THE COMPLEMENT OF A D-TREE IS PURE SHELLABLE

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Abstract. Let G be a simple undirected graph and let Δ_G be a simplicial complex whose faces correspond to the independent sets of G. A graph G is called shellable if Δ_G is a shellable simplicial complex. We prove that the complement of a d-tree is a pure shellable graph. This generalizes a recent result of Ferrarello who used a theorem due to R. Fröberg to prove that the complement of a d-tree is a Cohen-Macaulay graph.

1. Introduction

Let G be a finite simple graph on n vertices V(G) = \{x_1, ..., x_n\}. One can then associate to G a quadratic square-free monomial ideal I(G) in \( R = k[x_1, ..., x_n] \) by setting \( I(G) = (x_ix_j | \{x_i, x_j\} \in E(G)) \), where E(G) is the edge set of G. Besides to this algebraic object, a simplicial complex associates to a graph G which reflects many nice properties of the graph. The simplicial complex Δ_G of a graph G is defined by

\[ \Delta_G = \{ A \subseteq V | A \text{ is an independent set in } G \}, \]

where A is an independent set in G if none of its elements are adjacent.

A simplicial complex Δ is called shellable if the facets (maximal faces) of Δ can be ordered as \( F_1, \ldots, F_s \) such that for all \( 1 \leq i < j \leq s \), there

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exist some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j - 1\}$ with $F_j \setminus F_l = \{v\}$; cf. [2]. We call $F_1, \ldots, F_s$ a shelling of $\Delta$ when the facets have been ordered with respect to the definition of shellable. An important property of a pure shellable complex is that it is Cohen-Macaulay over every field which is due to Hochster [6] (see also [1, Theorem 5.1.13]). A graph $G$ is called shellable, if the simplicial complex $\Delta_G$ is a shellable simplicial complex.

A recent theme in commutative algebra is to understand how the properties of $G$ appear within the properties of $R/I(G)$, and vice versa. Cohen-Macaulay rings are of great interest to commutative algebraists. As a consequence, one particular stream of research has focused on the question of what graphs $G$ have the property that $R/I(G)$ is Cohen-Macaulay. Although a solution to the general problem is probably intractable, Herzog and Hibi [5] gave a combinatorial description of those graphs $G$ that are bipartite and Cohen-Macaulay (recall that a graph $G$ on the vertex set $[n]$ is bipartite if there exists a partition $[n] = X \cup Y$, with $X \cap Y = \emptyset$, such that each edge of $G$ is of the form $\{i, j\}$ with $i \in X$ and $j \in Y$).

We denote the complete graph on $t$ vertices by $K_t$. The following inductive definition of $d$-trees is customary:

(i) $K_{d+1}$ is a $d$-tree.

(ii) If $G$ is a $d$-tree and $v$ a new vertex, and $v$ is adjoined to $G$ via a sub-$K_d$ of $G$, then $\{v\} \cup G$ is a $d$-tree.

Note that a 0-tree is a graph that has a set of vertices but without any edge, and 1-tree is a usual tree.


Here, we prove a generalization of Ferrarello’s result by showing that the complement of a $d$-tree is a pure shellable graph and so it is Cohen-Macaulay.

2. Main result

First, we recall the definitions we use throughout the paper.
The complement of a $d$-tree is pure shellable.

**Definition 2.1.** the complement of a graph $G$ is a graph $H$ on the same vertices such that two vertices of $H$ are adjacent if and only if they are not adjacent in $G$.

**Definition 2.2.** A simplicial complex $\Delta$ over a set of vertices $V = \{v_1, ..., v_n\}$ is a collection of subsets of $V$, with the property that $\{v_i\} \in \Delta$ for all $i$, and if $F \in \Delta$, then all subsets of $F$ are also in $\Delta$ (including the empty set). An element of $\Delta$ is called a face of $\Delta$, and the dimension of a face $F$ of $\Delta$ is defined as $|F| - 1$, where $|F|$ is the number of vertices of $F$. The faces of dimensions 0 and 1 are called vertices and edges, respectively, and $\dim \emptyset = -1$.

The maximal faces of $\Delta$ under inclusion are called facets of $\Delta$. We denote the simplicial complex $\Delta$ with facets $F_1, ..., F_t$ by $\Delta = \langle F_1, ..., F_t \rangle$, and we call $\{F_1, ..., F_t\}$ the facet set of $\Delta$. A simplicial complex is called pure if all its facets have the same cardinality. A pure simplicial complex $\Delta$ is called shellable if the facets of $\Delta$ can be ordered as $F_1, \ldots, F_s$ such that for all $1 \leq i < j \leq s$, there exist some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j - 1\}$ with $F_j \setminus F_l = \{v\}$. We call $F_1, \ldots, F_s$ a shelling of $\Delta$ when the facets are been ordered with respect to the definition of shellable.

**Lemma 2.3.** Let $G$ be a simple graph and $F \subseteq V(G)$. Then, $F$ is a facet of $\Delta_G$ if and only if the induced subgraph of $G$ on the vertex set $F$ is a maximal complete subgraph of $G$.

**Proof.** First, let $F$ be a facet of $\Delta_G$: i.e., $F$ is a maximal independent subset of $V(G)$. For any $v, v' \in F$, we have $\{v, v'\} \notin E(G)$, and thus $\{v, v'\} \in E(G)$. So, the induced subgraph of $G$ on the vertex set $F$ is complete. The maximality of this induced subgraph follows from the maximality of $F$ in $G$.

Conversely, if $F$ is a maximal complete subgraph of $G$, then no two vertices of $F$ are adjacent in $\overline{G}$; i.e., $F$ is an independent subset of $V(\overline{G})$ and hence is a face of $\Delta_G$. The maximality again follows from the maximality of $F$ as a maximal complete subgraph of $G$. \qed

Now, we are ready to state and prove our main result.

**Theorem 2.4.** The complement of a $d$-tree is pure shellable.
Proof. Let $G$ be a $d$-tree with the vertex set $V(G) = \{v_1, ..., v_n\}$. By the definition of $d$-tree, all the maximal complete subgraphs of $G$ have $d+1$ vertices. It follows from Lemma 2.3 that $\Delta(G)$ is pure with $\dim \Delta(G) = d$.

By the inductive structure of $d$-tree, we may assume

$$G = \bigcup_{i=1}^{s} G_i,$$

where $G_i$ is a complete subgraph of $G$ isomorphic to $K_{d+1}$ (the complete graph over $d+1$ vertices), for all $i = 1, ..., s$, with the following properties:

1. $V(G_1) = \{v_1, ..., v_{d+1}\}$,
2. $V(G_i) = V(G'_i) \cup \{v_{d+i}\}$, where $G'_i$ is a complete subgraph of $\bigcup_{t=1}^{i-1} G_t$ isomorphic to $K_d$.

It is easy to see that $s = n - d$. Let $F_i = V(G_i)$ for all $i = 1, ..., s$. It follows from Lemma 2.3 that

$$\Delta(G) = \langle F_1, ..., F_s \rangle.$$

For all $1 \leq i < j \leq s$, we have $v_{d+j} \in F_j \setminus F_i$. Note that $V(G_j) = V(G'_j) \cup \{v_{d+j}\}$ and $G'_j$ is a complete subgraph of $\bigcup_{t=1}^{j-1} G_t$ isomorphic to $K_d$. So, there exists a maximal complete subgraph of $\bigcup_{t=1}^{j-1} G_t$, say $H$, such that $G'_j$ is a subgraph of $H$. But the only maximal complete subgraphs of $\bigcup_{t=1}^{j-1} G_t$ are $G_t$ for $t = 1, ..., j - 1$. Therefore, $G'_j$ is a subgraph of $G_l$ for some $l < j$, and hence $V(G'_j) \subseteq V(G_l) = F_l$. It is clear that

$$F_j \setminus F_i = V(G_j) \setminus V(G_i) = (V(G'_j) \cup \{v_{d+j}\}) \setminus V(G_i) = \{v_{d+j}\}.$$

It is known that any pure shellable simplicial complex is Cohen-Macaulay; i.e., $I_\Delta$ is a Cohen-Macaulay ideal; cf. [1, Theorem 5.1.13]. This fact together with Theorem 2.4 implies the next result.

**Corollary 2.5.** ([3, Theorem 3.3]) *The complement of a $d$-tree is Cohen-Macaulay.*

Proof. Let $G$ be a $d$-tree. Notice that $I(\overline{G}) = I_{\Delta(G)}$ and $\Delta(G)$ is pure shellable, and so $I_{\Delta(G)}$ is Cohen-Macaulay. □
The complement of a $d$-tree is pure shellable

It is natural to ask about the converse of Corollary 2.5. In the following we give an answer to this question in the case of chordal graphs. Recall that a graph $G$ is called chordal if every cycle of length greater than 3 of $G$ has a chord.

**Proposition 2.6.** Let $G$ be a chordal graph. The followings are equivalent.

(i) $G$ is a $d$-tree.
(ii) $\overline{G}$ is a pure shellable graph.
(iii) $\overline{G}$ is a Cohen-Macaulay graph.

**Proof.** (i)⇒(ii): this follows from Theorem 2.4.
(ii)⇒(iii): this follows from [1, Theorem 5.1.13].
(iii)⇒(i): this follows from [4, Corollary Page 63].

**Example 2.7.** The star graph $S_d$ of order $d$ is a tree on $d$ vertices with one vertex having vertex degree $d-1$ and the other $d-1$ vertices having vertex degree 1. The complement of $S_d$ has two connected components: a complete graph $K_{d-1}$ and an isolated vertex. It is easy to see that a graph is pure shellable if and only if its connected components are pure shellable. Thus, a complete graph is pure shellable. (Note that we can use 0-tree to show that a complete graph is pure shellable. Let $G$ be a 0-tree with $d$ vertices. Then, the complement of $G$ is the complete graph $K_d$.)

**Example 2.8.** Let $G$ be the complete graph $K_{d+1}$. Take $H$ as a graph obtained by connecting a new vertex of degree 1 to each vertex of $G$. Since the complement of $H$ is a $d$-tree, we have that $H$ is a pure shellable graph.

**References**


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