C}^n(\mathcal{A}, \mathcal{X})$ such that $\|\phi_1\| < K$ and $\psi = \delta(\phi_1)$ [7, Poroposition 1.1].

There are several reasons why one might wish to show that a cohomology group of a Banach algebra is a Banach space. The first reason is that if one can show that the algebraic cohomology group is trivial, then this often leads to the conclusion that the space of coboundaries is dense in the space of cocycles. If one can additionally prove that the space of coboundaries is closed, then one has a proof that the cohomology is trivial. The second reason for wanting cohomology groups to be Banach spaces, is that in more advanced calculations [2] one wishes to take projective tensor products of cohomology groups (this works well when the groups are Banach spaces). A third reason is that one can see showing the cohomology is a Banach space as a step to identify the Banach space and hence the cohomology group.

Johnson in [5], proved that for the free group on two generators, $\mathcal{H}^2(\ell^1(\mathbb{F}_2),\mathbb{C}) \neq 0$. Using [8, Theorem 8.3.1] we have that $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \ell^1(\mathbb{F}_2)) \neq 0$, and so $\mathcal{H}^2(\ell^1(\mathbb{F}_2),\ell^\infty(\mathbb{F}_2)) \neq 0$. Ivanov in [4], and Matsumoto and Morita in [6] showed that $\mathcal{H}^2(\ell^1(G),\mathbb{C})$ was a Banach space for every discrete group G with trivial action on $\mathbb{C}$. Pourabbas in [7] proved that the second-order cohomology groups of $\mathcal{H}^2(\ell^1(G),\ell^\infty(S))$ was a Banach space where G is a discrete group and let S be a G-set. Here, we will show that the second-order Hochschild cohomology groups $\mathcal{H}^2(M(G),\mathbb{C}_\varphi)$ and $\mathcal{H}^2(L^1(G),\mathbb{C}_\varphi)$ are Banach spaces where G is a locally compact group and φ is the augmentation character.

2. The structure of $\mathcal{H}^2(M(G),\mathbb{C}_\varphi)$

Let G be a locally compact group, and denote by $M(G)$ all complex-valued regular Borel measures μ on G such that $\|\mu\| = |\mu|(G)$. $M(G)$ with convolution multiplication is a unital Banach algebra and is the dual of $C_0(G)$. By the Jordan decomposition for every real valued measure $\mu \in M(G)$, there exist positive measures μ^+ and μ^- in $M(G)$ such that

$$\mu = \mu^+ + \mu^-, \qquad \|\mu\| = \|\mu^+\| + \|\mu^-\|.$$

Let $\mathcal{X}$ be a Banach G-bimodule. Then, $\mathcal{X}$ becomes an $M(G)$-bimodule [5, p. 24], by defining

$$\mu x = \int gx\,d\mu(g) \qquad x\mu = \int xg\,d\mu(g), \tag{2.1}$$

where $\mu \in M(G)$ and $x \in \mathcal{X}$. Consequently, $\mathcal{X}$ becomes a Banach $L^1(G)$-bimodule by restriction of these operations to the absolutely continuous measures. It is obvious that $\mathbb{C}$ is a Banach G-bimodule with trivial action

$$g\alpha = \alpha = \alpha g \qquad (g \in G, \alpha \in \mathbb{C}),$$

and hence $\mathbb{C}$ is a Banach $M(G)$-bimodule and from (2.1), we have

$$\mu\alpha = \alpha\mu = \alpha \int d\mu(g) = \alpha\varphi(\mu),$$

where $\varphi(\mu) = \mu(G)$ is a character on $M(G)$. Thus, $\mathbb{C}$ is a symmetric Banach $M(G)$-bimodule which is denoted by $\mathbb{C}_\varphi$.
For every $\psi \in \mathcal{C}^1(M(G), \mathbb{C}_\varphi)$ and for every $\mu_1, \mu_2 \in M(G)$, the boundary map gives

$$\delta\psi(\mu_1, \mu_2) = \varphi(\mu_1)\psi(\mu_2) - \psi(\mu_1\mu_2) + \varphi(\mu_2)\psi(\mu_1). \tag{2.2}$$

Since $\delta\psi$ is a bounded linear map, we have

$$|\varphi(\mu_1)\psi(\mu_2) - \psi(\mu_1\mu_2) + \varphi(\mu_2)\psi(\mu_1)| \leq \|\delta\psi\| \, \|\mu_1\| \, \|\mu_2\| \, . \tag{2.3}$$

Now we show that for every locally compact group G, the second-order Hochschild cohomology group of measure algebra $M(G)$ with coefficients in $\mathbb{C}_\varphi$ is a Banach space.

Theorem 2.1. *Let G be a locally compact group. Then, $\mathcal{H}^2(M(G), \mathbb{C}_\varphi)$ is a Banach space.*

Proof. It is enough to show that there exists a constant K such that $\|\psi\| \leq K \|\delta\psi\|$, for every $\psi \in \mathcal{C}^1(M(G), \mathcal{C}_\varphi)$. Let $\psi \in \mathcal{C}^1(M(G), \mathcal{C}_\varphi)$ and let μ be a positive measure of $M(G)$ such that $\|\mu\| = \mu(G) = \varphi(\mu)$. Let $\nu = \frac{\mu}{\|\mu\|}$ and replace $\mu_1 = \mu_2 = \nu$ in inequality (2.3). Then, $\varphi(\nu) = 1$ and we have

$$\left|\varphi(\nu)\psi(\nu) - \psi(\nu^2) + \varphi(\nu)\psi(\nu)\right| = \left|2\psi(\nu) - \psi(\nu^2)\right| \leq \|\delta\psi\| \, ,$$

and again for $\mu_1 = \nu^2$ and $\mu_2 = \nu$, we have

$$\left|\varphi(\nu^2)\psi(\nu) - \psi(\nu^3) + \varphi(\nu)\psi(\nu^2)\right| = \left|\psi(\nu) - \psi(\nu^3) + \psi(\nu^2)\right| \leq \|\delta\psi\| \, .$$

These inequalities and triangular inequality imply that

$$\left|3\psi(\nu) - \psi(\nu^3)\right| \leq 2 \|\delta\psi\| \, .$$

By induction for every $n \in \mathbb{Z}^+$, we have

$$|n\psi(\nu) - \psi(\nu^n)| \leq (n-1) \|\delta\psi\| \, ,$$

or

$$\left|\psi(\nu) - \frac{\psi(\nu^n)}{n}\right| \leq \frac{(n-1)}{n} \|\delta\psi\| \, .$$

Since ψ is a bounded operator, then $|\psi(\nu^n)| \leq \|\psi\|$, for all $n \in \mathbb{Z}^+$. Thus, $|\psi(\nu)| \leq \|\delta\psi\|$; that is,

$$|\psi(\mu)| \leq \|\delta\psi\| \, \|\mu\| .$$

For all $\mu \in M(G)$, there are two signed measures μ_1 and μ_2 in $M(G)$ such that $\mu = \mu_1 + i\mu_2$ and $\|\mu_i\| \leq \|\mu\|$, for $i = 1, 2$. Now, by the Jordan decomposition, we have

$$\begin{aligned} |\psi(\mu)| &\leq \left|\psi(\mu_1^+)\right| + \left|\psi(\mu_1^-)\right| + \left|\psi(\mu_2^+)\right| + \left|\psi(\mu_2^-)\right| \\ &\leq \|\delta\psi\| \left(\left\|\mu_1^+\right\| + \left\|\mu_1^-\right\| + \left\|\mu_2^+\right\| + \left\|\mu_2^-\right\|\right) \\ &= \|\delta\psi\| \left(\|\mu_1\| + \|\mu_2\|\right) \\ &\leq 2 \|\delta\psi\| \, \|\mu\| . \end{aligned} \tag{2.4}$$

Therefore, $|\psi(\mu)| \leq 2 \|\delta\psi\| \, \|\mu\|$, for every $\mu \in M(G)$; that is, $\|\psi\| \leq 2 \|\delta\psi\|$ and this complete the proof. □

3. **The structure of** $\mathcal{H}^2(L^1(G), \mathbb{C}_\varphi)$

Let G be a locally compact group. The same result is true for group algebra $L^1(G)$. As mentioned before, $\mathbb{C}$ is a Banach $L^1(G)$-bimodule with the restriction of module action (2.1) on $L^1(G)$; that is,

$$f\alpha = \alpha f = \int \alpha f(g) dg = \alpha\varphi(f) \qquad (f \in L^1(G), \alpha \in \mathbb{C}),$$

where $\varphi(f) = \int f(g) dg$, which is the augmentation character on $L^1(G)$. So, $\mathbb{C}$ with above module action is symmetric Banach $L^1(G)-$bimodule and it is denoted by $\mathbb{C}_\varphi$ too.

Theorem 3.1. *Let G be a locally compact group. Then, $\mathcal{H}^2(L^1(G), \mathbb{C}_\varphi)$ is a Banach space.*

Proof. Let S be the set of all simple functions in $L^1(G)$. Let $\psi \in \mathcal{C}^1(L^1(G), \mathbb{C}_\varphi)$ and let $s = \sum_{i=1}^k \alpha_i \chi_{I_i}$ be an arbitrary simple function in $L^1(G)$. Let $g = \frac{\chi_{I_i}}{\|\chi_{I_i}\|}$ and replace $\mu_1 = \mu_2 = g$ in inequality (2.3). Then, $\varphi(g) = 1$ and we have

$$\left|\varphi(g)\psi(g) - \psi(g^2) + \varphi(g)\psi(g)\right| = \left|2\psi(g) - \psi(g^2)\right| \leq \|\delta\psi\| .$$

The same induction as in the proof of Theorem (2.1) implies that

$$|\psi(\chi_{I_i})| \leq \|\delta\psi\| \, \|\chi_{I_i}\| .$$

Thus,

$$|\psi(s)| \leq \sum_{i=1}^{k} |\alpha_i| \, |\psi(\chi_{I_i})| \leq \sum_{i=1}^{k} |\alpha_i| \, \|\delta\psi\| \, \|\chi_{I_i}\| \tag{3.1}$$
$$\leq \sum_{i=1}^{k} (|\alpha_i| \, \|\chi_{I_i}\|) \, \|\delta\psi\| = \|s\|_1 \, \|\delta\psi\| \, .$$

For every $f \in L^1(G)$, since S is a dense subset of $L^1(G)$, there exists a sequence of simple functions $\{s_n\}$ in S such that $s_n \to f$ in $L^1(G)$ and for every n,

$$0 < |\|s_n\|_1 - \|f\|_1| \leq \|s_n - f\|_1 \to 0$$

as $n \to \infty$. By inequality (3.1), $|\psi(s_n)| \leq \|s_n\|_1 \, \|\delta\psi\|$, and hence by continuity of ψ, $|\psi(f)| \leq \|\delta\psi\| \, \|f\|_1$. It follows that $\|\psi\| \leq \|\delta\psi\|$, and so $\mathcal{H}^2(L^1(G), \mathbb{C}_\varphi)$ is a Banach space. □

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