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AN OPERATIONAL METHOD FOR THE NUMERICAL SOLUTION OF TWO DIMENSIONAL LINEAR FREDHOLM INTEGRAL EQUATIONS WITH AN ERROR ESTIMATION

M. Y. RAHIMI, S. SHAHMORAD*, F. TALATI AND A. TARI

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Abstract. We formulate the operational Tau method for the two dimensional linear Fredholm integral equations of the second kind. Some theoretical results are given to simplify application of the Tau method, and then existence and uniqueness of solution for these equations are investigated. We also estimate error of the proposed method and give some numerical examples to demonstrate its accuracy in finding solutions.

1. **Introduction**

As we know, suitable work has been done on the development and analysis of numerical methods for solving one dimensional integral equations of the second kind (for example, see [1-5] and their references). But, for the numerical solution of two dimensional integral equations, the work done is much less (for example, see [5, 7, 19]).
On the other hand, up to now, the operational Tau method (see Ortiz [8] and Ortiz and Samara [9]) has been developed for the numerical solution of ordinary differential equations (see, for instance, [10-12]) and for

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∗Corresponding author

partial differential equations (see, for example, [13-14]). Also, in recent years, this method has been formulated for the numerical solution of one dimensional integral and integro-differential equations (see [15-18]).
The aim of our work here is to formulate the operational Tau method for the two dimensional linear Fredholm integral equations (TDLFIE) of the form
(1.1)

$$\phi(x,t) - \int_c^d \int_a^b K(x,t,y,z)\phi(y,z)dydz = f(x,t), \quad x\epsilon[a,b], \quad t\epsilon[c,d].$$

where, $K(x,t,y,z)$ and $f(x,t)$ are continuous functions.
To this end, we replace different parts of TDLFIE by their matrix representations of the Tau method and hence convert it to the system of linear algebraic equations, and solve it to obtain an approximate solution of the problem.

2. Some preliminary results of the Tau method

The matrix Tau method, proposed by Ortiz and Samara [9] is based on using three simple matrices,

$$\mu = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \eta = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$, \iota = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1/2 & 0 & \cdots \\ 0 & 0 & 0 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

having the following properties.

Lemma 2.1. *If* $y_N(x) = \underline{a}_N X$ *with* $\underline{a}_N = (a_0, a_1, ..., a_N, 0, 0, ...)$ *and* $X = (1, x, x^2, ...)^T$*, then*

a. $\frac{d}{dx} y_N(x) = \underline{a}_N \eta X.$

b. $x y_N(x) = \underline{a}_N \mu X.$

c. $\int y_N(x)dx = \underline{a}_N \iota X.$

Proof. See [15].

Corollary 2.2. *Generally, under assumptions of Lemma 2.1, we have*

a. $x^i X = \mu^i X.$

b. $\int X dx = \iota X.$

For the remainder of the paper, we assume that $\underline{\mu}$, $\underline{\eta}$ and $\underline{\iota}$ are $(N+1) \times (N+1)$ matrices including the first $(N+1)$ rows and columns of μ, η and ι, respectively.

3. **Existence and uniqueness of solution**

Here, we give a proof for existence and uniqueness of solution of (1.1) based on [1].
We define the integral operator $A : C([a,b] \times [c,d]) \to C([a,b] \times [c,d])$ as

$$A\phi(x,t) = \int_c^d \int_a^b K(x,t,y,z)\phi(y,z)dydz, \quad x\epsilon[a,b], \quad t\epsilon[c,d] \tag{3.1}$$

Then, (1.1) can be written in the operator form

$$\phi - A\phi = f \tag{3.2}$$

The following theorem is an extension from one dimensional case (of [1]) to two dimensional case about the operator A.

Theorem 3.1. *Let $K : [a,b] \times [c,d] \times [a,b] \times [c,d] \to R$ be continuous. then, the operator A defined by (3.1) is bounded by the norm,*

$$\|A\|_\infty = \max_{x\in[a,b],t\in[c,d]} \int_c^d \int_a^b |K(x,t,y,z)|dydz \tag{3.3}$$

Proof. Let $\phi \in C([a,b] \times [c,d])$ and $\|\phi\|_\infty \le 1$. Then,

$$|(A\phi)(x,t)| \le \int_c^d \int_a^b |K(x,t,y,z)|dydz, \quad x\epsilon[a,b], \quad t\epsilon[c,d].$$

Hence,

$$\|A\|_\infty = \sup_{\|\phi\|_\infty \le 1} \|A\phi\|_\infty \le \max_{x\in[a,b],t\in[c,d]} \int_c^d \int_a^b |K(x,t,y,z)|dydz.$$

On the other hand, since K is continuous, there exist $x_0 \in [a,b]$ and $t_0 \in [c,d]$ such that

$$\int_c^d \int_a^b |K(x_0,t_0,y,z)|dydz = \max_{x\in[a,b],t\in[c,d]} \int_c^d \int_a^b |K(x,t,y,z)|dydz.$$

Let $\epsilon > 0$ be given, and define

$$\psi(y,z) = \frac{\overline{K(x_0,t_0,y,z)}}{|K(x_0,t_0,y,z)| + \epsilon}.$$

Then, $\|\psi\|_\infty \le 1$ and

$$\begin{aligned}\|A\psi\|_\infty \ge |(A\psi)(x_0,t_0)| &= \int_c^d \int_a^b \frac{|K(x_0,t_0,y,z)|^2}{|K(x_0,t_0,y,z)| + \epsilon} dydz \\ &\ge \int_c^d \int_a^b \frac{|K(x_0,t_0,y,z)|^2 - \epsilon^2}{|K(x_0,t_0,y,z)| + \epsilon} dydz \\ &= \int_c^d \int_a^b |K(x_0,t_0,y,z)|dydz - \epsilon(b-a)(d-c)\end{aligned}$$

Hence,

$$\|A\|_\infty = \sup_{\|\phi\|_\infty \le 1} \|A\phi\|_\infty \ge$$

$$\|A\psi\|_\infty \ge \int_c^d \int_a^b |K(x_0,t_0,y,z)|dydz - \epsilon(b-a)(d-c),$$

and since $\epsilon > 0$ is arbitrary, we have

$$\|A\|_\infty \ge \int_c^d \int_a^b |K(x_0,t_0,y,z)|dydz$$

$$= \max_{x\in[a,b],t\in[c,d]} \int_c^d \int_a^b |K(x,t,y,z)|dydz,$$

and hence the proof is complete. □

Now, we recall the following theorem from [1].

Theorem 3.2. *Let A be a bounded operator on $C([a,b]\times[c,d])$ with $\|A\|<1$ and I denote the identity operator. Then $I-A$ has a bounded inverse on $C([a,b]\times[c,d])$, which is given by the Neumann series*

$$(I-A)^{-1}=\sum_{k=0}^{\infty}A^k,$$

satisfying

$$\|(I-A)^{-1}\|\leq\frac{1}{1-\|A\|}.$$

Theorem 3.2 ensures that the following condition is a sufficient (not necessary) condition for existence and uniqueness of the solution of (3.2):

$$\max_{x\in[a,b],t\in[c,d]}\int_c^d\int_a^b|K(x,t,y,z)|dydz<1.$$

4. Description of the method

To solve Eq. (3.2) by operational approach, we assume that $K(x,t,y,z)$ and $f(x,t)$ are polynomials; otherwise, they can be approximated by suitable polynomials.
We assume the approximate solution has the truncated series form,

$$\phi(x,t)\simeq\sum_{i=0}^{N}\sum_{j=0}^{N}C_{ij}x^it^j=\underline{X}^TC\underline{T}, \tag{4.1}$$

where, $\underline{X}=(1,x,x^2,...,x^N)^T$, $\underline{T}=(1,t,t^2,...,t^N)^T$ and C is the following $(N+1)\times(N+1)$ matrix,

$$C=\begin{pmatrix} C_{00} & C_{01} & \cdots & C_{0N}\\ C_{10} & C_{11} & \cdots & C_{1N}\\ \vdots & \vdots & \vdots & \vdots\\ C_{N0} & C_{N1} & \cdots & C_{NN}\end{pmatrix}. \tag{4.2}$$

Theorem 4.1. *Let $K(x,t,y,z)=\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}k_{ijmn}x^it^jy^mz^n$. Then, we have*

$$\int_c^d \int_a^b K(x,t,y,z)\phi(y,z)dydz = \underline{X}^T \Pi_I \underline{T}, \tag{4.3}$$

where,

$$\Pi_I = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} k_{ijmn} P_{ijmn} \tag{4.4}$$

with

$$P_{ijmn} = e_{i+1}(\underline{\xi}^{(m)T}(b) - \underline{\xi}^{(m)T}(a))C(\underline{\xi}^{(n)}(d) - \underline{\xi}^{(n)}(c))e_{j+1}^T, \tag{4.5}$$

$\underline{\xi}^{(m)}(x) = \underline{\mu}^m \underline{\iota X}$, *and* $\underline{\xi}^{(n)}(x) = \underline{\mu}^n \underline{\iota X}$ *corresponding to the term* $x^i t^j y^m z^n$ *in the kernel.*

Proof. Since $\phi(x,t) = \underline{X}^T C \underline{T}$, we have

$$\int_c^d \int_a^b x^i t^j y^m z^n \phi(y,z)dydz = \int_c^d t^j \left[\int_a^b x^i y^m \underline{Y}^T dy\right] C \underline{Z} z^n dz, \tag{4.6}$$

where, $\underline{Y} = (1, y, y^2, ..., y^N)^T$ and $\underline{Z} = (1, z, z^2, ..., z^N)^T$.
By corollary 2.2,

$$\int_a^b x^i y^m \underline{Y}^T dy = x^i \int_a^b \underline{Y}^T (\underline{\mu^T})^m dy = x^i \{\int_a^b \underline{Y}^T dy\}(\underline{\mu^T})^m = x^i \{\underline{Y}^T \underline{\iota^T}\}_a^b (\underline{\mu^T})^m$$

$$= x^i\{(\underline{\mu}^m \underline{\iota Y}|_{y=b})^T - (\underline{\mu}^m \underline{\iota Y}|_{y=a})^T\} = \underline{X}^T e_{i+1}(\underline{\xi}^{(m)T}(b) - \underline{\xi}^{(m)T}(a)), \tag{4.7}$$

and similarly,

$$\int_c^d t^j \underline{Z} z^n dz = (\underline{\xi}^{(n)}(d) - \underline{\xi}^{(n)}(c))e_{j+1}^T \underline{T}. \tag{4.8}$$

By substituting (4.7) into the right side of (4.6), we obtain:

$$\int_c^d t^j \{\underline{X}^T e_{i+1}(\underline{\xi}^{(m)T}(b) - \underline{\xi}^{(m)T}(a))\} C \underline{Z} z^n dz$$

$$= \underline{X}^T e_{i+1}(\underline{\xi}^{(m)T}(b) - \underline{\xi}^{(m)T}(a)) C \left[\int_c^d t^j \underline{Z} z^n dz\right]. \tag{4.9}$$

Substituting (4.8) into (4.9), we obtain:

$$\int_c^d \int_a^b x^i t^j y^m z^n \phi(y,z) dy dz$$

$$= \underline{X}^T \{ e_{i+1} (\underline{\xi}^{(m)T}(b) - \underline{\xi}^{(m)T}(a)) C (\underline{\xi}^{(n)}(d) - \underline{\xi}^{(n)}(c)) e_{j+1}^T \} \underline{T} \tag{4.10}$$

Therefore, the proof is complete.

Lemma 4.2. *The elements of* $\underline{\xi}^{(m)}(x) = (\xi_1^{(m)}(x), ..., \xi_{N+1}^{(m)}(x))$ *are determined as:*

$$(\xi_k^{(m)}(x)) = \begin{cases} \frac{x^{k+m}}{k+m}, & k = 1,2,...,N-m \\ 0, & otherwise. \end{cases} \tag{4.11}$$

A similar result is obtained for $\underline{\xi}^{jn}(x)$ *by substituting* m *by* n.

Proof. By the definition of $\underline{\xi}^{im}(x)$, we have

$$\underline{\xi}^{(im)}(x) = \mu^m \iota \underline{X} = (\frac{1}{m+1} x^{m+1}, \frac{1}{m+2} x^{m+2}, \ldots, \frac{1}{N} x^N, 0, \ldots, 0)^T,$$

where, the last zero elements are repeated $(m+1)$ times.

Corollary 4.3. *The matrix* P_{ijmn} *in Theorem 4.1 has the form,*

$$P_{ijmn} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & p_{i+1,j+1} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

where the nonzero element is computed as:

$$p_{i+1,j+1} = \sum_{r=1}^{N-n} \sum_{k=1}^{N-m} \frac{d^{r+n} - c^{r+n}}{r+n} \frac{b^{k+m} - a^{k+m}}{k+m} C_{k-1,r-1}, \tag{4.12}$$

$$, \quad m,n = 0,1,\cdots,N-1.$$

Proof. By substituting (4.11) in (4.5), the result is at hand.
Note that if $m = N$ (or $n = N$), then $\underline{\xi}^{im} = \bar{0}$ ($\underline{\xi}^{jn} = \bar{0}$), and P_{ijmn} is a zero matrix.

Note that, up to now, we have written the left hand side of (1.1) in the matrix form by (4.1) and (4.3). At this time, we can write the right hand side of the problem (4.1) in the form,

$$f(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} f_{ij}x^i t^j = \underline{X}^T F \underline{T}, \tag{4.13}$$

where,

$$F = \begin{pmatrix} f_{00} & f_{01} & \cdots & f_{0N} \\ f_{10} & f_{11} & \cdots & f_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ f_{N0} & f_{N1} & \cdots & f_{NN} \end{pmatrix}. \tag{4.14}$$

Now, by substituting (4.1), (4.3) and (4.13) in (1.1), we obtain:

$$\underline{X}^T C \underline{T} - \underline{X}^T \Pi_I \underline{T} = \underline{X}^T F \underline{T},$$

or

$$\underline{X}^T (C - \Pi_I - F)\underline{T} = 0,$$

and therefore,

$$C - \Pi_I = F, \tag{4.15}$$

since $\underline{X}$ and $\underline{T}$ are bases.

By solving the system (4.15), we obtain $\phi(x,t)$ from (4.1).

5. **Error estimation**

A complete error analysis and convergence of the Tau method are investigated in [6]. Here, we give a way of estimating error for our proposed method. Indeed, this error estimation ensures that the method can be applied to problems with areasonable confidence unknowns solutions.

Define the error function as:

$$e(x,t) = \phi(x,t) - \phi_N(x,t), \tag{5.1}$$

where, $\phi(x,t)$ and $\phi_N(x,t)$ are the exact and approximate solutions of the integral equation (1.1), respectively.

Substituting $\phi_N(x,t)$ in (1.1) leads to:

$$\phi_N(x,t) - \int_c^d \int_a^b K(x,t,y,z)\phi_N(y,z)dydz = f(x,t) + p_N(x,t) \tag{5.2}$$

where, $p_N(x,t)$ is a perturbation term and can be obtained by substituting the computed solution $\phi_N(x,t)$ into the equation,

$$p_N(x,t) = \phi_N(x,t) - \int_c^d \int_a^b K(x,t,y,z)\phi_N(y,z)dydz - f(x,t). \tag{5.3}$$

We proceed to find an approximation $e_N(x,t)$ to the error function $e(x,t)$ in the same way as we did before for the solution of equation (1.1).
Now, by subtracting (5.2) from (1.1) and using (5.1), the error function $e(x,t)$ satisfies:

$$e(x,t) - \int_c^d \int_a^b K(x,t,y,z)e(y,z)dydz = -p_N(x,t). \tag{5.4}$$

It should be noted that in order to construct the approximation $e_N(x,t)$ to $e(x,t)$, only the right hand side of the equation (5.1) needs to be recomputed, and thus by solving the integral equation (5.4) we obtain an estimate for the error function (5.1).
We also get the following error bound from (5.4), which enable us to control the estimated errors:

$$\|e\|_\infty \leq \frac{\|p_N\|_\infty}{1 - \|K\|_\infty} \tag{5.5}$$

6. **Numerical examples**

Here, we give some examples to show accuracy of the solutions obtained by our proposed method.
Note that, as we mentioned previously, whenever $K(x,t,y,z)$ or $f(x,t)$ are not polynomials, they must be approximated by polynomials of suitable degrees. Therefore, in the following examples, we approximate non-polynomial parts of $K(x,t,y,z)$ and $f(x,t)$ by the Taylor polynomial.

Example 1. Consider the integral equation,

$$\phi(x,t) - \int_{-1}^{1}\int_{-1}^{1}(xsiny + tsinz)\phi(y,z)dydz$$
$$= xcost + t + 4(xsin(1) - t)(cos(1) - sin(1)),$$
$$x, t \in [-1, 1],$$

with the exact solution $\phi(x,t) = xcost + t$.
First, we expand $sinz$ and $cost$ in Taylor series on $z_0 = 0$ and $t_0 = 0$. Tables 1 and 2 show the absolute errors ($e(x,t)$) and their estimations ($e_N(x,t)$) at the points $(x,t) = ((0.25)i, (0.25)i), \quad i = -4, \ldots, 4$, with $N = 12$ and $N = 14$, respectively.

TABLE 1

(x,t)	$e(x,t)$	$e_N(x,t)$
$(-1,-1)$	$0.680166e-7$	$0.680104e-7$
$(-0.75,-0.75)$	$0.510209e-7$	$0.510163e-7$
$(-0.5,-0.5)$	$0.340140e-7$	$0.340109e-7$
$(-0.25,-0.25)$	$0.170070e-7$	$0.170055e-7$
$(0,0)$	0	0
$(0.25,0.25)$	$0.170070e-7$	$0.170055e-7$
$(0.5,0.5)$	$0.340140e-7$	$0.340109e-7$
$(0.75,0.75)$	$0.510209e-7$	$0.510163e-7$
$(1,1)$	$0.680166e-7$	$0.680104e-7$

TABLE 2

(x,t)	$e(x,t)$	$e_N(x,t)$
$(-1,-1)$	$0.379145e-9$	$0.379124e-9$
$(-0.75,-0.75)$	$0.284394e-9$	$0.284378e-9$
$(-0.5,-0.5)$	$0.189596e-9$	$0.189586e-9$
$(-0.25,-0.25)$	$0.947983e-10$	$0.947928e-10$
$(0,0)$	0	0
$(0.25,0.25)$	$0.947983e-10$	$0.947928e-10$
$(0.5,0.5)$	$0.189596e-9$	$0.189586e-9$
$(0.75,0.75)$	$0.284394e-9$	$0.284378e-9$
$(1,1)$	$0.379145e-9$	$0.379124e-9$

Example 2. The integral equation,

$$\phi(x,t) - \int_{-1}^{1}\int_{-1}^{1}(xy + te^{z})\phi(y,z)dydz$$
$$= xe^{-t} - \frac{1}{3}x(1 + 2e - \frac{2}{e}) + t(1 - \frac{4}{e}),\ x,t \in [-1,1],$$

has the exact solution $\phi(x,t) = xe^{-t} + x + t$.
Similar to Example 1, we estimate e^{-t} by Taylor series on $t_0 = 0$. Numerical results are reported in tables 3 and 4 for the absolute errors and their estimations at the points $(x,t) = ((0.25)i, (0.25)i)$, $i = -4, ..., 4$, with $N = 12$ and $N = 14$, respectively.

Table 3

(x,t)	$e(x,t)$	$e_N(x,t)$
$(-1,-1)$	$0.169117e-7$	$0.169090e-7$
$(-0.75,-0.75)$	$0.128104e-7$	$0.128078e-7$
$(-0.5,-0.5)$	$0.854228e-8$	$0.854055e-8$
$(-0.25,-0.25)$	$0.427114e-8$	$0.427028e-8$
$(0,0)$	$0.212730e-20$	$0.212730e-20$
$(0.25,0.25)$	$0.427114e-8$	$0.427028e-8$
$(0.5,0.5)$	$0.854229e-8$	$0.854057e-8$
$(0.75,0.75)$	$0.128161e-7$	$0.128135e-7$
$(1,1)$	$0.172344e-7$	$0.172302e-7$

Table 4

(x,t)	$e(x,t)$	$e_N(x,t)$
$(-1,-1)$	$0.934594e-10$	$0.934498e-10$
$(-0.75,-0.75)$	$0.706981e-10$	$0.706887e-10$
$(-0.5,-0.5)$	$0.471374e-10$	$0.471311e-10$
$(-0.25,-0.25)$	$0.235687e-10$	$0.235656e-10$
$(0,0)$	$0.425459e-20$	$0.425459e-20$
$(0.25,0.25)$	$0.235687e-10$	$0.235656e-10$
$(0.5,0.5)$	$0.471374e-10$	$0.471312e-10$
$(0.75,0.75)$	$0.707135e-10$	$0.707040e-10$
$(1,1)$	$0.949944e-10$	$0.949792e-10$

Example 3. Consider the third example as:

$$\phi(x,t) - \int_0^1 \int_0^1 (siny + tcosz - 1)\phi(y,z)dydz = xsin(x-t) + \frac{7}{4}t - \frac{5}{4}cos(1)$$

$$-\frac{3}{2}sin(1) - \frac{1}{2}tcos(1) - tsin(1) + \frac{1}{8}sin(2) - \frac{1}{4}cos(2) + \frac{1}{4}tcos(2) + 2,$$
$$x,t \in [-1,1],$$

which has the exact solution: $\phi(x,t) = xsin(x-t) + t$.
Tables 5 and 6 show absolute errors and their estimations with $N = 12$ and $N = 14$, respectively.

TABLE 5

(x,t)	$e(x,t)$	$e_N(x,t)$
$(0,0)$	$0.315868e-6$	$0.315900e-6$
$(0.25,0.25)$	$0.454994e-6$	$0.455033e-6$
$(0.5,0.5)$	$0.594120e-6$	$0.594166e-6$
$(0.75,0.75)$	$0.733212e-6$	$0.733265e-6$
$(1,1)$	$0.870615e-6$	$0.870665e-6$

TABLE 6

(x,t)	$e(x,t)$	$e_N(x,t)$
$(0,0)$	$0.748611e-8$	$0.748626e-8$
$(0.25,0.25)$	$0.106630e-7$	$0.106632e-7$
$(0.5,0.5)$	$0.138399e-7$	$0.138402e-7$
$(0.75,0.75)$	$0.170168e-7$	$0.170170e-7$
$(1,1)$	$0.201841e-7$	$0.201844e-7$

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M. Y. Rahimi, S. Shahmorad
Department of Applied Mathematics, University of Tabriz, Tabriz, Iran.
Email: m_rahimi@tabrizu.ac.ir
Email: shahmorad@tabrizu.ac.ir

F. Talati
Department of Mechanical Engineering, University of Tabriz, Tabriz, Iran.
Email: talati@tabrizu.ac.ir

A. Tari
Department of Mathematics, Shahed University, Tehran, Iran.
Email: tari@shahed.ac.ir