

CONSTRUCTION OF COMPACTLY SUPPORTED NONSEPARABLE ORTHOGONAL WAVELETS OF $L^2(\mathbb{R}^n)$

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ABSTRACT. We present a method for the construction of nonseparable and compactly supported orthogonal wavelet bases of $L^2(\mathbb{R}^n)$, $n \geq 2$. The orthogonal wavelets are associated with dilation matrix $3I_n$, where I_n is the identity matrix of order n . An example is given to illustrate how to use our method to construct nonseparable orthogonal wavelet bases.

1. Introduction

In recent years, multivariate nonseparable wavelets have attracted the interest of many mathematicians. The details can be seen in [1]-[3] and [9]-[11]. Although separable wavelet bases have a lot of advantages, they have a number of drawbacks. They are so special that they have very little design freedom, and separability imposes an unnecessary product structure on the plane which is artificial for natural images. One way to avoid this is through the construction of nonseparable wavelets.

Nonseparable wavelet bases have enough degrees of freedom to construct bases having several properties simultaneously such as orthogonality, symmetry and compact support. The theory and the design of 1-D

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compactly supported wavelet bases are well understood. Although the theory and analysis of multivariate wavelet bases have been extensively studied, the design of n -D compactly supported nonseparable orthogonal wavelet basis is still a challenging problem. As we know, the construction of nonseparable wavelets with dilation matrix $2I_n$ is possible (see [5]-[8]). In [5], the author has given a brief description of a fairly general method for constructing compactly nonseparable and orthonormal wavelet bases of $L^2(R^n)$. In [6] and [7], the author has studied the construction of nonseparable biorthogonal and orthogonal wavelets of $L^2(R^n)$ respectively, for $n \geq 2$. Currently, it also turns out that many researchers proceed to study the nonseparable wavelets with dilation matrix M , specially the matrix M satisfying $M^2 = 2I$ (see [2], [10]-[11]). Such dilation matrices make the MRA involve a unique wavelet which is easy to construct from the scaling function.

Here, based on [6] and [7], we use a set of matrices D_i , $i = 1, \dots, n-1$, satisfying some conditions to give the construction of n -D nonseparable orthogonal wavelets, and give a proof that the constructed orthogonal wavelets are nonseparable. Finally, we give an example.

2. Design of n -D low-pass orthogonal wavelet filters

In this section, we first provide the reader with some definitions to be used frequently in our work. In the following, we denote the point $(\omega_1, \omega_2, \dots, \omega_k) \in R^k$, $\pi_k = (\pi, \pi, \dots, \pi) \in R^k$, and D_0 by the identity matrix I_n . Finally, if $D \in Z^{n \times n}$ is a square matrix, then $D(\omega)$ denotes the product $D \cdot \omega^T$.

Definition 2.1. A ladder of closed subspaces $\{V_j\}_{j \in Z}$ of $L^2(R^n)$ is called a multiresolution analysis (MRA) if the following conditions hold:

- (i) $V_j \subset V_{j+1}$ for $j \in Z$;
- (ii) $\bigcap_{j \in Z} V_j = \{0\}$, $\overline{\bigcup_{j \in Z} V_j} = L^2(R^n)$;
- (iii) $f(x) \in V_j \iff f(3x) \in V_{j+1}$;
- (iv) there exists a function $\phi(x)$ in V_0 such that the set $\{\phi(x-k)\}_{k \in Z^n}$ is a Riesz basis for V_0 .

Definition 2.2. A matrix $D \in Z^{n \times n}$ is said to be a dilation matrix if all its singular values σ_i , $i = 1, \dots, n$, are larger than 1 in modulus.

Definition 2.3. Define a set $E_n = \{0, \frac{2}{3}\pi, \frac{4}{3}\pi\}^n$ and let η_i be an element of E_n ; if $\eta_i^1 = \frac{2}{3}\pi_n - \eta_i$ and $\eta_i^2 = \frac{4}{3}\pi_n - \eta_i$, then η_i^1 and η_i^2 are said to be symmetric of η_i in E_n . A subset A of E_n is said to be symmetric if $\forall \eta_i \in A, \exists \eta_i^1 \in A \bmod (2\pi Z^n)$, and $\eta_i^2 \in A \bmod (2\pi Z^n)$, where η_i^1 and η_i^2 are symmetric of η_i in A .

Definition 2.4. An n -D wavelet filter $H_n(\omega_1, \dots, \omega_n)$ is said to be separable if $H_n(\cdot)$ can be written in the following form:

$$H_n(\omega_1, \dots, \omega_n) = \prod_{i=1}^k m_i(a_{i1}\omega_1 + \dots + a_{in}\omega_n),$$

for some integer $1 \leq k \leq n$. Here, $m_i(\cdot)$ are some 1-D wavelet filters and $(a_{i1}, \dots, a_{in}) \in Z^n$.

Using the properties of MRA, we conclude that the scaling function $\Phi(x)$ has to satisfy the following dilation equation,

$$(2.1) \quad \Phi(x) = \sum_{k \in Z^n} \alpha_k \Phi(3x - k).$$

By taking the Fourier transform on both sides of (2.1), we get

$$(2.2) \quad \widehat{\Phi}(\omega) = \prod_{j=1}^{\infty} \mathcal{H}_0\left(\frac{\omega}{3^j}\right),$$

where $\mathcal{H}_0(\omega) = 3^{-n} \sum_{k \in Z^n} \alpha_k e^{-ik \cdot \omega}$. To construct compactly supported n -D orthogonal scaling functions, we need to construct n -D trigonometric polynomials $H_n(\omega)$ satisfying the following condition,

$$(2.3) \quad |H_n(\omega)|^2 + |H_n(\omega + \frac{2}{3}\pi_n)|^2 + |H_n(\omega + \frac{4}{3}\pi_n)|^2 = 1, \quad \forall \omega \in R^n.$$

The construction is given by the following lemma.

Lemma 2.5. Let $H_1(\omega)$ and $G_1(\omega), G_1'(\omega)$ be a 1-D low-pass and two high-pass orthogonal filters. Define the n -D filter $H_n(\omega_1, \dots, \omega_n)$ by the following iterative process: $\forall 2 \leq k \leq n$, choose an integer $1 \leq \ell_k \leq k-1$ such that

$$\begin{aligned}
P_{k-1}(\omega_1, \dots, \omega_{k-1}) &= H_{k-1}(3\omega_1, \dots, 3\omega_{k-1}) \\
Q_{k-1}(\omega_1, \dots, \omega_{k-1}) &= H_{k-1}(3\omega_1 + \frac{2}{3}\pi, \dots, 3\omega_{k-1} + \frac{2}{3}\pi) \\
R_{k-1}(\omega_1, \dots, \omega_{k-1}) &= H_{k-1}(3\omega_1 + \frac{4}{3}\pi, \dots, 3\omega_{k-1} + \frac{4}{3}\pi) \\
H_k(\omega_1, \dots, \omega_k) &= P_{k-1}(\omega_1, \dots, \omega_{k-1})H_{\ell_k}(\omega_{k-\ell_k+1}, \dots, \omega_k) \\
&\quad + Q_{k-1}(\omega_1, \dots, \omega_{k-1})G_{\ell_k}(\omega_{k-\ell_k+1}, \dots, \omega_k) \\
&\quad + R_{k-1}(\omega_1, \dots, \omega_{k-1})G'_{\ell_k}(\omega_{k-\ell_k+1}, \dots, \omega_k),
\end{aligned}$$

where $G_\ell(\omega_1, \dots, \omega_\ell)$ and $G'_\ell(\omega_1, \dots, \omega_\ell)$ satisfy the following equation,

$$(2.4) \quad MM^* = I_3,$$

$$M = \begin{pmatrix} H_\ell(\omega) & H_\ell(\omega + \frac{2}{3}\pi) & H_\ell(\omega + \frac{4}{3}\pi) \\ G_\ell(\omega) & G_\ell(\omega + \frac{2}{3}\pi) & G_\ell(\omega + \frac{4}{3}\pi) \\ G'_\ell(\omega) & G'_\ell(\omega + \frac{2}{3}\pi) & G'_\ell(\omega + \frac{4}{3}\pi) \end{pmatrix}, \quad \forall \omega \in \mathbb{R}^\ell. \quad \text{Then,}$$

$H_n(0, \dots, 0) = 1$. Moreover, $H_n(\omega_1, \dots, \omega_n)$ satisfies the condition (2.3).

Proof. The proof is carried out by induction. First, we check the result for $k = 2$. In this case, $\ell_2 = 1$, $P_1(\omega_1) = H_1(3\omega_1)$, $Q_1(\omega_1) = H_1(3\omega_1 + \frac{2}{3}\pi)$, and $R_1(\omega_1) = H_1(3\omega_1 + \frac{4}{3}\pi)$. Since $H_1(0) = 1$, and $H_1(\omega)$, $G_1(\omega)$, $G'_1(\omega)$ satisfy the orthogonal condition (2.4), then $G_1(0) = G'_1(0) = 0$. Hence, $H_2(0, 0) = 1$. Since $P_1(\omega_1) = P_1(\omega_1 + \frac{2}{3}\pi) = P_1(\omega_1 + \frac{4}{3}\pi)$, $Q_1(\omega_1) = Q_1(\omega_1 + \frac{2}{3}\pi) = Q_1(\omega_1 + \frac{4}{3}\pi)$, $R_1(\omega_1) = R_1(\omega_1 + \frac{2}{3}\pi) = R_1(\omega_1 + \frac{4}{3}\pi)$, we get

$$\begin{aligned}
&|H_2(\omega_1, \omega_2)|^2 + |H_2(\omega_1 + \frac{2}{3}\pi, \omega_2 + \frac{2}{3}\pi)|^2 + |H_2(\omega_1 + \frac{4}{3}\pi, \omega_2 + \frac{4}{3}\pi)|^2 \\
= &|P_1(\omega_1)|^2[|H_1(\omega_2)|^2 + |H_1(\omega_2 + \frac{2}{3}\pi)|^2 + |H_1(\omega_2 + \frac{4}{3}\pi)|^2] \\
&+ |Q_1(\omega_1)|^2[|G_1(\omega_2)|^2 + |G_1(\omega_2 + \frac{2}{3}\pi)|^2 + |G_1(\omega_2 + \frac{4}{3}\pi)|^2] \\
&+ |R_1(\omega_1)|^2[|G'_1(\omega_2)|^2 + |G'_1(\omega_2 + \frac{2}{3}\pi)|^2 + |G'_1(\omega_2 + \frac{4}{3}\pi)|^2]
\end{aligned}$$

$$\begin{aligned}
& +P_1(\omega_1)\overline{Q_1}(\omega_1)[H_1(\omega_2)\overline{G_1}(\omega_2) + H_1(\omega_2 + \frac{2}{3}\pi)\overline{G_1}(\omega_2 + \frac{2}{3}\pi) \\
& + H_1(\omega_2 + \frac{4}{3}\pi)\overline{G_1}(\omega_2 + \frac{4}{3}\pi)] + P_1(\omega_1)\overline{R_1}(\omega_1)[H_1(\omega_2)\overline{G'_1}(\omega_2) \\
& + H_1(\omega_2 + \frac{2}{3}\pi)\overline{G'_1}(\omega_2 + \frac{2}{3}\pi) + H_1(\omega_2 + \frac{4}{3}\pi)\overline{G'_1}(\omega_2 + \frac{4}{3}\pi)] \\
& + Q_1(\omega_1)\overline{P_1}(\omega_1)[G_1(\omega_2)\overline{H_1}(\omega_2) + G_1(\omega_2 + \frac{2}{3}\pi)\overline{H_1}(\omega_2 + \frac{2}{3}\pi) \\
& + G_1(\omega_2 + \frac{4}{3}\pi)\overline{H_1}(\omega_2 + \frac{4}{3}\pi)] + Q_1(\omega_1)\overline{R_1}(\omega_1)[G_1(\omega_2)\overline{G'_1}(\omega_2) \\
& + G_1(\omega_2 + \frac{2}{3}\pi)\overline{G'_1}(\omega_2 + \frac{2}{3}\pi) + G_1(\omega_2 + \frac{4}{3}\pi)\overline{G'_1}(\omega_2 + \frac{4}{3}\pi)] \\
& + R_1(\omega_1)\overline{P_1}(\omega_1)[G'_1(\omega_2)\overline{H_1}(\omega_2) + G'_1(\omega_2 + \frac{2}{3}\pi)\overline{H_1}(\omega_2 + \frac{2}{3}\pi) \\
& + G'_1(\omega_2 + \frac{4}{3}\pi)\overline{H_1}(\omega_2 + \frac{4}{3}\pi)] + R_1(\omega_1)\overline{Q_1}(\omega_1)[G'_1(\omega_2)\overline{G_1}(\omega_2) \\
& + G'_1(\omega_2 + \frac{2}{3}\pi)\overline{G_1}(\omega_2 + \frac{2}{3}\pi) + G'_1(\omega_2 + \frac{4}{3}\pi)\overline{G_1}(\omega_2 + \frac{4}{3}\pi)] \\
= & |P_1(\omega_1)|^2 + |Q_1(\omega_1)|^2 + |R_1(\omega_1)|^2 + P_1(\omega_1)\overline{Q_1}(\omega_1) \cdot 0 \\
& + P_1(\omega_1)\overline{R_1}(\omega_1) \cdot 0 + Q_1(\omega_1)\overline{P_1}(\omega_1) \cdot 0 + Q_1(\omega_1)\overline{R_1}(\omega_1) \cdot 0 \\
& + R_1(\omega_1)\overline{P_1}(\omega_1) \cdot 0 + R_1(\omega_1)\overline{Q_1}(\omega_1) \cdot 0 \\
= & |H_1(3\omega_1)|^2 + |H_1(3\omega_1 + \frac{2}{3}\pi)|^2 + |H_1(3\omega_1 + \frac{4}{3}\pi)|^2 = 1.
\end{aligned}$$

Next, we assume that the lemma holds for all $2 \leq \ell \leq k < n$. For $2 \leq \ell \leq k$, we have $H_\ell(0, \dots, 0) = 1$, and $G_\ell(0, \dots, 0) = G'_\ell(0, \dots, 0) = 0$. Since $\ell_{k+1} \leq k$, then

$$\begin{aligned}
& H_{k+1}(0, \dots, 0) \\
= & P_k(0, \dots, 0)H_{\ell_{k+1}}(0, \dots, 0) + Q_k(0, \dots, 0)G_{\ell_{k+1}}(0, \dots, 0) \\
& + R_k(0, \dots, 0)G'_{\ell_{k+1}}(0, \dots, 0) = 1.
\end{aligned}$$

The induction hypothesis also implies that for $2 \leq \ell \leq k$, H_ℓ , G_ℓ , G'_ℓ satisfy the equation (2.4). Next, we check the result for $k+1$. For the sake of simplicity, we let $P_k(\cdot)$ denote $P_k(\omega_1, \dots, \omega_k)$, $P_k(\cdot + \frac{2}{3}\pi_k)$ denote $P_k(\omega_1 + \frac{2}{3}\pi, \dots, \omega_k + \frac{2}{3}\pi)$, and $P_k(\cdot + \frac{4}{3}\pi_k)$ denote $P_k(\omega_1 + \frac{4}{3}\pi, \dots, \omega_k + \frac{4}{3}\pi)$. Similarly, Q_k , R_k , H_{k+1} and $G_{\ell_{k+1}}$, $G'_{\ell_{k+1}}$ will be denoted as follows. Since $P_k(\cdot) = P_k(\cdot + \frac{2}{3}\pi_k) = P_k(\cdot + \frac{4}{3}\pi_k)$, $Q_k(\cdot) = Q_k(\cdot + \frac{2}{3}\pi_k) = Q_k(\cdot + \frac{4}{3}\pi_k)$, $R_k(\cdot) = R_k(\cdot + \frac{2}{3}\pi_k) = R_k(\cdot + \frac{4}{3}\pi_k)$, then by using the induction

hypothesis, we get

$$\begin{aligned}
& |H_{k+1}(\cdot)|^2 + |H_{k+1}(\cdot + \frac{2}{3}\pi_{k+1})|^2 + |H_{k+1}(\cdot + \frac{4}{3}\pi_{k+1})|^2 \\
= & |P_k(\cdot)|^2 \left[|H_{\ell_{k+1}}(\cdot)|^2 + |H_{\ell_{k+1}}(\cdot + \frac{2}{3}\pi_{\ell_{k+1}})|^2 + |H_{\ell_{k+1}}(\cdot + \frac{4}{3}\pi_{\ell_{k+1}})|^2 \right] \\
& + |Q_k(\cdot)|^2 \left[|G_{\ell_{k+1}}(\cdot)|^2 + |G_{\ell_{k+1}}(\cdot + \frac{2}{3}\pi_{\ell_{k+1}})|^2 + |G_{\ell_{k+1}}(\cdot + \frac{4}{3}\pi_{\ell_{k+1}})|^2 \right] \\
& + |R_k(\cdot)|^2 \left[|G'_{\ell_{k+1}}(\cdot)|^2 + |G'_{\ell_{k+1}}(\cdot + \frac{2}{3}\pi_{\ell_{k+1}})|^2 + |G'_{\ell_{k+1}}(\cdot + \frac{4}{3}\pi_{\ell_{k+1}})|^2 \right] \\
& + P_k \overline{Q_k}(\cdot) \left[H_{\ell_{k+1}} \overline{G_{\ell_{k+1}}}(\cdot) + H_{\ell_{k+1}} \overline{G_{\ell_{k+1}}}(\cdot + \frac{2}{3}\pi_{\ell_{k+1}}) \right. \\
& \left. + H_{\ell_{k+1}} \overline{G'_{\ell_{k+1}}}(\cdot + \frac{4}{3}\pi_{\ell_{k+1}}) \right] + P_k \overline{R_k}(\cdot) \left[H_{\ell_{k+1}} \overline{G'_{\ell_{k+1}}}(\cdot) \right. \\
& \left. + H_{\ell_{k+1}} \overline{G'_{\ell_{k+1}}}(\cdot + \frac{2}{3}\pi_{\ell_{k+1}}) + H_{\ell_{k+1}} \overline{G'_{\ell_{k+1}}}(\cdot + \frac{4}{3}\pi_{\ell_{k+1}}) \right] \\
& + Q_k \overline{P_k}(\cdot) \left[G_{\ell_{k+1}} \overline{H_{\ell_{k+1}}}(\cdot) + G_{\ell_{k+1}} \overline{H_{\ell_{k+1}}}(\cdot + \frac{2}{3}\pi_{\ell_{k+1}}) \right. \\
& \left. + G_{\ell_{k+1}} \overline{H_{\ell_{k+1}}}(\cdot + \frac{4}{3}\pi_{\ell_{k+1}}) \right] + Q_k \overline{R_k}(\cdot) \left[G_{\ell_{k+1}} \overline{G'_{\ell_{k+1}}}(\cdot) \right. \\
& \left. + G_{\ell_{k+1}} \overline{G'_{\ell_{k+1}}}(\cdot + \frac{2}{3}\pi_{\ell_{k+1}}) + G_{\ell_{k+1}} \overline{G'_{\ell_{k+1}}}(\cdot + \frac{4}{3}\pi_{\ell_{k+1}}) \right] \\
& + R_k \overline{P_k}(\cdot) \left[G'_{\ell_{k+1}} \overline{H_{\ell_{k+1}}}(\cdot) + G'_{\ell_{k+1}} \overline{H_{\ell_{k+1}}}(\cdot + \frac{2}{3}\pi_{\ell_{k+1}}) \right. \\
& \left. + G'_{\ell_{k+1}} \overline{H_{\ell_{k+1}}}(\cdot + \frac{4}{3}\pi_{\ell_{k+1}}) \right] + R_k \overline{Q_k}(\cdot) \left[G'_{\ell_{k+1}} \overline{G_{\ell_{k+1}}}(\cdot) \right. \\
& \left. + G'_{\ell_{k+1}} \overline{G_{\ell_{k+1}}}(\cdot + \frac{2}{3}\pi_{\ell_{k+1}}) + G'_{\ell_{k+1}} \overline{G_{\ell_{k+1}}}(\cdot + \frac{4}{3}\pi_{\ell_{k+1}}) \right] \\
= & |P_k(\cdot)|^2 + |Q_k(\cdot)|^2 + |R_k(\cdot)|^2 \\
= & |H_k(3\cdot)|^2 + |H_k(3\cdot + \frac{2}{3}\pi_k)|^2 + |H_k(3\cdot + \frac{4}{3}\pi_k)|^2 = 1.
\end{aligned}$$

Then, the induction hypothesis holds for $k + 1$. Hence, we get $|H_n(\cdot)|^2 + |H_n(\cdot + \frac{2}{3}\pi_n)|^2 + |H_n(\cdot + \frac{4}{3}\pi_n)|^2 = 1$. \square

It is well known that to design a compactly supported orthogonal wavelet basis of $L^2(R^n)$, it is necessary to construct one low-pass filter \mathcal{H}_0 and $3^n - 1$ high-pass filters \mathcal{H}_i , $i = 1, \dots, 3^n - 1$. Consequently, a set of special matrices is required for the design of \mathcal{H}_0 .

For $i = 1, \dots, n-1$, we consider a set of $n-1$ matrices $D_i \in \mathbb{Z}^{n \times n}$ satisfying the following three conditions:

(c₁) $\forall \eta_j \in E_n, \exists \eta_j^1 \in E_n$ and $\eta_j^2 \in E_n$ such that $D_i(\eta_j) = D_i(\eta_j^1) \pmod{(2\pi\mathbb{Z}^n)}$, and $D_i(\eta_j) = D_i(\eta_j^2) \pmod{(2\pi\mathbb{Z}^n)}$, where $\eta_j^1 = \frac{2}{3}\pi_n - \eta_j$, $\eta_j^2 = \frac{4}{3}\pi_n - \eta_j$.

(c₂) If $\eta_{j'} \neq \eta_j$, $\eta_{j'} \neq \eta_j^1$ and $\eta_{j'} \neq \eta_j^2$, then $D_i(\eta_j) \neq D_i(\eta_{j'}) \pmod{(2\pi\mathbb{Z}^n)}$.

(c₃) If $F_i = D_i D_{i-1} \cdots D_1(E_n) \pmod{(2\pi\mathbb{Z}^n)}$, then F_i is a symmetric subset of E_n ; i.e., $\forall \eta \in F_i, \eta^1 \in F_i, \eta^2 \in F_i$.

By Lemma 2.5, we prove the following theorem providing us with n -D low-pass orthogonal wavelet filters.

Theorem 2.6. *Let $H_n(\omega_1, \dots, \omega_n)$ be the n -D filter of Lemma 2.1. Let D_1, D_2, \dots, D_{n-1} be the dilation matrices that satisfy the above three conditions (c₁), (c₂) and (c₃). Define an n -D filter \mathcal{H}_0 by*

$$(2.5) \quad \mathcal{H}_0(\omega_1, \dots, \omega_n) = \prod_{k=0}^{n-1} H_n(D_k \cdots D_0(\omega_1, \dots, \omega_n)).$$

Then, $\mathcal{H}_0(0, \dots, 0) = 1$. Moreover, \mathcal{H}_0 satisfies the following orthogonality condition,

$$(2.6) \quad \sum_{i=0}^{3^n-1} |\mathcal{H}_0(\omega + \eta_i)|^2 = 1, \quad \forall \omega \in \mathbb{R}^n,$$

where $\eta_i, i = 0, \dots, 3^n - 1$ are the different points of the set $E_n = \{0, \frac{2}{3}\pi, \frac{4}{3}\pi\}^n$.

Proof. Since $H_n(0, \dots, 0) = 1$, then $\mathcal{H}_0(0, \dots, 0) =$

$\prod_{k=0}^{n-1} H_n(D_k \cdots D_0(0, \dots, 0)) = 1$. We first let $\eta_i^1 = \eta_{3^{n-1}+i}, \eta_i^2 = \eta_{2 \cdot 3^{n-1}+i}, i = 0, \dots, 3^{n-1} - 1$. Since $D_1(\eta_i) = D_1(\eta_i^1) \pmod{(2\pi\mathbb{Z}^n)}$, and $D_1(\eta_i) = D_1(\eta_i^2) \pmod{(2\pi\mathbb{Z}^n)}$, $\forall \eta_i \in E_n$, we conclude that

$$\begin{aligned} & \sum_{i=0}^{3^n-1} |\mathcal{H}_0(\omega + \eta_i)|^2 \\ &= \sum_{i=0}^{3^n-1} |H_n(\omega + \eta_i)|^2 \prod_{k=1}^{n-1} \left| H_n(D_k \cdots D_1(\omega + \eta_i)) \right|^2 \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{3^{n-1}-1} \left[|H_n(\omega + \eta_i)|^2 + |H_n(\omega + \eta_i^1)|^2 + |H_n(\omega + \eta_i^2)|^2 \right] \\
&\quad \times \prod_{k=1}^{n-1} \left| H_n \left(D_k \cdots D_1(\omega + \eta_i) \right) \right|^2 \\
&= \sum_{i=0}^{3^{n-1}-1} \prod_{k=1}^{n-1} \left| H_n \left(D_k \cdots D_1(\omega + \eta_i) \right) \right|^2.
\end{aligned}$$

Again, we let $D_1^1(\eta_i) = D_1(\eta_{3^{n-2}+i})$, $D_1^2(\eta_i) = D_1(\eta_{2 \cdot 3^{n-2}+i})$, where $D_1^1(\eta_i)$ and $D_1^2(\eta_i)$ are symmetric of $D_1(\eta_i)$, $i = 0, \dots, 3^{n-2} - 1$. Then $D_2[D_1(\eta_i)] = D_2[D_1^1(\eta_i)] \pmod{(2\pi Z^n)}$, and $D_2[D_1(\eta_i)] = D_2[D_1^2(\eta_i)] \pmod{(2\pi Z^n)}$, $\forall \eta_i \in E_n$, obtaining

$$\begin{aligned}
&\sum_{i=0}^{3^{n-1}-1} \prod_{k=1}^{n-1} \left| H_n \left(D_k \cdots D_1(\omega + \eta_i) \right) \right|^2 \\
&= \sum_{i=0}^{3^{n-1}-1} \left| H_n \left(D_1(\omega + \eta_i) \right) \right|^2 \prod_{k=2}^{n-1} \left| H_n \left(D_k \cdots D_2 D_1(\omega + \eta_i) \right) \right|^2 \\
&= \sum_{i=0}^{3^{n-2}-1} \left[\left| H_n \left(D_1(\omega + \eta_i) \right) \right|^2 + \left| H_n \left(D_1^1(\omega + \eta_i) \right) \right|^2 \right. \\
&\quad \left. + \left| H_n \left(D_1^2(\omega + \eta_i) \right) \right|^2 \right] \times \prod_{k=2}^{n-1} \left| H_n \left(D_k \cdots D_1(\omega + \eta_i) \right) \right|^2 \\
&= \sum_{i=0}^{3^{n-2}-1} \prod_{k=2}^{n-1} \left| H_n \left(D_k \cdots D_1(\omega + \eta_i) \right) \right|^2 \\
&= \vdots \\
&= \left| H_n \left([D_{n-1} \cdots D_1](\omega + \eta_0) \right) \right|^2 + \left| H_n \left([D_{n-1} \cdots D_1]^1(\omega + \eta_0) \right) \right|^2 \\
&\quad + \left| H_n \left([D_{n-1} \cdots D_1]^2(\omega + \eta_0) \right) \right|^2 = 1,
\end{aligned}$$

where $[D_{n-1} \cdots D_1]^1(\eta_i)$ and $[D_{n-1} \cdots D_1]^2(\eta_i)$ are symmetric of $[D_{n-1} \cdots D_1](\eta_i)$. Hence, (2.6) holds. \square

Theorem 2.7. *The wavelet filters $\mathcal{H}_0(\omega_1, \omega_2)$ given by Theorem 2.6 are nonseparable.*

Proof. According to Theorem 2.6,

$$(2.7) \quad \mathcal{H}_0(\omega_1, \omega_2) = H_2(\omega_1, \omega_2)H_2\left(D_1(\omega_1, \omega_2)\right),$$

where $H_2(\omega_1, \omega_2)$ is defined as given in given in Lemma 2.5. To show that $\mathcal{H}_0(\omega_1, \omega_2)$ is nonseparable, we only need to check that $H_2(\omega_1, \omega_2)$ is nonseparable. By Lemma 2.5, we get

$$(2.8) \quad H_2(\omega_1, \omega_2) = H_1(3\omega_1)H_1(\omega_2) + H_1(3\omega_1 + \frac{2}{3}\pi)G_1(\omega_2) + H_1(3\omega_1 + \frac{4}{3}\pi)G_1'(\omega_2),$$

where $H_1(\omega_1)$ is a 1-D orthogonal filter. We assume that $H_2(\omega_1, \omega_2)$ is separable.

First case: we prove that

$$(2.9) \quad H_2(\omega_1, \omega_2) = m_1(a_{11}\omega_1 + a_{12}\omega_2)m_2(a_{21}\omega_1 + a_{22}\omega_2)$$

is not possible, where $m_1(\cdot)$, $m_2(\cdot)$ are two 1-D orthogonal filters, $(a_{11}, a_{12}), (a_{21}, a_{22}) \in \mathbb{Z}^2$.

Next we discuss the following nine cases:

(i) $a_{11} + a_{12} = 0 \pmod{3}$ and $a_{21} + a_{22} = 1 \pmod{3}$. Since

$$(2.10) \quad |H_2(\omega_1, \omega_2)|^2 + |H_2(\omega_1 + \frac{2}{3}\pi, \omega_2 + \frac{2}{3}\pi)|^2 + |H_2(\omega_1 + \frac{4}{3}\pi, \omega_2 + \frac{4}{3}\pi)|^2 = 1.$$

for the sake of simplicity, we let $m_1(A)$ denote $m_1(a_{11}\omega_1 + a_{12}\omega_2)$, $m_2(B)$ denote $m_2(a_{21}\omega_1 + a_{22}\omega_2)$. Then, by substituting (2.9) into (2.10), we get

$$\begin{aligned} & |m_1(A)|^2|m_2(B)|^2 + \left| m_1\left[A + \frac{2}{3}\pi(a_{11} + a_{12})\right] \right|^2 \\ & \times \left| m_2\left[B + \frac{2}{3}\pi(a_{21} + a_{22})\right] \right|^2 + \left| m_1\left[A + \frac{4}{3}\pi(a_{11} + a_{12})\right] \right|^2 \\ & \times \left| m_2\left[B + \frac{4}{3}\pi(a_{21} + a_{22})\right] \right|^2 \\ & = |m_1(A)|^2 \left[|m_2(B)|^2 + |m_2(B + \frac{2}{3}\pi)|^2 + |m_2(B + \frac{4}{3}\pi)|^2 \right] \\ & = |m_1(A)|^2 \cdot 1 = 1, \quad \forall (\omega_1, \omega_2) \in \mathbb{R}^2, \end{aligned}$$

which is a contradiction. The same result holds in the following cases:

- (ii) $a_{11} + a_{12} = 0 \pmod{3}$ and $a_{21} + a_{22} = 2 \pmod{3}$;
- (iii) $a_{11} + a_{12} = 1 \pmod{3}$ and $a_{21} + a_{22} = 0 \pmod{3}$;
- (iv) $a_{11} + a_{12} = 2 \pmod{3}$ and $a_{21} + a_{22} = 0 \pmod{3}$;

(v) $a_{11} + a_{12} = 0 \pmod{3}$ and $a_{21} + a_{22} = 0 \pmod{3}$.

Next, we consider the case:

(vi) $a_{11} + a_{12} = 1 \pmod{3}$ and $a_{21} + a_{22} = 1 \pmod{3}$. Similarly, by substituting (2.9) into (2.10), we get

$$\begin{aligned}
& |m_1(A)|^2 |m_2(B)|^2 + \left| m_1 \left[A + \frac{2}{3}\pi(a_{11} + a_{12}) \right] \right|^2 \\
& \times \left| m_2 \left[B + \frac{2}{3}\pi(a_{21} + a_{22}) \right] \right|^2 + \left| m_1 \left[A + \frac{4}{3}\pi(a_{11} + a_{12}) \right] \right|^2 \\
& \times \left| m_2 \left[B + \frac{4}{3}\pi(a_{21} + a_{22}) \right] \right|^2 \\
= & |m_1(A)|^2 |m_2(B)|^2 + |m_1(A + \frac{2}{3}\pi)|^2 |m_2(B + \frac{2}{3}\pi)|^2 \\
& + [1 - |m_1(A)|^2 - |m_1(A + \frac{2}{3}\pi)|^2][1 - |m_1(B)|^2 - |m_1(B + \frac{2}{3}\pi)|^2] \\
= & |m_1(A)|^2 [|m_1(B)|^2 + |m_1(B + \frac{2}{3}\pi)|^2 - 1] \\
& + |m_1(B)|^2 [|m_1(A)|^2 + |m_1(A + \frac{2}{3}\pi)|^2 - 1] \\
= & |m_1(A + \frac{2}{3}\pi)|^2 [|m_1(B + \frac{2}{3}\pi)|^2 - 1] \\
& + |m_1(B + \frac{2}{3}\pi)|^2 [|m_1(A + \frac{2}{3}\pi)|^2 - 1] + 1 \\
= & 1.
\end{aligned}$$

Since $|m_1(A + \frac{2}{3}\pi)|^2, |m_1(B + \frac{2}{3}\pi)|^2 \leq 1$, by the previous equality, we get $|m_1(A + \frac{2}{3}\pi)|^2 = |m_1(B + \frac{2}{3}\pi)|^2 = 1$, which is a contradiction. The same result holds in the following three cases:

- (vii) $a_{11} + a_{12} = 1 \pmod{3}$ and $a_{21} + a_{22} = 2 \pmod{3}$;
- (viii) $a_{11} + a_{12} = 2 \pmod{3}$ and $a_{21} + a_{22} = 1 \pmod{3}$;
- (viiii) $a_{11} + a_{12} = 2 \pmod{3}$ and $a_{21} + a_{22} = 2 \pmod{3}$.

Hence, we have proved that (2.9) not to be possible.

Second case: we prove that

$$(2.11) \quad H_2(\omega_1, \omega_2) = m_0(a_1\omega_1 + a_2\omega_2)$$

is not possible either, where $m_0(\cdot)$ is a 1-D orthogonal wavelet filter.

Next, we discuss the following three cases:

(i) $a_1 + a_2 = 0 \pmod{3}$. By substituting (2.11) into (2.10), we get $|m_0|^2(a_1\omega_1 + a_2\omega_2) = 1/3, \forall (\omega_1, \omega_2) \in R^2$, which is a contradiction with $m_0(0) = 1$.

(ii) $a_1 + a_2 = 1 \pmod{3}$. First, we assume that $a_1 = 1 \pmod{3}$ and $a_2 = 0 \pmod{3}$. According to (2.11) and (2.8), we get $H_2(\frac{2}{9}\pi, 0) = m_0(\frac{2}{9}\pi a_1) = 0$, $H_2(\frac{2}{9}\pi, \frac{2}{3}\pi) = m_0(\frac{2}{9}\pi a_1 + \frac{2}{3}\pi a_2) = G'_1(\frac{2}{3}\pi)$, and $H_2(\frac{2}{9}\pi, \frac{4}{3}\pi) = m_0(\frac{2}{9}\pi a_1 + \frac{4}{3}\pi a_2) = G'_1(\frac{4}{3}\pi)$. On the other hand, since $a_2 = 0 \pmod{3}$, we obtain $m_0(\frac{2}{9}\pi a_1) = m_0(\frac{2}{9}\pi a_1 + \frac{2}{3}\pi a_2) = m_0(\frac{2}{9}\pi a_1 + \frac{4}{3}\pi a_2)$. Hence, we get $G'_1(\frac{2}{3}\pi) = G'_1(\frac{4}{3}\pi) = 0$, which is a contradiction with $G'_1(0) = 0$. Second, we assume that $a_1 = 0 \pmod{3}$ and $a_2 = 1 \pmod{3}$. According to (2.11) and (2.8), we get $H_2(\frac{2}{3}\pi, 0) = m_0(\frac{2}{3}\pi a_1) = 1$, $H_2(\frac{4}{3}\pi, 0) = m_0(\frac{4}{3}\pi a_1) = 1$. Since $a_1 = 0 \pmod{3}$, we get $m_0(\frac{2}{3}\pi) = m_0(\frac{4}{3}\pi) = 1$, which is a contradiction with $m_0(0) = 1$.

(iii) $a_1 + a_2 = 2 \pmod{3}$. The proof is similar to (ii). Collecting everything together, we conclude that $H_2(\omega_1, \omega_2)$ is nonseparable. Furthermore, $\mathcal{H}_0(\omega_1, \omega_2)$ is also nonseparable. \square

Similarly, the previous proof can be easily extended to the n -D case. Then, we get the following corollary.

Corollary 2.8. *The wavelet filters $\mathcal{H}_0(\omega_1, \dots, \omega_n)$ given by Theorem 2.6 are nonseparable.*

3. Design of n -D high-pass orthogonal wavelet filters

As we have previously mentioned, the construction of the $3^n - 1$ mother wavelets Ψ^i , $i = 1, \dots, 3^n - 1$, requires the construction of $3^n - 1$, n -D high-pass filters \mathcal{H}_i , $i = 1, \dots, 3^n - 1$. These high-pass filters together with the previously defined filters \mathcal{H}_0 have to satisfy the equations,

$$(3.1) \quad \sum_{j=0}^{3^n-1} \mathcal{H}_i(\omega + \eta_j) \overline{\mathcal{H}_{i'}(\omega + \eta_j)} = \delta_{ii'}, \quad \forall 0 \leq i, i' \leq 3^n - 1, \quad \omega \in R^n,$$

where η_j , $j = 0, \dots, 3^n - 1$, are the different points of the set $E_n = \{0, \frac{2}{3}\pi, \frac{4}{3}\pi\}^n$.

In our case, a solution to (3.1) is given by the following theorem.

Theorem 3.1. Let $H_0(\omega) = H_n(\omega_1, \dots, \omega_n)$, where $\omega \in R^n$, and $H_n(\cdot)$ is the wavelet filter as specified in Lemma 2.5. Let D_1, D_2, \dots, D_{n-1} be the dilation matrices that satisfy the three conditions $(c_1), (c_2)$ and (c_3) . If \mathcal{H}_i is the filter defined by:

$$(3.2) \quad \mathcal{H}_i(\omega) = \prod_{k=0}^{n-1} \left[\varepsilon_k^i H_0(D_k \cdots D_0 \omega) + \frac{1 - \varepsilon_k^i}{\sqrt{2}} G_0(D_k \cdots D_0 \omega) + \frac{1 - \varepsilon_k^i}{\sqrt{2}} G'_0(D_k \cdots D_0 \omega) \right],$$

where G_0, G'_0 together with H_0 satisfy the following equation,

$$\begin{pmatrix} H_0(\omega) & H_0(\omega + \frac{2}{3}\pi_n) & H_0(\omega + \frac{4}{3}\pi_n) \\ G_0(\omega) & G_0(\omega + \frac{2}{3}\pi_n) & G_0(\omega + \frac{4}{3}\pi_n) \\ G'_0(\omega) & G'_0(\omega + \frac{2}{3}\pi_n) & G'_0(\omega + \frac{4}{3}\pi_n) \end{pmatrix} \times \begin{pmatrix} H_0(\omega) & H_0(\omega + \frac{2}{3}\pi_n) & H_0(\omega + \frac{4}{3}\pi_n) \\ G_0(\omega) & G_0(\omega + \frac{2}{3}\pi_n) & G_0(\omega + \frac{4}{3}\pi_n) \\ G'_0(\omega) & G'_0(\omega + \frac{2}{3}\pi_n) & G'_0(\omega + \frac{4}{3}\pi_n) \end{pmatrix}^* = I_3,$$

$(\varepsilon_0^i, \varepsilon_1^i, \dots, \varepsilon_n^i)_{i=1, \dots, 3^n-1}$ are the different points of $\{0, 1\}^n \setminus (0, \dots, 0)$. Then, \mathcal{H}_i , $i = 1, \dots, 3^n - 1$, is a solution of (3.1).

Proof. First, we consider two integers $i, i' \in \{1, \dots, 3^n - 1\}$ such that $i \neq i'$ and prove that

$$\sum_{j=0}^{3^n-1} \mathcal{H}_i(\omega + \eta_j) \overline{\mathcal{H}_{i'}(\omega + \eta_j)} = 0.$$

Since $i \neq i'$, then there exists $0 \leq \ell \leq n - 1$ such that $\varepsilon_k^i = \varepsilon_k^{i'}$, $\forall 0 \leq k \leq \ell - 1$ and $\varepsilon_\ell^i \neq \varepsilon_\ell^{i'}$. We assume that $\varepsilon_\ell^i = 1$, $\varepsilon_\ell^{i'} = 0$. We first study the case where $0 \leq \ell < n - 1$. By using the factorization technique in

the proof of Theorem 2.6, we get

$$\begin{aligned}
& \sum_{j=0}^{3^n-1} \mathcal{H}_i(\omega + \eta_j) \overline{\mathcal{H}_{i'}}(\omega + \eta_j) \\
= & \sum_{j=0}^{3^n-1} \left[\varepsilon_0^i H_0(\omega + \eta_j) + \frac{1 - \varepsilon_0^i}{\sqrt{2}} G_0(\omega + \eta_j) + \frac{1 - \varepsilon_0^i}{\sqrt{2}} G'_0(\omega + \eta_j) \right] \\
& \times \left[\varepsilon_0^{i'} \overline{H_0}(\omega + \eta_j) + \frac{1 - \varepsilon_0^{i'}}{\sqrt{2}} \overline{G_0}(\omega + \eta_j) + \frac{1 - \varepsilon_0^{i'}}{\sqrt{2}} \overline{G'_0}(\omega + \eta_j) \right] \\
& \times \prod_{k=1}^{n-1} \left[\varepsilon_k^i H_0(D_k \cdots D_0(\omega + \eta_j)) + \frac{1 - \varepsilon_k^i}{\sqrt{2}} G_0(D_k \cdots D_0(\omega + \eta_j)) \right. \\
& \quad \left. + \frac{1 - \varepsilon_k^i}{\sqrt{2}} G'_0(D_k \cdots D_0(\omega + \eta_j)) \right] \\
& \times \left[\varepsilon_k^{i'} \overline{H_0}(D_k \cdots D_0(\omega + \eta_j)) + \frac{1 - \varepsilon_k^{i'}}{\sqrt{2}} \overline{G_0}(D_k \cdots D_0(\omega + \eta_j)) \right. \\
& \quad \left. + \frac{1 - \varepsilon_k^{i'}}{\sqrt{2}} \overline{G'_0}(D_k \cdots D_0(\omega + \eta_j)) \right] \\
= & \sum_{j=0}^{3^{n-1}-1} \left\{ \left[\varepsilon_0^i H_0(\omega + \eta_j) + \frac{1 - \varepsilon_0^i}{\sqrt{2}} G_0(\omega + \eta_j) + \frac{1 - \varepsilon_0^i}{\sqrt{2}} G'_0(\omega + \eta_j) \right] \right. \\
& \times \left[\varepsilon_0^{i'} \overline{H_0}(\omega + \eta_j) + \frac{1 - \varepsilon_0^{i'}}{\sqrt{2}} \overline{G_0}(\omega + \eta_j) + \frac{1 - \varepsilon_0^{i'}}{\sqrt{2}} \overline{G'_0}(\omega + \eta_j) \right] \\
& + \left[\varepsilon_0^i H_0(\omega + \eta_j^1) + \frac{1 - \varepsilon_0^i}{\sqrt{2}} G_0(\omega + \eta_j^1) + \frac{1 - \varepsilon_0^i}{\sqrt{2}} G'_0(\omega + \eta_j^1) \right] \\
& \times \left[\varepsilon_0^{i'} \overline{H_0}(\omega + \eta_j^1) + \frac{1 - \varepsilon_0^{i'}}{\sqrt{2}} \overline{G_0}(\omega + \eta_j^1) + \frac{1 - \varepsilon_0^{i'}}{\sqrt{2}} \overline{G'_0}(\omega + \eta_j^1) \right] \\
& + \left[\varepsilon_0^i H_0(\omega + \eta_j^2) + \frac{1 - \varepsilon_0^i}{\sqrt{2}} G_0(\omega + \eta_j^2) + \frac{1 - \varepsilon_0^i}{\sqrt{2}} G'_0(\omega + \eta_j^2) \right] \\
& \times \left[\varepsilon_0^{i'} \overline{H_0}(\omega + \eta_j^2) + \frac{1 - \varepsilon_0^{i'}}{\sqrt{2}} \overline{G_0}(\omega + \eta_j^2) + \frac{1 - \varepsilon_0^{i'}}{\sqrt{2}} \overline{G'_0}(\omega + \eta_j^2) \right] \left. \right\} \\
& \times \prod_{k=1}^{n-1} \left[\varepsilon_k^i H_0(D_k \cdots D_0(\omega + \eta_j)) + \frac{1 - \varepsilon_k^i}{\sqrt{2}} \right. \\
& \times G_0(D_k \cdots D_0(\omega + \eta_j)) + \frac{1 - \varepsilon_k^i}{\sqrt{2}} G'_0(D_k \cdots D_0(\omega + \eta_j)) \left. \right] \\
& \times \left[\varepsilon_k^{i'} \overline{H_0}(D_k \cdots D_0(\omega + \eta_j)) + \frac{1 - \varepsilon_k^{i'}}{\sqrt{2}} \overline{G_0}(D_k \cdots D_0(\omega + \eta_j)) \right. \\
& \quad \left. + \frac{1 - \varepsilon_k^{i'}}{\sqrt{2}} \overline{G'_0}(D_k \cdots D_0(\omega + \eta_j)) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1 - \varepsilon_k^{i'}}{\sqrt{2}} \overline{G_0'}(D_k \cdots D_0(\omega + \eta_j)) \Big] \\
= & \sum_{j=0}^{3^{n-1}-1} \prod_{k=1}^{n-1} \left[\varepsilon_k^i H_0(D_k \cdots D_0(\omega + \eta_j)) + \frac{1 - \varepsilon_k^i}{\sqrt{2}} \right. \\
& \times G_0(D_k \cdots D_0(\omega + \eta_j)) + \frac{1 - \varepsilon_k^i}{\sqrt{2}} G_0'(D_k \cdots D_0(\omega + \eta_j)) \Big] \\
& \times \left[\varepsilon_k^{i'} \overline{H_0}(D_k \cdots D_0(\omega + \eta_j)) + \frac{1 - \varepsilon_k^{i'}}{\sqrt{2}} \overline{G_0}(D_k \cdots D_0(\omega + \eta_j)) \right. \\
& \left. + \frac{1 - \varepsilon_k^{i'}}{\sqrt{2}} \overline{G_0'}(D_k \cdots D_0(\omega + \eta_j)) \right] \\
= & \vdots \\
= & \sum_{j=0}^{3^{n-\ell}-1} \prod_{k=\ell}^{n-1} \left[\varepsilon_k^i H_0(D_k \cdots D_0(\omega + \eta_j)) + \frac{1 - \varepsilon_k^i}{\sqrt{2}} \right. \\
& \times G_0(D_k \cdots D_0(\omega + \eta_j)) + \frac{1 - \varepsilon_k^i}{\sqrt{2}} G_0'(D_k \cdots D_0(\omega + \eta_j)) \Big] \\
& \times \left[\varepsilon_k^{i'} \overline{H_0}(D_k \cdots D_0(\omega + \eta_j)) + \frac{1 - \varepsilon_k^{i'}}{\sqrt{2}} \overline{G_0}(D_k \cdots D_0(\omega + \eta_j)) \right. \\
& \left. + \frac{1 - \varepsilon_k^{i'}}{\sqrt{2}} \overline{G_0'}(D_k \cdots D_0(\omega + \eta_j)) \right] \\
= & \sum_{j=0}^{3^{n-\ell-1}-1} \left\{ H_0([D_\ell \cdots D_0](\omega + \eta_j)) \left[\frac{1}{\sqrt{2}} \overline{G_0}([D_\ell \cdots D_0](\omega + \eta_j)) \right. \right. \\
& \left. \left. + \frac{1}{\sqrt{2}} \overline{G_0'}([D_\ell \cdots D_0](\omega + \eta_j)) \right] + H_0([D_\ell \cdots D_0]^1(\omega + \eta_j)) \right. \\
& \times \left[\frac{1}{\sqrt{2}} \overline{G_0}([D_\ell \cdots D_0]^1(\omega + \eta_j)) + \frac{1}{\sqrt{2}} \overline{G_0'}([D_\ell \cdots D_0]^1(\omega + \eta_j)) \right] \\
& \left. + H_0([D_\ell \cdots D_0]^2(\omega + \eta_j)) \left[\frac{1}{\sqrt{2}} \overline{G_0}([D_\ell \cdots D_0]^2(\omega + \eta_j)) + \frac{1}{\sqrt{2}} \right. \right. \\
& \left. \left. \times \overline{G_0'}([D_\ell \cdots D_0]^2(\omega + \eta_j)) \right] \right\} \times \prod_{k=\ell+1}^{n-1} \left[\varepsilon_k^i H_0(D_k \cdots D_0(\omega + \eta_j)) \right. \\
& \left. + \frac{1 - \varepsilon_k^i}{\sqrt{2}} G_0(D_k \cdots D_0(\omega + \eta_j)) + \frac{1 - \varepsilon_k^i}{\sqrt{2}} G_0'(D_k \cdots D_0(\omega + \eta_j)) \right] \\
& \times \left[\varepsilon_k^{i'} \overline{H_0}(D_k \cdots D_0(\omega + \eta_j)) + \frac{1 - \varepsilon_k^{i'}}{\sqrt{2}} \overline{G_0}(D_k \cdots D_0(\omega + \eta_j)) \right. \\
& \left. + \frac{1 - \varepsilon_k^{i'}}{\sqrt{2}} \overline{G_0'}(D_k \cdots D_0(\omega + \eta_j)) \right] = 0,
\end{aligned}$$

where $[D_\ell \cdots D_0]^1(\eta_j)$ and $[D_\ell \cdots D_0]^2(\eta_j)$ are symmetric of $[D_\ell \cdots D_0](\eta_j)$, $\ell = 0, \dots, n-2$. For the second case where $\ell = n-1$, one can easily check that

$$\begin{aligned} & \sum_{j=0}^{3^n-1} \mathcal{H}_i(\omega + \eta_j) \overline{\mathcal{H}_{i'}}(\omega + \eta_j) \\ = & H_0\left([D_{n-1} \cdots D_0](\omega + \eta_0)\right) \left[\frac{1}{\sqrt{2}} \overline{G_0}\left([D_{n-1} \cdots D_0](\omega + \eta_0)\right) \right. \\ & \left. + \frac{1}{\sqrt{2}} \overline{G'_0}\left([D_{n-1} \cdots D_0](\omega + \eta_0)\right) \right] + H_0\left([D_{n-1} \cdots D_0]^1(\omega + \eta_0)\right) \\ & \times \left[\frac{1}{\sqrt{2}} \overline{G_0}\left([D_{n-1} \cdots D_0]^1(\omega + \eta_0)\right) \right. \\ & \left. + \frac{1}{\sqrt{2}} \overline{G'_0}\left([D_{n-1} \cdots D_0]^1(\omega + \eta_0)\right) \right] + H_0\left([D_{n-1} \cdots D_0]^2(\omega + \eta_0)\right) \\ & \times \left[\frac{1}{\sqrt{2}} \overline{G_0}\left([D_{n-1} \cdots D_0]^2(\omega + \eta_0)\right) \right. \\ & \left. + \frac{1}{\sqrt{2}} \overline{G'_0}\left([D_{n-1} \cdots D_0]^2(\omega + \eta_0)\right) \right] = 0. \end{aligned}$$

Finally, the case $i = i'$ has a proof similar to that of Theorem 2.6 and we conclude that $\sum_{j=0}^{3^n-1} \mathcal{H}_i(\omega + \eta_j) \overline{\mathcal{H}_i}(\omega + \eta_j) = 1$. \square

Remark 3.2. It is well known that condition (3.1) does not ensure that \mathcal{H}_i , $i = 1, \dots, 3^n - 1$, generate an orthogonal wavelet basis of $L^2(\mathbb{R}^n)$. In fact, we need to study the stability of the wavelet functions $\Psi_{j,k}^i$ generated by \mathcal{H}_i . The stability of $\Psi_{j,k}^i$ can be similarly established as in [7].

4. An example

Example. Let $H_1(\omega) = \frac{1}{3}(1 + z + z^2)$, $z = e^{-i\omega}$, $\omega \in \mathbb{R}$. According to [4], $H_1(\omega)$ generates an orthogonal Haar scaling function ϕ with scale=3. Then, we have $\sum_{k \in \mathbb{Z}} |\hat{\phi}(\omega + 2\pi k)|^2 = 1$. Hence, the translates of ϕ are stable. Furthermore, $H_1(\omega)$ also satisfies the condition,

$$|H_1(\omega)|^2 + |H_1(\omega + \frac{2}{3}\pi)|^2 + |H_1(\omega + \frac{4}{3}\pi)|^2 = 1, \quad \forall \omega \in \mathbb{R}.$$

Let $H_2(\omega_1, \omega_2)$ be the 2-D filter given by $H_2(\omega_1, \omega_2) = H_1(\omega_1)$. It is easy to see that $H_2(\omega_1, \omega_2)$ satisfies the condition (2.3). Consequently, we consider the matrix $D_1 = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$. It is easy to check that D_1 satisfies the three conditions (c_1) , (c_2) and (c_3) . Hence, by applying Theorem 2.6, we conclude that the 2-D wavelet filter \mathcal{H}_0 given by

$$\begin{aligned} \mathcal{H}_0(\omega_1, \omega_2) &= H_2(\omega_1, \omega_2)H_2\left(D_1(\omega_1, \omega_2)\right) \\ &= H_1(\omega_1)H_1(2\omega_1 + \omega_2) \\ &= \left[\frac{1}{3}(1 + e^{-i\omega_1} + e^{-2i\omega_1})\right] \left[\frac{1}{3}(1 + e^{-i(2\omega_1 + \omega_2)} + e^{-i(2\omega_1 + \omega_2)})\right] \\ &= \frac{1}{9}(1 + z_1 + z_1^2 + z_1^2 z_2 + z_1^3 z_2 + z_1^4 z_2 + z_1^4 z_2^2 + z_1^5 z_2^2 \\ &\quad + z_1^6 z_2^2) \end{aligned}$$

satisfies the orthogonality condition (2.6), where $z_1 = e^{-i\omega_1}$, $z_2 = e^{-i\omega_2}$. Note that if Φ denotes the scaling function generated by \mathcal{H}_0 , then according to [7], we conclude that the translates of Φ are stable and \mathcal{H}_0 generates a stable orthogonal wavelet basis of $L^2(\mathbb{R}^2)$.

Furthermore, let $G_1(\omega) = -\frac{\sqrt{2}}{6} + \frac{\sqrt{2}}{3}z - \frac{\sqrt{2}}{6}z^2$, $G'_1(\omega) = -\frac{\sqrt{6}}{6} + \frac{\sqrt{6}}{6}z^2$, $z = e^{-i\omega}$, be the corresponding high-pass filters of $H_1(\omega)$ (see [4]). We can check that H_1, G_1, G'_1 satisfy the equation (2.4). Let $G_2(\omega_1, \omega_2)$ and $G'_2(\omega_1, \omega_2)$ be the 2-D filters given by $G_2(\omega_1, \omega_2) = G_1(\omega_1)$ and $G'_2(\omega_1, \omega_2) = G'_1(\omega_1)$, respectively. Then, it is easy to see that $H_2(\omega_1, \omega_2), G_2(\omega_1, \omega_2)$ and $G'_2(\omega_1, \omega_2)$ satisfy the equation (2.4). According to Theorem 3.1, the corresponding high-pass filters \mathcal{H}_i $i = 1, \dots, 8$, are obtained via (3.2). Since the translates of Φ are stable, we conclude that the $\Psi_{j,k}^i$ form a stable orthogonal wavelet basis of $L^2(\mathbb{R}^2)$.

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