

## GENERATION OF THE AHLFORS FIVE ISLANDS THEOREM

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Communicated by Saeid Azam

ABSTRACT. By using the Ahlfors theory of covering surface, a normality criterion for families of meromorphic functions is proved. The normality criterion extends the Ahlfors's five islands theorem.

### 1. Introduction

Since 1968, when Baker [1] used the Ahlfors five islands theorem to prove that repelling periodic points were dense in the Julia set of an entire function, the Ahlfors five islands theorem has become an important tool in complex dynamics. Most recently, Bergweiler discussed its role in [2], describing how it could be used to deal with a variety of problems. This includes questions concerning the Hausdorff dimension of Julia set, the existence of singleton components of Julia sets, and the existence of repelling periodic (see [3]). Because the Ahlfors five islands theorem is so valid in the study of complex dynamics, we shall deduce the Ahlfors five islands theorem from a normality criterion for families of meromorphic functions - a special case of Ahlfors's Scheibensatz - which will be proved here by using the Ahlfors theory of covering surface. In order to state the Ahlfors five islands theorem and the main results in this paper, we shall recall the following definitions and notations (see Schiff [5] and Tsuji [9]).

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MSC(2000): Primary: 30D35; Secondary: 30D30.

Keywords: Normality criterion , covering surface, multiple values.

Received: 18 October 2008, Accepted: 14 April 2009.

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Let  $D$  be a domain in the complex plane  $\mathbb{C}$  and  $\Gamma$  be a family of meromorphic functions from  $D$  (endowed with the Euclidean metric) to the Riemann sphere  $\mathbb{V}$  of diameter 1, endowed with the spherical metric  $\chi$ , given by  $\chi(z, z')$ , which is the lower bound of lengths of curves joining  $z$  and  $z'$  on  $\mathbb{V}$ . The value  $\chi(z, z')$  is the Euclidean length of shortest arc of the great circles through  $z$  and  $z'$  on  $\mathbb{V}$ . A family  $\Gamma$  of meromorphic functions on  $D$  is said to be normal on  $D$ , if each sequence  $\{f_n\} \subset \Gamma$  has a subsequence which converges  $\chi$ - uniformly on any compact subset of  $D$ .

Denote a connected domain on  $\mathbb{V}$  by  $F_0$ . The boundary of  $F_0$  consists of a finite number of mutually disjoint closed Jordan curves  $\{\Lambda_j\}$ , and the spherical distance between any two distinct closed curves  $\Lambda_i$  and  $\Lambda_j$  in  $\{\Lambda_j\}$  is  $\chi(\Lambda_i, \Lambda_j) \geq \delta \in (0, \frac{1}{2})$ .

Let  $F$  be a finite covering surface of  $F_0$ , consisting of a finite number of sheets, and be bounded by a finite number of analytic Jordan curves. The part of the boundary of  $F$ , which does not lie above the boundary of  $F_0$ , is called the relative boundary of  $F$ , and we denote it by  $L$ . Let  $D$  be a domain on  $F_0$ , its boundary be consisted of a finite number of points or analytic closed Jordan curves, and  $F(D)$  be the part of  $F$ , which lies above  $D$ . We denote the area of  $F, F(D)$  and  $F_0$  by  $|F|, |F(D)|$  and  $|F_0|$ , respectively. Let

$$S = \frac{|F|}{|F_0|}, S(D) = \frac{|F(D)|}{|D|}.$$

Let  $D$  be a simply connected domain on  $\mathbb{V}$  and  $F(D)$  be the part of  $F$ , which lies above  $D$ . Suppose that  $F(D)$  consists of a finite number of connected surfaces  $\{F_{\mathbb{V}}(D)\}$ . For some  $F_{\mathbb{V}} \in \{F_{\mathbb{V}}(D)\}$ , we call  $F_{\mathbb{V}}$  is an island if  $F_{\mathbb{V}}$  has no relative boundary, with respect to  $D$ . Denote by  $n_{\mathbb{V}}^l$ , the number of simply connected islands in  $F(D)$ , which consists of not more than  $l$  sheets. When  $l = 1$ , then we call it a simple island. Here, we shall give a new proof of the following case of Ahlfors's Scheibensatz (see, e.g., [3]).

**Theorem 1.1.** *Denote  $\Gamma$  by the family of meromorphic functions  $\{f : D \rightarrow \mathbb{C}_{\infty} := \mathbb{C} \cup \{\infty\}\}$ , and let  $D_i, i = 1, 2, \dots, q$ , be Jordan domains on the Riemann sphere  $\mathbb{V}$  with pairwise disjoint closures. Suppose that no  $f \in \Gamma$  has islands with no more than  $l$  sheets for  $q = \lfloor \frac{2l+2}{l} \rfloor + 1$  distinct domains  $D_i, i = 1, 2, \dots, q$ , in  $D$ , where  $[x]$  is the integer part of  $x$  and  $l$  is any positive integer. Then,  $\Gamma$  is normal.*

When  $l = 1$  in Theorem 1.1, we have the following Ahlfors five islands theorem.

**Corollary 1.2.** *Denote  $\Gamma$  by the family of meromorphic functions  $\{f : D \rightarrow \mathbb{C}_\infty\}$ . Suppose that every  $f \in \Gamma$  does not have simple islands for 5 distinct Jordan domains  $D_i, i = 1, 2, \dots, 5$ , on the Riemann sphere  $\mathbb{V}$  with pairwise disjoint closures. Then,  $\Gamma$  is normal.*

## 2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following lemmas. Under the notations in Section 1, we have the following improved Ahlfors type covering theorems due to D. C. Sun [6], which gives an explicit estimation of the constant in the theorem.

**Lemma 2.1.** *For any simply connected finite covering surface  $F$  of  $F_0$ , we have*

$$|S - S(D)| < AL,$$

where  $A = \max\{\frac{2}{\delta}, \frac{\pi^2}{|D|}\}$ .

**Lemma 2.2.** [7] *Suppose that  $F$  is a simply connected finite covering surface of a sphere surface  $\mathbb{V}$ , and  $\{D_v\}$  are  $q(> 2)$  disjoint disks on  $\mathbb{V}$ , where the spherical distance of any pair of  $\{D_v\}$  is at least  $\delta$ . Let  $n_v$  be the number of simply connected islands in  $F(D_v)$ . Then,*

$$\sum_{v=1}^q n_v \geq (q - 2)S - \frac{C}{\delta^3}L,$$

where  $L$  is the length of the relative boundary of  $F$  and  $C$  is a constant.

Here, we shall prove two other lemmas for later use.

**Lemma 2.3.** [10] *Let  $F$  be a simply connected finite covering surface of a sphere surface  $\mathbb{V}$ , and let  $\{D_v\}_{v=1}^q$ , with  $q(> [\frac{2l+2}{l}])$ , be disjoint disks with radius  $\frac{\delta}{3}$  on  $\mathbb{V}$  and without a pair of  $\{D_v\}$  such that their spherical distance is less than  $\delta$ . Denote by  $n_v^l$  the number of simply connected*

islands in  $F(D_v)$ , which consist of not more than  $l$  sheets. Then,

$$\sum_{v=1}^q n_v^{(l)} \geq (q - 2 - \frac{2}{l})S - \frac{C(l+1) + 18\pi^2 q}{l\delta^3} L.$$

**Proof.** It is easy to verify that

$$n_v = n_v^{(l)} + n_v^{(l)}, \quad S(D_v) \geq n_v^{(l)} + (l+1)n_v^{(l)},$$

where  $n_v^{(l)}$  means the number of simply connected islands in  $F(D_v)$ , which consist of not less than  $l+1$  sheets.

Thus,

$$S(D_v) \geq (l+1)(n_v^{(l)} + n_v^{(l)}) - ln_v^{(l)} = (l+1)n_v - ln_v^{(l)}.$$

Since  $|D_v| \geq \frac{\delta^2}{9}$  and  $\delta \in (0, \frac{1}{2})$ , it follows from Lemma 2.1 that  $S(D_v) - S \leq |S(D_v) - S| < AL < \frac{2}{\delta} \frac{\pi^2}{|D_v|} = \frac{18\pi^2}{\delta^3} L$ . Thus,

$$S + \frac{18\pi^2}{\delta^3} L > S(D_v) \geq (l+1)n_v - ln_v^{(l)}.$$

Adding from  $v = 1$  to  $q$ , we have

$$qS + \frac{18q\pi^2}{\delta^3} L + l \sum_{v=1}^q n_v^{(l)} > (l+1) \sum_{v=1}^q n_v.$$

Then, Lemma 2.3 follows from Lemma 2.2 and the above expression.  $\square$

**Lemma 2.4.** Let  $f(z)$  be a meromorphic function in  $|z| < R, 0 < R < \infty$ , and  $F$  be the Riemann surface generated by  $f$  on  $\mathbb{V}$ . Let  $\{D_v\}_{v=1}^q$ , with  $q (> [\frac{2l+2}{l}])$ , be disjoint disks with radius  $\frac{\delta}{3}$  on  $\mathbb{V}$  and without a pair of  $\{D_v\}$  such that their spherical distance is less than  $\delta$ . Suppose that  $n_v^{(l)}$  is the number of simply connected islands in  $F(D_v)$ , which consists of not more than  $l$  sheets. Then, for any  $r \in (0, R)$ , we have

$$\sum_{v=1}^q n_v^{(l)} \geq (q - 2 - \frac{2}{l})S(r, f) - \left\{ \frac{C(l+1) + 18\pi^2 q}{l\delta^3} \right\}^2 2\pi^2 \frac{1}{\log R - \log r},$$

where (see Tsuji [9]),

$$S(r, f) = \frac{1}{\pi} \int_{|z|<r} \left( \frac{|f'(z)|}{(1 + |f(z)|^2)} \right)^2 r d\theta dr, \quad z = re^{i\theta}.$$

**Proof.** By Lemma 2.3, we have

$$\sum_{v=1}^q n_v^l \geq (q - 2 - \frac{2}{l})S(r, f) - \frac{C(l+1) + 18\pi^2q}{l\delta^3}L(r, f),$$

where (see Tsuji [9]),

$$(2.1) \quad L(r, f) = \frac{1}{\pi} \int_{|z|=r} \frac{|f'(re^{i\theta})|}{(1 + |f(re^{i\theta})|^2)} r d\theta.$$

By the Schwarz's inequality, we can obtain the following inequality (see Tsuji [9] or Hayman [4]),

$$L^2(r, f) \leq 2\pi^2 r \frac{dS(r, f)}{dr}.$$

For any  $r' \in (r, R)$ , if  $(q - 2 - \frac{2}{l})S(r, f) \geq \sum_{v=1}^q n_v^l$ , then

$$0 \leq (q - 2 - \frac{2}{l})S(r, f) - \sum_{v=1}^q n_v^l \leq \frac{C(l+1)+18\pi^2q}{l\delta^3}L(r, f).$$

Thus,

$$\begin{aligned} \{(q - 2 - \frac{2}{l})S(r', f) - \sum_{v=1}^q n_v^l\}^2 &\leq \{\frac{C(l+1)+18\pi^2q}{l\delta^3}\}^2 L^2(r', f) \\ &\leq \{\frac{C(l+1)+18\pi^2q}{l\delta^3}\}^2 2\pi^2 r' \frac{dS(r', f)}{dr'}. \end{aligned}$$

Therefore,

$$\begin{aligned} \log \frac{R}{r} = \int_r^R \frac{dr'}{r'} &\leq \{\frac{C(l+1)+18\pi^2q}{l\delta^3}\}^2 2\pi^2 \int_r^R \frac{dS(r', f)}{[(q-2-\frac{2}{l})S(r', f) - \sum_{v=1}^q n_v^l]^2} \\ &\leq \{\frac{C(l+1)+18\pi^2q}{l\delta^3}\}^2 2\pi^2 \frac{1}{(q-2-\frac{2}{l})S(r, f) - \sum_{v=1}^q n_v^l}. \end{aligned}$$

Hence,

$$(q - 2 - \frac{2}{l})S(r, f) \leq \sum_{v=1}^q n_v^l + \{\frac{C(l+1) + 18\pi^2q}{l\delta^3}\}^2 2\pi^2 \frac{1}{\log R - \log r}$$

□

**Definition 2.5.** [5] A family  $\Gamma$  of functions defined on a domain  $D$  is said to be spherically equicontinuous at a point  $z' \in D$  if, for each  $\varepsilon > 0$ , there is a  $\delta > 0$  such that for any  $z \in D, |z - z'| < \delta$ , for every  $f \in \Gamma$ , we

have  $\chi(f(z), f(z')) < \varepsilon$ . Moreover,  $\Gamma$  is spherically equicontinuous on a subset  $E \subseteq D$  if it is spherically equicontinuous at each point of  $E$ .

**Lemma 2.6.** [5] *A family  $\Gamma$  of meromorphic functions in a domain  $D$  is normal if and only if  $\Gamma$  is spherically equicontinuous in  $D$ .*

We are now in the position to prove Theorem 1.1 by referring the method in [8].

**Proof of Theorem 1.1.** For every  $f \in \Gamma$  and any  $z_0 \in D$ , let  $B(z_0, t) = \{z \in \mathbb{V}; \chi(z, z_0) < t\}$  and  $B(z_0, t) \subset D$ . For the sake of simplicity, we put  $f(z + z_0) = F(z)$ . Denote by  $l_r$  the relative boundary of covering surface  $f(B(z_0, r)) = F(B(0, r))$  ( $r < t$ ) and denote by  $L(r)$  the length of  $l_r$ . Let

$$\tilde{a}(r) = \sup\{\chi(F(re^{i\theta}), F(0)); 0 \leq \theta < 2\pi\}, a(r) = \min\{\tilde{a}(r), 1\}.$$

Since no  $f \in \Gamma$  has islands with no more than  $l$  sheets for  $q = [\frac{2l+2}{l}] + 1$  distinct domains  $D_i$ ,  $i = 1, 2, \dots, q$ , in  $D$  we conclude that every corresponding  $F$  does not have islands with number of sheets no more than  $l$  for  $q = [\frac{2l+2}{l}] + 1$  distinct domains  $D_i$ ,  $i = 1, 2, \dots, q$ , in  $B(0, t)$ . By Lemma 2.4, we have for any  $r < t$ ,

$$(2.2) \quad S(r, F) \leq \frac{H}{\log \frac{t}{r}},$$

where  $H = [\frac{C(l+1)+18\pi^2q}{l\delta^3}]^2 2\pi^2 (q - 2 - \frac{2}{l})^{-1}$ . We put  $r_0 = te^{-4H}$ . Then,

$$(2.3) \quad S(r_0, F) \leq \frac{1}{4}.$$

For any  $r < r_0$ , we have  $a(r) \leq L(r)$ . In fact, when  $L(r) \geq 1$ ,  $a(r) \leq L(r)$ , in general. When  $L(r) < 1$ , then  $l_r$  lies in some hemisphere of  $\mathbb{V}$  which we write as  $\mathbb{V}_1$ . By (3), we have  $F(B(0, r)) \subset \mathbb{V}_1$ . Suppose that  $z_1 = re^{i\theta}$  satisfies  $\chi(F(z_1), F(0)) = \tilde{a}(r)$ . We draw a great circle  $\Gamma$  which passes the points  $F(z_1), F(0)$  on  $\mathbb{V}$ , and denote by  $aF(z_1)F(0)b$  the part of  $\Gamma$  which lies inside of  $\mathbb{V}_1$ . Here,  $a$  and  $b$  are two intersection points of the circle  $\Gamma$  and the boundary  $\partial\mathbb{V}_1$ . Let  $d$  be an intersection point of the arcs  $\Gamma$  and  $l_r$ . Then,

$$(2.4) \quad a(r) \leq \tilde{a}(r) \leq \chi(F(z_1), F(0)) + \chi(F(0), d) = \chi(F(z_1), d) \leq L(r).$$

Note that the area of  $F(\{|z| \leq r\})$  is a nondecreasing function of  $r$ , and thus  $\tilde{a}(r) = \sup\{\chi(F(re^{i\theta}), F(0))\}$  and  $a(r)$  are also nondecreasing. Hence, for any  $r < \frac{1}{2}r_0$ , by (2.1), (2.2) and (2.4), we have

$$\begin{aligned}
a^2(r) \log 2 &= \int_r^{2r} \frac{a^2(t)}{t} dt \leq \int_r^{2r} \frac{a^2(t)}{t} dt \leq \int_r^{2r} \frac{L^2(t, F)}{t} dt \\
&\leq 2\pi^2 \int_r^{2r} dS(t, F) < 2\pi^2 S(2r, F) \\
&\leq \frac{2\pi^2 H}{\log \frac{t}{2r}}.
\end{aligned}$$

Hence, for any  $\varepsilon > 0$ , when  $r < \exp(-\frac{2\pi^2 H \varepsilon^{-2}}{\log 2})t/2 = \delta(\varepsilon)$ , we have  $a(r) < \varepsilon$ . Therefore, for any  $z \in B(0, t)$ ,  $\chi(z, 0) < \delta(\varepsilon)$  and any  $F$ , we have

$$(2.5) \quad \chi(F(z - z_0), F(0)) < \varepsilon.$$

We may deduce from (2.5) that  $\Gamma$  is spherically equicontinuous on  $D$ . By using Lemma 2.6, we know that  $\Gamma$  is normal and then The result follows.  $\square$

### Acknowledgments

The author thanks the referees very much for their valuable suggestions. This research was supported by the research foundational of Hubei Educational Committee (No:B20092809) and the research foundational of Xianning University (BK0714).

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