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# A RELAXED EXTRAGRADIENT APPROXIMATION METHOD OF TWO INVERSE-STRONGLY MONOTONE MAPPINGS FOR A GENERAL SYSTEM OF VARIATIONAL INEQUALITIES, FIXED POINT AND EQUILIBRIUM PROBLEMS<sup>†</sup>

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ABSTRACT. We introduce and study an iterative sequence for finding the common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the solutions of the general system of variational inequality for two inverse-strongly monotone mappings. Under suitable conditions, some strong convergence theorems for approximating a common element of the above three sets are obtained. Moreover, using the above theorem, we also find solutions of a general system of variational inequalities and a zero of a maximal monotone operator in a real Hilbert space. As applications, we utilize our results to study the zeros of the maximal monotone and some convergence problem for strictly pseudocontractive mappings. Our results include the previous results as special cases extending and improving the results of Ceng et al. [4], Yao and Yao [18] and some others.

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## 1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Recall that a mapping T of H into itself is called nonexpansive if  $||Tx - Ty|| \le ||x - y||$  for all  $x, y \in H$ . A point  $x \in C$  is a fixed point of T provided Tx = x. Denote by F(T) the set of fixed points of T; that is,  $F(T) = \{x \in C : Tx = x\}$ . Let f be a bifunction of  $C \times C$ into  $\mathbf{R}$ , where  $\mathbf{R}$  is the set of real numbers. The equilibrium problem for  $f : C \times C \to \mathbf{R}$  is to find  $x \in C$  such that

(1.1) 
$$f(x,y) \ge 0$$
 for all  $y \in C$ .

The set of solutions of (1.1) is denoted by EP(f). Given a mapping  $T: C \to H$ , let  $f(x, y) = \langle Tx, y - x \rangle$  for all  $x, y \in C$ . Then,  $z \in EP(f)$  if and only if  $\langle Tz, y - z \rangle \geq 0$  for all  $y \in C$ ; i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to finding a solution of (1.1). In 1997, Combettes and Hirstoaga [5] introduced an iterative scheme for finding the best approximation to initial data when EP(f) is nonempty and proved a strong convergence theorem.

Let  $A: C \to H$  be a mapping. The classical variational inequality, denoted by VI(A, C), is to find  $u \in C$  such that

(1.2) 
$$\langle Au, v-u \rangle \ge 0,$$

for all  $v \in C$ . The variational inequality has been extensively studied in the literature; see, e.g., [1, 6, 17, 19, 20] and the references therein. A mapping A of C into H is called *monotone* if

(1.3) 
$$\langle Au - Av, u - v \rangle \ge 0,$$

for all  $u, v \in C$ . A mapping A of C into H is called  $\alpha$ -inverse-stronglymonotone if there exists a positive real number  $\alpha$  such that

(1.4) 
$$\langle Au - Av, u - v \rangle \ge \alpha \|Au - Av\|^2$$

for all  $u, v \in C$ . It is obvious that any  $\alpha$ -inverse-strongly-monotone mapping A is monotone and Lipschitz continuous. For finding an element of  $F(S) \cap VI(A, C)$ , Takahashi and Toyoda [12] introduced the following iterative scheme:

(1.5) 
$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n),$$

for every n = 0, 1, 2, ..., where  $x_0 = x \in C, \alpha_n$  is a sequence in (0, 1), and  $\lambda_n$  is a sequence in  $(0, 2\alpha)$ . Recently, Nadezhkina and Takahashi [7] and Zeng and Yao [20] proposed some new iterative schemes for finding elements in  $F(S) \cap VI(A, C)$ . In 1976, Korpelevich [2] introduced the following so-called extragradient method:

(1.6) 
$$\begin{cases} x_0 = x \in C, \\ \bar{x}_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = P_C(x_n - \lambda_n A \bar{x}_n), \end{cases}$$

for all  $n \geq 0$ , where  $\lambda_n \in (0, \frac{1}{k}), C$  is a closed convex subset of  $\mathbb{R}^n$  and A is a monotone and k-Lipschitz continuous mapping of C into  $\mathbb{R}^n$ . He proved that if VI(C, A) is nonempty, then the sequences  $\{x_n\}$  and  $\{\bar{x}_n\}$ , generated by (1.6), converge to the same point  $z \in VI(C, A)$ .

Motivated by the idea of Korpelevichs extragradient method, Zeng and Yao [20] introduced a new extragradient method for finding an element of  $F(S) \cap VI(C, A)$  and obtained the following strong convergence theorem under some suitable conditions. Let  $\{x_n\}$  and  $\{y_n\}$  be sequences in C defined as follows:

(1.7) 
$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n u + (1 - \alpha_n) SP_C(x_n - \lambda_n A y_n), \ \forall n \ge 0. \end{cases}$$

Then, the sequence  $\{x_n\}$  and  $\{y_n\}$  converge strongly to the same point  $P_{\mathcal{F}(S)\cap VI(C,A)}x_0$  provided that  $\lim_{n\to\infty} ||x_{n+1}-x_n|| = 0$ . Later, Nadezhkina and Takahashi [7] and Zeng and Yao [20] proposed some new iterative schemes for finding elements in  $F(S)\cap VI(C, A)$ . In the same year, Yao and Yao [18] introduced the following iterative scheme: Let C be a closed convex subset of real Hilbert space H. Let A be an  $\alpha$ -inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that  $F(S) \cap VI(C, A) \neq \emptyset$ . Suppose  $x_1 = u \in C$ and  $\{x_n\}, \{y_n\}$  are given by (1.7) where  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  are three sequences in [0, 1] and  $\{\lambda_n\}$  is a sequence in  $[0, 2\alpha]$ . They proved that the sequence  $\{x_n\}$  defined by (1.7) converges strongly to common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for  $\alpha$ -inverse-strongly monotone mappings under some parameter controlling conditions. After that, Plubtieng and Punpaeng [9] introduced an iterative scheme,

(1.8) 
$$\begin{cases} f(y_n, u) + \frac{1}{r_n} \langle u - y_n, y_n - x_n \rangle \ge 0, & \forall u \in C, \\ y_n = P_C(x_n - \lambda_n A x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda_n A y_n), \end{cases}$$

for approximating a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the equilibrium problem and obtained a strong convergence theorem in a real Hilbert space.

Let C be a closed convex subset of real Hilbert space H. Let  $A, B : C \to H$  be two mappings. We consider the following problem of finding  $(x^*, y^*) \in C \times C$  such that

(1.9) 
$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \ge 0, \quad \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^* \rangle \ge 0, \quad \forall x \in C, \end{cases}$$

which is called a general system of variational inequalities where  $\lambda > 0$ and  $\mu > 0$  are two constants. In particular, if A = B, then problem (1.9) reduces to finding  $(x^*, y^*) \in C \times C$  such that

(1.10) 
$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \ge 0, \quad \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^* \rangle \ge 0, \quad \forall x \in C, \end{cases}$$

which is defined by Verma [14] and Verma [15], and is called the new system of variational inequalities. Furthermore, if  $x^* = y^*$ , then problem (1.10) reduces to the classical variational inequality VI(C, A).

Recently, Ceng et al. [4] introduced the following iterative scheme by a relaxed extragradient method. Let the mappings  $A, B : C \longrightarrow H$ be  $\alpha$ -inverse-strongly monotone and  $\beta$ -inverse-strongly monotone, respectively. Let  $S : C \longrightarrow C$  be a nonexpansive mapping and suppose  $x_1 = u \in C$  and  $\{x_n\}$  is generated by

(1.11) 
$$\begin{cases} y_n = P_C(x_n - \mu B x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + SP_C(y_n - \lambda A y_n), & n \ge 1, \end{cases}$$

where  $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$ , and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  are three sequences in [0, 1] with  $\alpha_n + \beta_n + \gamma_n = 1, \forall n \ge 1$ . Then, they proved that the iterative sequence  $\{x_n\}$  converges strongly to some point  $x_0 \in C$ .

Here, motivated and inspired by the above results, we will introduce a new iterative scheme (3.1) below for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solutions of a general system of variational inequality problem for two inverse-strongly-monotone mappings in a Hilbert space. Then, we prove some strong convergence theorems which are connected with Ceng et al.'s result [4], Takahashi and Takahashi's result [13] and Zeng and Yao's result [20]. Our results extends and improve the corresponding results of Ceng et al. [4], Plubtieng and Punpaeng [9], Su et al. [10] and several others.

# 2. Preliminaries

Let *H* be a real Hilbert space with norm  $\|\cdot\|$  and inner product  $\langle\cdot,\cdot\rangle$  and let *C* be a closed convex subset of *H*. Let *H* be a real Hilbert space. Then,

(2.1) 
$$||x - y||^2 = ||x||^2 - ||y||^2 - 2\langle x - y, y \rangle$$

and

(2.2) 
$$\|\lambda x + (1-\lambda)y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all  $x, y \in H$  and  $\lambda \in [0, 1]$ . For every point  $x \in H$ , there exists a unique nearest point in C, denoted by  $P_C x$ , such that

$$||x - P_C x|| \le ||x - y|| \quad \text{for all } y \in C$$

 $P_C$  is called the metric projection of H onto C. It is well known that  $P_C$  is a nonexpansive mapping of H onto C and satisfies

(2.3) 
$$\langle x - y, P_C x - P_C y \rangle \ge \|P_C x - P_C y\|^2,$$

for every  $x, y \in H$ . Moreover,  $P_C x$  is characterized by the following properties:  $P_C x \in C$  and

(2.4) 
$$\langle x - P_C x, y - P_C x \rangle \le 0,$$

(2.5) 
$$||x - y||^2 \ge ||x - P_C x||^2 + ||y - P_C x||^2,$$

for all  $x \in H, y \in C$ . It is easy to see that the following is true:

(2.6) 
$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda Au), \lambda > 0.$$

The following lemmas will be useful for proving our convergence result.

**Lemma 2.1.** (Osilike and Igbokwe [8]) Let  $(E, \langle ., . \rangle)$  be an inner product space. Then, for all  $x, y, z \in E$  and  $\alpha, \beta, \gamma \in [0, 1]$  with  $\alpha + \beta + \gamma = 1$ , we have,

$$\|\alpha x + \beta y + \gamma z\|^{2} = \alpha \|x\|^{2} + \beta \|y\|^{2} + \gamma \|z\|^{2} - \alpha \beta \|x - y\|^{2} - \alpha \gamma \|x - z\|^{2} - \beta \gamma \|y - z\|^{2}.$$

**Lemma 2.2.** (Suzuki [11]) Let  $\{x_n\}$  and  $\{y_n\}$  be bounded sequences in a Banach space X and let  $\{\beta_n\}$  be a sequence in [0,1] with  $0 < \lim \inf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$ . Suppose  $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers  $n \geq 0$  and  $\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$ . Then,  $\lim_{n \to \infty} \|y_n - x_n\| = 0$ . **Lemma 2.3.** (Goebel and Kirk [3]) Let H be a Hilbert space, C a closed convex subset of H, and  $T : C \rightarrow C$  a nonexpansive mapping with  $F(T) \neq \emptyset$ . If  $\{x_n\}$  is a sequence in C weakly converging to  $x \in C$  and if  $\{(I-T)x_n\}$  converges strongly to y, then (I-T)x = y.

**Lemma 2.4.** (Xu [16]). Assume  $\{a_n\}$  is a sequence of nonnegative real numbers such that

$$a_{n+1} \le (1 - \alpha_n)a_n + \delta_n, \ n \ge 0,$$

where  $\{\alpha_n\}$  is a sequence in (0,1) and  $\{\delta_n\}$  is a sequence in **R** such that

(1)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , (2)  $\limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \le 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty$ .

Then,  $\lim_{n\to\infty} a_n = 0$ .

For solving the equilibrium problem for a bifunction  $f: C \times C \to \mathbf{R}$ , let us assume that F satisfies the following conditions:

- (A1) f(x, x) = 0, for all  $x \in C$ ;
- (A2) f is monotone; i.e.,  $f(x, y) + F(y, x) \le 0$ , for all  $x, y \in C$ ;
- (A3) for each  $x, y, z \in C$ ,  $\lim_{t\to 0} f(tz + (1-t)x, y) \le f(x, y)$ ;
- (A4) for each  $x \in C, y \mapsto f(x, y)$  is convex and lower semicontinuous.

The following lemma appears implicitly in [1]

**Lemma 2.5.** (Blum and Oettli [1]) Let C be a nonempty closed convex subset of H and let f be a bifunction of  $C \times C$  into **R** satisfying (A1)-(A4). Let r > 0 and  $x \in H$ . Then, there exists  $z \in C$  such that

$$f(z,y) + \frac{1}{r} \langle y - z, z - x \rangle \ge 0$$
 for all  $y \in C$ .

The following lemma is given in [5].

**Lemma 2.6.** (Combettes and Hirstoaga [5]) Assume that  $f: C \times C \to \mathbf{R}$ satisfies (A1)-(A4). For r > 0 and  $x \in H$ , define a mapping  $T_r : H \to C$ as follows:

$$T_r(x) = \{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \ge 0, \forall y \in C \},\$$

for all  $z \in H$ . Then, the followings hold:

(1)  $T_r$  is single- valued;

- (2)  $T_r$  is firmly nonexpansive; i.e., for any  $x, y \in H$ ,  $||T_r x T_r y||^2 \le \langle T_r x T_r y, x y \rangle$ ;
- (3)  $F(T_r) = EP(f);$
- (4) EP(f) is closed and convex.

**Lemma 2.7.** (Ceng et al. [4, Lemma 2.1]) For given  $x^*, y^* \in C \times C$ ,  $(x^*, y^*)$  is a solution of problem (1.9) if and only if  $x^*$  is a fixed point of the mapping  $G: C \to C$  defined by

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \quad \forall x \in C,$$

where  $y^* = P_C(x^* - \mu B x^*)$ ,  $\lambda, \mu$  are positive constants and  $A, B : C \to H$  are two mappings.

**Remark 2.8.** Let  $\mathcal{A} : C \to H$  be  $\alpha$ -inverse-strongly-monotone. For each  $u, v \in C$  and  $\lambda > 0$ , we have,

$$\|(I - \lambda \mathcal{A})u - (I - \lambda \mathcal{A})v\|^2 = \|(u - v) - \lambda(\mathcal{A}u - \mathcal{A}v)\|^2$$
  
$$= \|u - v\|^2 - 2\lambda\langle u - v, \mathcal{A}u - \mathcal{A}v\rangle$$
  
$$+ \lambda^2 \|\mathcal{A}u - \mathcal{A}v\|^2$$
  
$$\leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|\mathcal{A}u - \mathcal{A}v\|^2.$$

So, if  $\lambda \leq 2\alpha$ , then  $I - \lambda \mathcal{A}$  is a nonexpansive mapping from C to H.

We note that the mapping  $G : C \to C$  is a nonexpansive mapping provided  $\lambda \in (0, 2\alpha)$  and  $\mu \in (0, 2\beta)$ .

Throughout this paper, the set of fixed points of the mapping G is denoted by  $\Omega$ .

# 3. Main results

Here, we introduce an iterative scheme by the relaxed extragradient approximation method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solution set of the general system of variational inequality problem for two inverse-strongly monotone mappings in a real Hilbert space. We prove that the iterative sequences converge strongly to a common element of the above three sets. **Theorem 3.1.** Let C be a closed convex subset of a real Hilbert space H. Let f be a bifunction from  $C \times C$  to **R** satisfying (A1)-(A4) and A, B :  $C \longrightarrow H$  be  $\alpha$ - and  $\beta$ -inverse-strongly monotone mappings, respectively. Let S be a nonexpansive mapping of C into itself such that  $F(S) \cap \Omega \cap$  $EP(f) \neq \emptyset$ , given  $x_1 = u \in H$  arbitrary. Let the sequences  $\{x_n\}, \{y_n\}$ and  $\{u_n\}$  be given by

(3.1) 
$$\begin{aligned} f(u_n, y) &+ \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C; \\ y_n &= P_C(u_n - \mu B u_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), \forall n \in \mathbf{N}, \end{aligned}$$

where  $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  are three sequences in [0, 1] and  $\{r_n\} \subset (0, \infty)$  satisfying the following conditions:

 $\begin{array}{ll} (\mathrm{i}) & \alpha_n + \beta_n + \gamma_n = 1, \\ (\mathrm{ii}) & \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \\ (\mathrm{iii}) & 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1, \\ (\mathrm{iv}) & \liminf_{n \to \infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \end{array}$ 

Then,  $\{x_n\}$  converges strongly to  $z \in F(S) \cap \Omega \cap EP(f)$ , where  $z = P_{F(S)\cap \Omega \cap EP(f)}u$  and (z, y) is a solution of problem (1.9), where  $y = P_C(z - \mu Bz)$ .

**Proof.** Let  $x^* \in F(S) \cap \Omega \cap EP(f)$ , and let  $\{T_{r_n}\}$  be a sequence of mappings defined as in Lemma 2.6 and  $u_n = T_{r_n} x_n$ . Then,  $x^* = Sx^*$ ,  $x^* = T_{r_n} x^*$  and

$$x^* = P_C[P_C(x^* - \mu B x^*) - \lambda A P_C(x^* - \mu B x^*)],$$

where we put  $y^* = P_C(x^* - \mu Bx^*)$  and  $v_n = P_C(y_n - \lambda Ay_n)$ . Then,  $x^* = P_C(y^* - \lambda Ay^*)$  and

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C v_n.$$

For any  $n \in \mathbb{N}$ , we have,

(3.2) 
$$||u_n - x^*|| = ||T_{r_n}x_n - T_{r_n}x^*|| \le ||x_n - x^*||.$$

Since  $P_C$  is nonexpansive and from Remark 2.8, we obtain that  $I - \lambda A$  and  $I - \mu B$  are nonexpansive. Then, it follows:

$$\begin{aligned} \|v_n - x^*\|^2 &= \|P_C(y_n - \lambda A y_n) - P_C(y^* - \lambda A y^*)\|^2 \\ &\leq \|(I - \lambda A)y_n - (I - \lambda A)y^*\|^2 \\ &\leq \|y_n - y^*\|^2 \\ &= \|P_C(u_n - \mu B u_n) - P_C(x^* - \mu B x^*)\|^2 \\ &\leq \|(u_n - \mu B u_n) - (x^* - \mu B x^*)\|^2 \\ &\leq \|u_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2. \end{aligned}$$

Thus, we also have,

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n u + \beta_n x_n + \gamma_n S v_n - x^*\| \\ &\leq \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|v_n - x^*\| \\ &\leq \alpha_n (\|u - x^*\|) + \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| \\ &\leq \alpha_n (\|u - x^*\|) + (1 - \alpha_n) \|x_n - x^*\| \\ &\leq \max\{\|u - x^*\|, \|x_1 - x^*\|\} \\ &= \|u - x^*\|. \end{aligned}$$

Therefore, the sequence  $\{x_n\}$  is bounded. Hence, we also have that the sets  $\{u_n\}$ ,  $\{v_n\}$   $\{Ay_n\}$ ,  $\{Bx_n\}$  and  $\{Sv_n\}$  are bounded. Moreover, by nonexpansiveness of  $I - \lambda A$ ,  $I - \mu B$  and  $P_C$ , we get

$$\|v_{n+1} - v_n\| = \|P_C(y_{n+1} - \lambda A y_{n+1}) - P_C(y_n - \lambda A y_n)\|$$
  

$$\leq \|(y_{n+1} - \lambda A y_{n+1}) - (y_n - \lambda A y_n)\|$$
  

$$\leq \|(I - \lambda A) y_{n+1} - (I - \lambda A) y_n\|$$
  

$$\leq \|y_{n+1} - y_n\|$$
  

$$= \|P_C(u_{n+1} - \mu B u_{n+1}) - P_C(u_n - \mu B u_n)\|$$
  

$$\leq \|(I - \mu B) u_{n+1} - (I - \mu B) u_n\|$$
  

$$\leq \|u_{n+1} - u_n\|.$$
(3.3)

On the other hand, from  $u_j = T_{r_j} x_j$ , where j = n, n + 1, we have,

(3.4) 
$$f(u_j, y) + \frac{1}{r_j} \langle y - u_j, u_j - x_j \rangle \ge 0 \text{ for all } y \in C.$$

Putting  $y = u_{n+1}$  and  $y = u_n$  in (3.4), we get

$$f(u_n, u_{n+1}) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \ge 0$$

and

$$f(u_{n+1}, u_n) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \ge 0$$

From (A2) we have,

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \rangle \ge 0,$$

and hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \rangle \ge 0.$$

Since  $\liminf_{n\to\infty} r_n > 0$ , without loss of generality, assume that there exists a real number c such that  $r_n > c > 0$  for all  $n \in \mathbb{N}$ . Then, we have,

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1}) \rangle \\ &\leq \|u_{n+1} - u_n\| \{ \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}| \|u_{n+1} - x_{n+1}\| \} \end{aligned}$$

and hence,

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ (3.5) &\leq \|x_{n+1} - x_n\| + \frac{L}{c} |r_{n+1} - r_n|, \end{aligned}$$

where  $L = \sup\{||u_n - x_n|| : n \in \mathbb{N}\}$ . Substituting (3.5) into (3.3), we obtain:

(3.6) 
$$||v_{n+1} - v_n|| \leq ||x_{n+1} - x_n|| + \frac{L}{c} |r_{n+1} - r_n|.$$

Let  $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ . Thus, we get

$$z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n u + \gamma_n SP_C(y_n - \lambda Ay_n)}{1 - \beta_n} = \frac{\alpha_n u + \gamma_n Sv_n}{1 - \beta_n}$$

It follows:

$$z_{n+1} - z_n = \frac{\alpha_{n+1}u + \gamma_{n+1}Sv_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n u + \gamma_n Sv_n}{1 - \beta_n}$$
  
=  $(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n})u + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(Sv_{n+1} - Sv_n)$   
 $+ (\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n})Sv_n.$ 

Combining (3.6) and (3.7), we obtain:

$$\begin{split} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ &\leq |\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}| \|u\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|v_{n+1} - v_n\| \\ &+ |\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}| \|Sv_n\| - \|x_{n+1} - x_n\| \\ &\leq |\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}| \|u\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|x_{n+1} - x_n\| \\ &+ \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \frac{L}{c} |r_{n+1} - r_n| \\ &+ |\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}| \|Sv_n\| - \|x_{n+1} - x_n\| \\ &\leq |\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}| (\|u\| + \|Sv_n\|) + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \frac{L}{c} |r_{n+1} - r_n|. \end{split}$$

From (ii), (iv) and (v), we get

$$\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Thus, by Lemma 2.2, we have,

$$\lim_{n \to \infty} \|z_n - x_n\| = 0.$$

Consequently,

(3.8) 
$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \beta_n) \|z_n - x_n\| = 0.$$

By (iv), (v), (3.3) and (3.5), we also have,  $||v_{n+1} - v_n|| \to 0, ||u_{n+1} - u_n|| \to 0 \text{ and } ||y_{n+1} - y_n|| \to 0 \text{ as } n \to \infty.$ Since

$$x_{n+1} - x_n = \alpha_n u + \beta_n x_n + \gamma_n S v_n - x_n = \alpha_n (u - x_n) + \gamma_n (S v_n - x_n),$$
  
it follows from (ii) and (2.8) that

it follows from (ii) and (3.8) that

(3.9) 
$$\lim_{n \to \infty} \|x_n - Sv_n\| = 0.$$

Since  $x^* \in F(S) \cap \Omega \cap EP(f)$ , we observe:

$$\|v_n - x^*\| = \|P_C(y_n - \lambda A y_n) - P_C(y^* - \lambda A y^*)\|$$
  

$$\leq \|(y_n - \lambda A y_n) - (y^* - \lambda A y^*)\|$$
  

$$\leq \|y_n - y^*\| = \|P_C(u_n - \lambda A u_n) - P_C(x^* - \lambda A x^*)\|$$
  

$$\leq \|(u_n - \lambda A u_n) - (x^* - \lambda A x^*)\|$$
  

$$\leq \|u_n - x^*\|$$
  

$$(3.10)$$

and

$$\begin{aligned} \|u_n - x^*\|^2 &= \|T_{r_n} x_n - T_{r_n} x^*\|^2 \le \langle T_{r_n} x_n - T_{r_n} x^*, x_n - x^* \rangle \\ &= \langle u_n - x^*, x_n - x^* \rangle \\ &= \frac{1}{2} (\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2), \end{aligned}$$

and then  $||u_n - x^*||^2 \le ||x_n - x^*||^2 - ||x_n - u_n||^2$ . From (3.10), we have,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n u + \beta_n x_n + \gamma_n S v_n - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|S v_n - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|v_n - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|u_n - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\ &\quad + \gamma_n (\|x_n - x^*\|^2 - \|x_n - u_n\|^2) \\ &= \alpha_n \|u - x^*\|^2 + (\beta_n + \gamma_n) \|x_n - x^*\|^2 - \gamma_n \|x_n - u_n\|^2 \\ &= \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) \|x_n - x^*\|^2 - \gamma_n \|x_n - u_n\|^2 \\ \end{aligned}$$

$$(3.11) \qquad \leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n \|x_n - u_n\|^2 \end{aligned}$$

and hence,

$$\gamma_n \|x_n - u_n\|^2 \le \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2$$

$$(3.12) \le \alpha_n \|u - x^*\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|).$$

Using (ii) and (3.8), we get

(3.13) 
$$\lim_{n \to \infty} \|x_n - u_n\| = 0.$$

Since  $\liminf_{n\to\infty} r_n > 0$ , we have,

(3.14) 
$$\lim_{n \to \infty} \left\| \frac{x_n - u_n}{r_n} \right\| = \lim_{n \to \infty} \frac{1}{r_n} \|x_n - u_n\| = 0.$$

Again, since  $\alpha_n \to 0$  and (3.8) imply that  $||u_n - x_n|| \to 0$ , as  $n \to \infty$ . From (3.2), (3.10) and Lemma 2.1, we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|v_n - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \{\|(y_n - \lambda A y_n) - (y^* - \lambda A y^*)\|^2\} \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \{\|y_n - y^*\|^2 \\ &+ \lambda (\lambda - 2\alpha) \|Ay_n - Ay^*\|^2 \} \\ &= \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|u_n - x^*\|^2 \\ &+ \gamma_n \lambda (\lambda - 2\alpha) \|Ay_n - Ay^*\|^2 \\ &= \alpha_n \|u - x^*\|^2 + (\beta_n + \gamma_n) \|x_n - x^*\|^2 \\ &+ \gamma_n \lambda (\lambda - 2\alpha) \|Ay_n - Ay^*\|^2 \\ &= \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) \|x_n - x^*\|^2 + \gamma_n \lambda (\lambda - 2\alpha) \|Ay_n - Ay^*\|^2 \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|v_n - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|y_n - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \{\|(u_n - \mu B u_n) \\ &- (x^* - \mu B x^*)\|^2 \} \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \{\|u_n - x^*\|^2 \\ &+ \mu(\mu - 2\beta) \|B u_n - B x^*\|^2 \} \\ &= \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 \\ &+ \gamma_n \mu(\mu - 2\beta) \|B u_n - B x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 + \gamma_n \mu(\mu - 2\beta) \|B u_n - B x^*\|^2. \end{aligned}$$

Hence, by (3.15) and (3.15), we obtain:

$$\begin{aligned} &\gamma_n \lambda (2\alpha - \lambda) \|Ay_n - Ay^*\|^2 \\ &\leq & \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &= & \alpha_n \|u - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)(\|x_n - x^*\| - \|x_{n+1} - x^*\|) \\ &\leq & \alpha_n \|u - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \end{aligned}$$

$$\gamma_{n}\mu(2\beta - \mu) \|Bu_{n} - Bx^{*}\|^{2}$$

$$\leq \alpha_{n}\|u - x^{*}\|^{2} + \|x_{n} - x^{*}\|^{2} - \|x_{n+1} - x^{*}\|^{2}$$

$$= \alpha_{n}\|u - x^{*}\|^{2} + (\|x_{n} - x^{*}\| + \|x_{n+1} - x^{*}\|)(\|x_{n} - x^{*}\| - \|x_{n+1} - x^{*}\|)$$

$$\leq \alpha_{n}\|u - x^{*}\|^{2} + (\|x_{n} - x^{*}\| + \|x_{n+1} - x^{*}\|)\|x_{n} - x_{n+1}\|.$$

From (ii), (iii), (3.8), (3.15) and (3.15), respectively, we also have,

(3.15) 
$$||Ay_n - Ay^*|| \to 0 \text{ and } ||Bu_n - Bx^*|| \to 0, \text{ as } n \to \infty.$$

By (2.3), we obtain:

$$\begin{aligned} \|y_n - y^*\|^2 &= \|P_C(u_n - \mu Bu_n) - P_C(x^* - \mu Bx^*)\|^2 \\ &\leq \langle (u_n - \mu Bu_n) - (x^* - \mu Bx^*), y_n - y^* \rangle \\ &= \frac{1}{2} \{ \|(u_n - \mu Bu_n) - (x^* - \mu Bx^*)\|^2 + \|y_n - y^*\|^2 \\ &- \|(u_n - \mu Bu_n) - (x^* - \mu Bx^*) - (y_n - y^*)\|^2 \} \\ &\leq \frac{1}{2} \{ \|u_n - x^*\|^2 + \|y_n - y^*\|^2 - \|(u_n - y_n) - \mu(Bu_n - Bx^*) \\ &- (x^* - y^*)\|^2 \} \\ &= \frac{1}{2} \{ \|u_n - x^*\|^2 + \|y_n - x^*\|^2 - \|(u_n - y_n) - (x^* - y^*)\|^2 \\ &+ 2\mu \langle (u_n - y_n) - (x^* - y^*), Bu_n - Bx^* \rangle - \mu^2 \|Bu_n - Bx^*\|^2 \}, \end{aligned}$$

which implies:

$$||y_n - y^*||^2 \le ||u_n - x^*||^2 - ||(u_n - y_n) - (x^* - y^*)||^2 + 2\mu \langle (u_n - y_n) - (x^* - y^*), Bu_n - Bx^* \rangle - \mu^2 ||Bu_n - Bx^*||^2.$$

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and

Thus, we observe:

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|v_n - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|y_n - y^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\ &+ \gamma_n \{\|u_n - x^*\|^2 - \|(u_n - y_n) - (x^* - y^*)\|^2 \\ &+ 2\mu \langle (u_n - y_n) - (x^* - y^*), Bu_n - Bx^* \rangle - \mu^2 \|Bu_n - Bx^*\|^2 \} \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\ &+ \gamma_n \|x_n - x^*\|^2 - \gamma_n \|(u_n - y_n) - (x^* - y^*)\|^2 \\ &+ 2\gamma_n \mu \|(u_n - y_n) - (x^* - y^*)\| \|Bu_n - Bx^*\| - \gamma_n \mu^2 \|Bu_n - Bx^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) \|x_n - x^*\|^2 - \gamma_n \|(u_n - y_n) - (x^* - y^*)\|^2 \\ &+ 2\gamma_n \mu \|(u_n - y_n) - (x^* - y^*)\| \|Bu_n - Bx^*\|. \end{aligned}$$

It follows:

$$\gamma_{n} \| (u_{n} - y_{n}) - (x^{*} - y^{*}) \|^{2}$$

$$\leq \alpha_{n} \| u - x^{*} \|^{2} + \| x_{n} - x^{*} \|^{2} - \| x_{n+1} - x^{*} \|^{2}$$

$$+ 2\gamma_{n} \mu \| (u_{n} - y_{n}) - (x^{*} - y^{*}) \| \| Bu_{n} - Bx^{*} \|$$

$$\leq \alpha_{n} \| u - x^{*} \|^{2} + \| x_{n+1} - x_{n} \| (\| x_{n} - x^{*} \| + \| x_{n+1} - x^{*} \|^{2})$$

$$+ 2\gamma_{n} \mu \| (u_{n} - y_{n}) - (x^{*} - y^{*}) \| \| Bu_{n} - Bx^{*} \|.$$

From (ii), (3.15), (3.8) and  $||Bu_n - Bx^*|| \to 0$ , as  $n \to \infty$ , we have,

(3.17) 
$$||(u_n - y_n) - (x^* - y^*)|| \to 0, \text{ as } n \to \infty.$$

We observe:

$$\begin{aligned} \|(y_n - v_n) + (x^* - y^*)\|^2 \\ &= \|(y_n - y^*) - [P_C(y_n - \lambda Ay_n) - x^*]\|^2 \\ &= \|y_n - \lambda Ay_n - (y^* - \lambda Ay^*) \\ - [P_C(y_n - \lambda Ay_n) - x^*] + \lambda (Ay_n - Ay^*)\|^2 \\ &\leq \|y_n - \lambda Ay_n - (y^* - \lambda Ay^*)\|^2 \\ - \|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\|^2 \\ + 2\lambda \langle Ay_n - Ay^*, (y_n - v_n) + (x^* - y^*) \rangle \\ &\leq \|y_n - \lambda Ay_n - (y^* - \lambda Ay^*)\|^2 \\ - \|SP_C(y_n - \lambda Ay_n) - SP_C(y^* - \lambda Ay^*)\|^2 \\ + 2\lambda \|Ay_n - Ay^*\|\|(y_n - v_n) + (x^* - y^*)\| \\ &\leq \|y_n - \lambda Ay_n - (y^* - \lambda Ay^*)\|^2 - \|Sv_n - Sx^*\|^2 \\ + 2\lambda \|Ay_n - Ay^*\|\|(y_n - v_n) + (x^* - y^*)\| \\ &\leq \|y_n - \lambda Ay_n - (y^* - \lambda Ay^*) - Sv_n - Sx^*\| \\ \times (\|y_n - \lambda Ay_n - (y^* - \lambda Ay^*)\| + \|Sv_n - Sx^*\|) \\ + 2\lambda \|Ay_n - Ay^*\|\|(y_n - v_n) + (x^* - y^*)\| \\ &\leq \|x_n - Sv_n + x^* - y^* - (x_n - y_n) - \lambda (Ay_n - Ay^*)\| \\ \times (\|y_n - \lambda Ay_n - (y^* - \lambda Ay^*)\|^2\| + \|Sv_n - Sx^*\|) \\ + 2\lambda \|Ay_n - Ay^*\|\|(y_n - v_n) + (x^* - y^*)\|. \end{aligned}$$

From (3.9), (3.17) and  $||Ay_n - Ay^*|| \to 0$ , as  $n \to \infty$ , it follows:

$$||(y_n - v_n) + (x^* - y^*)|| \to 0, \quad (n \to \infty).$$

We note that

$$||Sv_n - v_n|| \leq ||Sv_n - x_n|| + ||x_n - u_n|| + ||(u_n - y_n) - (x^* - y^*)|| + ||(y_n - v_n) + (x^* - y^*)||.$$

We then obtain:

(3.18) 
$$\lim_{n \to \infty} \|Sv_n - v_n\| = 0.$$

Next, we show that

$$\limsup_{n \to \infty} \langle u - z_0, x_n - z_0 \rangle \le 0,$$

where  $z_0 = P_{F(S)\cap\Omega\cap EP(f)}u$ . To show this inequality, we choose a subsequence  $\{v_{n_i}\}$  of  $\{v_n\}$  such that

$$\limsup_{n \to \infty} \langle u - z_0, Sv_n - z_0 \rangle = \lim_{i \to \infty} \langle u - z_0, Sv_{n_i} - z_0 \rangle.$$

Since  $\{v_{n_i}\}$  is bounded, there exists a subsequence  $\{v_{n_i}\}$  of  $\{v_{n_i}\}$  which converges weakly to z. Without loss of generality, we can assume that  $v_{n_i} \rightarrow z$ . From  $||Sv_n - v_n|| \rightarrow 0$ , we obtain  $Sv_{n_i} \rightarrow z$ . Let us show  $z \in EP(f)$ . Since  $u_n = T_{r_n} x_n$ , we have,

$$f(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \forall y \in C.$$

From (A2), it follows:

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \ge f(y, u_n)$$

and hence  $\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq f(y, u_{n_i})$ . From  $||u_n - x_n|| \to 0, ||x_n - Sv_n|| \to 0$  and  $||Sv_n - v_n|| \to 0$ , we get  $u_{n_i} \rightharpoonup z$ . Since  $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ , it follows by (A4) that  $0 \geq f(y, z)$ , for all  $y \in C$ . For t with  $0 < t \leq 1$  and  $y \in C$ , let  $y_t = ty + (1 - t)z$ . Since  $y \in C$  and  $z \in C$ , we have  $y_t \in C$  and hence  $f(y_t, z) \leq 0$ . So, from (A1) and (A4) we have  $0 = f(y_t, y_t) \leq tf(y_t, y) + (1 - t)f(y_t, z) \leq tf(y_t, y)$  and hence  $0 \leq f(y_t, y)$ . From (A3), we have  $f(z, y) \geq 0$ , for all  $y \in C$  and hence  $z \in EP(f)$ . By Opial's condition, we obtain  $z \in F(S)$ . Finally, by the same argument as that in the proof of [4, Theorem 3.1, p. 384-385], we can show that  $z \in \Omega$ . Hence,  $z \in F(S) \cap \Omega \cap EP(f)$ . Now from (2.4), we have,

$$\limsup_{n \to \infty} \langle u - z_0, x_n - z_0 \rangle = \limsup_{n \to \infty} \langle u - z_0, Sv_n - z_0 \rangle$$
$$= \lim_{i \to \infty} \langle u - z_0, Sv_{n_i} - z_0 \rangle$$
$$= \langle u - z_0, z - z_0 \rangle \le 0.$$

Finally, we show that  $x_n \to z_0$ , where  $z_0 = P_{F(S) \cap VI(A,C) \cap EP(f)}u$ . We observe:

$$\begin{aligned} \|x_{n+1} - z_0\|^2 &= \langle \alpha_n u + \beta_n x_n \\ &+ \gamma_n S v_n - z_0, x_{n+1} - z_0 \rangle \\ &= \alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle + \beta_n \langle x_n - z_0, x_{n+1} - z_0 \rangle \\ &+ \gamma_n \langle S v_n - z_0, x_{n+1} - z_0 \rangle \\ &\leq \frac{1}{2} \beta_n (\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &+ \frac{1}{2} \gamma_n (\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\ &\leq \frac{1}{2} \{ (1 - \alpha_n) \|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2 \} \\ &+ \alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle, \end{aligned}$$

which implies:

$$||x_{n+1} - z_0||^2 \le (1 - \alpha_n) ||x_n - z_0||^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle.$$

Finally, by (3.19) and Lemma 2.4, we get that  $\{x_n\}$  converges to  $z_0$ , where  $z_0 = P_{F(S) \cap \Omega \cap EP(f)} u$ . This completes the proof. 

By Theorem 3.1, we obtain the following corollaries.

**Corollary 3.2.** Let C be a closed convex subset of a real Hilbert space H. Let f be a bifunction from  $C \times C$  to **R** satisfying (A1)-(A4) and  $A: C \longrightarrow H$  be  $\alpha$ -inverse-strongly monotone. Let S be a nonexpansive mapping of C into itself such that  $F(S) \cap \Omega \cap EP(f) \neq \emptyset$ . Let f be a contraction of H into itself, given  $x_0 \in H$  arbitrary. Suppose that  $x_1 = u \in C$  and  $\{x_n\}, \{y_n\}$  and  $\{u_n\}$  be given by

(3.20) 
$$\begin{aligned} f(u_n, y) &+ \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C; \\ y_n &= P_C(u_n - \lambda A u_n) \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda A y_n), \forall n \in \mathbf{N}, \end{aligned}$$

where  $\lambda \in (0, 2\alpha)$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  are three sequences in [0, 1]. If  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  and  $\lambda \in [a, b]$  for some a, b with  $0 < a < b < 2\alpha$  and  $\{r_n\} \subset (0,\infty)$  satisfy the following conditions:

- $\begin{array}{l} \text{(i)} \quad \alpha_n + \beta_n + \gamma_n = 1, \\ \text{(ii)} \quad \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \\ \text{(iii)} \quad 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1, \\ \text{(iv)} \quad \liminf_{n \to \infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} r_n| < \infty, \end{array}$

then  $\{x_n\}$  converges strongly to  $u = P_{F(S) \cap \Omega \cap EP(f)}u$ .

**Proof.** By letting A = B and  $\lambda = \mu$ , for  $n \in \mathbb{N}$ , in Theorem 3.1, we obtain the desired result. 

Setting  $P_H = I$ , we obtain the following corollary.

**Corollary 3.3.** Let C be a closed convex subset of a real Hilbert space H. Let f be a bifunction from  $C \times C$  to **R** satisfying (A1)-(A4) and  $A: C \longrightarrow H$  be  $\alpha$ -inverse-strongly monotone. Let S be a nonexpansive mapping of C into itself such that  $F(S) \cap VI(A,C) \cap EP(f) \neq \emptyset$ . Let f be a contraction of H into itself, given  $x_0 \in H$  arbitrary. Suppose  $x_1 = u \in C$  and  $\{x_n\}, \{y_n\}$  and  $\{u_n\}$  be given by

(3.21) 
$$\begin{aligned} f(u_n, y) &+ \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C; \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n SP_C(u_n - \lambda A u_n), \forall n \in \mathbf{N}, \end{aligned}$$

where  $\lambda \in (0, 2\alpha)$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  are three sequences in [0, 1]. If  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  and  $\lambda \in [a, b]$  for some a, b with  $0 < a < b < 2\alpha$  and  $\{r_n\} \subset (0,\infty)$  satisfy the following conditions:

- (i)  $\alpha_n + \beta_n + \gamma_n = 1$ , (ii)  $\lim_{n \to \infty} \alpha_n = 0$ ,  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , (iii)  $0 < \liminf_{n \to \infty} \beta_n \le \limsup_{n \to \infty} \beta_n < 1$ ,
- (iv)  $\liminf_{n \to \infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} r_n| < \infty,$

then  $\{x_n\}$  converges strongly to  $z \in F(S) \cap VI(C, A) \cap EP(f)$ , where  $z = P_{F(S) \cap VI(C,A) \cap EP(f)}u.$ 

Using Theorem 3.1, we obtain the following two corollaries in Hilbert space.

Corollary 3.4. (Ceng et. al [4, Theorem 3.1]) Let C be a closed convex subset of a real Hilbert space H. Let A and B be  $\alpha$ - and  $\beta$ -inversestrongly monotone mappings of C into H, respectively, and let S be a nonexpansive mapping of C into itself such that  $F(S) \cap \Omega \neq \emptyset$ . Suppose  $x_1 = u \in C$  and  $\{x_n\}, \{y_n\}$  are given by

$$y_n = P_C(x_n - \mu B x_n)$$
  
$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n),$$

where  $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  are three sequences in [0,1] and  $\{r_n\} \subset (0,\infty)$  satisfying the following conditions:

- $\begin{array}{ll} (\mathrm{i}) & \alpha_n + \beta_n + \gamma_n = 1, \\ (\mathrm{ii}) & \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \\ (\mathrm{iii}) & 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1. \end{array}$

Then,  $\{x_n\}$  converges strongly to  $P_{F(S)\cap\Omega}u$  and  $(x^*, y^*)$  is a solution of problem (1.9), where  $y^* = P_C(x^* - \mu B x^*)$ .

**Proof.** Put F(x,y) = 0, for all  $x, y \in C$ , and  $r_n = 1$ , for all  $n \in \mathbb{N}$ , in Theorem 3.1 . Then, we have  $u_n = P_C x_n = x_n$ . So, from Theorem 3.1, the sequence  $\{x_n\}$  converges strongly to  $P_{F(S)\cap\Omega}u$ . 

Corollary 3.5. Let C be a closed convex subset of a real Hilbert space H. Let A be an  $\alpha$ -inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that  $F(S) \cap VI(A, C) \neq I(A, C)$  $\emptyset$ . Suppose that  $x_1 = u \in C$  and  $\{x_n\}, \{y_n\}$  be given by

$$y_n = P_C(x_n - \lambda A x_n)$$
  
$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n),$$

where  $\lambda \in [0, 2\alpha]$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  are three sequences in [0, 1] satisfying

(i) 
$$\alpha_n + \beta_n + \gamma_n = 1$$
,

(ii)  $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$ (iii)  $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$ 

Then,  $\{x_n\}$  converges strongly to  $P_{F(S)\cap\Omega}u$ . Moreover, we also have  $(x^*, y^*)$  is a solution of problem (1.10), where  $y^* = P_C(x^* - \lambda Ax^*)$ .

**Proof.** By taking A = B and  $\lambda = \mu$  in Corollary 3.4, we get the desired result. 

## 4. Applications

A mapping  $T: C \to C$  is called strictly pseudocontractive on C if there exists k with  $0 \le k < 1$  such that

$$||Tx - Ty||^{2} \le ||x - y||^{2} + k||(I - T)x + (I - T)y||^{2}, \text{ for all } x, y \in C.$$

If k = 0, then T is nonexpansive. Put A = I - T, where  $T : C \to C$ is a strictly pseudocontractive mapping with k. Then, we have, for all  $x, y \in C$ ,

$$||(I-A)x - (I-A)y||^2 \le ||x-y||^2 + k||Ax - Ay||^2.$$

On the other hand, we have,

 $||(I - A)x - (I - A)y||^{2} = ||x - y||^{2} - 2\langle x - y, Ax - Ay \rangle + ||Ax - Ay||^{2}.$ Hence, we have,

$$\langle x-y, Ax-Ay \rangle \ge \frac{1-k}{2} \|Ax-Ay\|^2.$$

Then, A is  $\frac{1-k}{2}$ -inverse strongly monotone.

Now, using Theorem 3.1, we state a strong convergence theorem for a pair of nonexpansive mappings and strictly pseudocontractive mappings.

**Theorem 4.1.** Let C be a closed convex subset of a real Hilbert space H. Let f be a bifunction from  $C \times C$  to **R** satisfy (A1)-(A4) and let S be a nonexpansive mappings of C into itself and let T, V be strictly pseudocontractive mapping with constant k of C into itself such that  $F(S) \cap F(T) \cap EP(F) \neq \emptyset$ . Suppose  $x_1 = u \in C$  and  $\{x_n\}, \{y_n\}$  and  $\{u_n\}$  are given by

$$f(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \qquad \forall y \in C;$$
  

$$y_n = (1 - \mu)u_n + \mu V u_n$$
  

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S((1 - \lambda)y_n + \lambda T y_n),$$

for all  $n \in \mathbf{N}$ , where  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  are three sequences in  $[0, 1], \lambda \in$ [0, 1-k] and  $\mu \in [0, 1-l]$ . If  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0, \infty)$  satisfy

- (i)  $\alpha_n + \beta_n + \gamma_n = 1$ , (ii)  $\lim_{n \to \infty} \alpha_n = 0$ ,  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , (iii)  $0 < \liminf_{n \to \infty} \beta_n \le \limsup_{n \to \infty} \beta_n < 1$ , (iv)  $\liminf_{n \to \infty} r_n > 0$ ,  $\sum_{n=1}^{\infty} |r_{n+1} r_n| < \infty$ ,

then  $\{x_n\}$  converges strongly to  $z = P_{F(S) \cap F(T) \cap EP(f)}u$ .

**Proof.** Put A = I - T and B = I - V. Then, A is  $\frac{1-k}{2}$ -inverse-strongly monotone and B is  $\frac{1-l}{2}$ -inverse-strongly monotone. We have that F(T)is the solution set of VI(A, C) and  $\Omega$ ; i.e.,  $F(T) = VI(A, C) \Leftrightarrow$ problem (1.9)  $\Leftrightarrow$  problem (1.10) (see cf. Ceng et al. [4, Theorem 4.1 pp. 388–389]) and

 $P_C(u_n - \mu B u_n) = (1 - \mu)u_n + \mu V u_n \text{ and } P_C(y_n - \lambda A y_n) = (1 - \lambda)y_n + \lambda T y_n.$ Therefore, by Theorem 3.1, the result follows. 

Therefore, the following Corollary immediately from Theorem 4.1.

**Corollary 4.2.** (Ceng et al. [4, Corollary 3.3]) Let C be a closed convex subset of a real Hilbert space H. Let S be a nonexpansive mapping of C into itself and let T, V be strictly pseudocontractive mappings with constant k of C into itself such that  $F(S) \cap F(T) \neq \emptyset$ . Suppose  $x_1 = u \in$ C and  $\{x_n\}$  is given by

$$y_n = (1 - \mu)x_n + \mu V x_n$$
  
$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S((1 - \lambda)y_n + \lambda T y_n),$$

for all  $n \in \mathbf{N}$ , where  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  are three sequences in  $[0, 1], \lambda \in$ [0, 1-k] and  $\mu \in [0, 1-l]$ . If  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  satisfy the following conditions:

- (i)  $\alpha_n + \beta_n + \gamma_n = 1$ , (ii)  $\lim_{n \to \infty} \alpha_n = 0$ ,  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (iii)  $0 < \liminf_{n \to \infty} \beta_n \le \limsup_{n \to \infty} \beta_n < 1$ ,

then  $\{x_n\}$  converges strongly to  $x^* = P_{F(S) \cap \Omega}u$  and  $(x^*, y^*)$  is a solution of problem (1.10), where  $y^* = (1 - \mu)x^* - \mu Vx^*$ ).

**Corollary 4.3.** Let C be a closed convex subset of a real Hilbert space H. Let S be a nonexpansive mapping of C into itself and let T be a strictly pseudocontractive mapping with constant k of C into itself such that  $F(S) \cap F(T) \neq \emptyset$ . Suppose  $x_1 = u \in C$  and  $\{x_n\}$  is given by

$$y_n = (1 - \lambda)x_n + \lambda T x_n$$
  
$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S((1 - \lambda)y_n + \lambda T y_n),$$

for all  $n \in \mathbf{N}$ , where  $\lambda \in [0, 1-k]$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  are three sequences in [0, 1] satisfying

- (i)  $\alpha_n + \beta_n + \gamma_n = 1$ , (ii)  $\lim_{n \to \infty} \alpha_n = 0$ ,  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , (iii)  $0 < \liminf_{n \to \infty} \beta_n \le \limsup_{n \to \infty} \beta_n < 1$ .

Then,  $\{x_n\}$  converges strongly to  $P_{F(S)\cap F(T)}u$ .

The following three theorems are connected with the problem of obtaining a common element of the sets of zeroes of a maximal monotone operator and an  $\alpha$ -inverse-strongly monotone operator.

**Theorem 4.4.** Let C be a nonempty closed convex subset of H. Let fbe a bifunction from  $C \times C$  to **R** satisfying (A1) - (A4) and let A be an  $\alpha$ -inverse-strongly monotone operator in H and  $B: H \to 2^H$  be a maximal monotone operator such that  $A^{-1}(0) \cap B^{-1}(0) \cap EP(f) \neq .$  Let

 $J_r^B$  be the resolvent of B for each r > 0. Let  $\{x_n\}$  and  $\{u_n\}$  be sequences generated by  $x_1 = u \in H$  and

(4.1) 
$$\begin{cases} f(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, & \forall y \in H, \\ y_n = (u_n - \lambda A u_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_r^B(y_n - \lambda A y_n), \end{cases}$$

where  $\{\lambda\} \subset [c,d]$  for some  $[c,d] \subset (0,2\alpha)$ ,  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  and  $\{r_n\}$ satisfy the following conditions:

- (i)  $\alpha_n + \beta_n + \gamma_n = 1$ , (ii)  $\{\alpha_n\} \subset [0,1], \sum_{n=0}^{\infty} \alpha_n = \infty, \alpha_n \to 0$ ; (iii)  $\{r_n\} \subset (0,\infty)$ ,  $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} r_n| < \infty$ , (iv)  $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ .

Then,  $\{x_n\}$  and  $\{u_n\}$  converge strongly to  $z \in A^{-1}(0) \cap B^{-1}(0) \cap EP(f)$ , where  $z = P_{A^{-1}(0) \cap B^{-1}(0) EP(f)} x_1$ .

**Proof.** Since  $A^{-1}0 = V(I, A)$  and  $F(J_r^B) = B^{-1}(0)$ , putting  $P_H = I$ , then, by Theorem 3.1, we obtain the desired result easily.

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