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ERGODIC THEORETIC CHARACTERIZATION OF LEFT AMENABLE LAU ALGEBRAS

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ABSTRACT. This paper deals with the notion of left amenability for a large class of Banach algebras known as Lau algebras. It makes a study of left amenability in the framework of ergodicity considering the antirepresentations of a Lau algebra on a Banach space.

1. Introduction

Recall that a *Lau algebra* (the same as F-algebra in Lau [4]) is a complex Banach algebra \mathcal{A} which is the (unique) predual of a W^* algebra \mathcal{M} and the identity element u of \mathcal{M} is a multiplicative linear functional on \mathcal{A} ; see Pier [13]. Note that \mathcal{M} need not be unique [4]. We shall identify the continuous dual \mathcal{A}^* with a fixed W^* -algebra whose identity is multiplicative on \mathcal{A} .

Example of Lau algebras include the Fourier algebra A(G), the Fourier-Stieltjes algebra B(G), the group algebra $L^1(G)$ of a locally compact group G, and the measure algebra $M(\Omega)$ of a locally compact semigroup or hypergroup Ω .

The Lau algebra \mathcal{A} is called *left amenable* if for each two-sided Banach \mathcal{A} -module X with $a.x = u(a) \ x \ (a \in \mathcal{A}, \ x \in X)$, every bounded derivation $D : \mathcal{A} \to X^*$ is inner. The notion of left amenability for Lau algebras was introduced by Lau [4]. In the same paper he extended several characterizations of amenable locally compact groups

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to left amenable Lau algebras; see also Ghahramani and Lau [3], Lau [5], Lau and Wong [6], and the recent papers [7]-[10] of t he author.

In this paper, we establish a characterization of left amenable Lau algebras. In order to find this result, we investigate some relations between left amenability and ergodic theory.

2. Ergodic antirepresentations

Throughout, let \mathcal{A} denote a Lau algebra and let u be the identity of the dual W^* -algebra \mathcal{A}^* of \mathcal{A} . Also, set

$$P_1(\mathcal{A}) = \{ a \in \mathcal{A} : \| a \| = u(a) = 1 \},\$$

and note that $P_1(\mathcal{A})$ is the set of all elements a in \mathcal{A} that induce positive functionals on \mathcal{A}^* with norm one [14], 1.5.1 and 1.5.2.

By an antirepresentation T of \mathcal{A} on a Banach space X, we shall mean a norm continuous map $T : a \mapsto T_a$ from \mathcal{A} into $\mathcal{B}(X)$, the Banach space of all bounded operators on X, such that $T_{ab} = T_b T_a$ for all $a, b \in \mathcal{A}$. In this case, we put

$$\begin{aligned} X_{\kappa} &= & \cap \{ \text{ kernel } (T_a - I) : a \in P_1(\mathcal{A}) \}, \\ X_{\rho} &= & \text{ The closure of the span of } \cup \{ \text{ range } (T_a - I) : a \in P_1(\mathcal{A}) \}, \\ X_{\sigma} &= & X_{\kappa} + X_{\rho}, \text{ and} \\ C_x &= & \text{ The closure of } \{ T_a(x) : a \in P_1(\mathcal{A}) \}. \end{aligned}$$

Note that X_{κ} and X_{ρ} are closed subspaces of X. By the strong operator topology on $\mathcal{B}(X)$, we shall mean the locally convex topology determined by the family $\{\mathcal{P}_x : x \in X\}$ of seminorms on $\mathcal{B}(X)$, where $\mathcal{P}_x(S) = || S(x) ||$ for all $x \in X$ and $S \in \mathcal{B}(X)$.

We say that an antirepresentation T of \mathcal{A} on a Banach space X is *ergodic* if there is a net $(E_{\gamma})_{\gamma \in \Gamma}$ in $\mathcal{B}(X)$ such that

 $(\mathcal{E}_1) \ E_{\gamma}(T_a - I) \to 0$ in the strong operator topology for all $a \in P_1(\mathcal{A})$,

$$(\mathcal{E}_2) E_{\gamma}(x) \in C_x$$
 for all $x \in X$ and $\gamma \in \Gamma$.

We commence with the following version of the Mean Ergodic Theorem for Lau algebras which helps us to get a grip on these concepts. A characterization of Lau algebras

Theorem 2.1. Let T be an ergodic antirepresentation of \mathcal{A} on a Banach space X with $(E_{\gamma})_{\gamma \in \Gamma}$ satisfying (\mathcal{E}_1) and (\mathcal{E}_2) . Then the following hold:

(a) $|| E_{\gamma} || \leq || T ||$ for all $\gamma \in \Gamma$.

(b) $(E_{\gamma}(x))_{\gamma \in \Gamma}$ is norm convergent to an element of $X_{\kappa} \cap C_x$ for all $x \in X_{\sigma}$. In particular, $E_{\gamma}(x) \to x$ (resp. 0) for all $x \in X_{\kappa}$ (resp. $x \in X_{\rho}$).

(c) $X_{\sigma} = X_{\kappa} \oplus X_{\rho}, T_a(X_{\sigma}) \subseteq X_{\sigma}$ for all $\gamma \in \Gamma, C_x \subseteq X_{\sigma}$ for all $x \in X_{\sigma}$, and $E_{\gamma}(X_{\sigma}) \subseteq X_{\sigma}$ for all $\gamma \in \Gamma$.

(d) If $P: X_{\sigma} \to X_{\kappa}$ is the projection associated with the direct sum $X_{\sigma} = X_{\kappa} \oplus X_{\rho}$, then $E_{\gamma}(x) \to P(x)$ and $X_{\kappa} \cap C_x = \{P(x)\}$ for all $x \in X_{\sigma}$.

Proof. (a). This follows from (\mathcal{E}_2) and the fact that $|| T_a(x) || \le || T ||$ for all $a \in P_1(\mathcal{A})$ and $x \in X$ with $|| x || \le 1$.

(b). By (\mathcal{E}_1) , $E_{\gamma}(y) \to 0$ for all $y \in \bigcup \{ \text{ range } (T_a - I) : a \in P_1(\mathcal{A}) \}$. This together with (a) imply that $E_{\gamma}(x_{\rho}) \to 0$ for all $x_{\rho} \in X_{\rho}$.

Now let $x_{\kappa} \in X_{\kappa}$ and $x_{\rho} \in X_{\rho}$, and put $x = x_{\kappa} + x_{\rho}$. Then, since $E_{\gamma}(x_{\kappa}) \in C_{x_{\kappa}} = \{x_{\kappa}\}$, we conclude that

$$E_{\gamma}(x) = x_{\kappa} + E_{\gamma}(x_{\rho}) \to x_{\kappa} \in X_{\kappa} \cap C_x.$$

(c). $X_{\kappa} \cap X_{\rho} = \{0\}$ by (b), and hence $X_{\sigma} = X_{\kappa} \oplus X_{\rho}$. Now, if $a, b \in P_1(\mathcal{A})$ and $x \in X$, then $ba \in P_1(\mathcal{A})$ and

$$T_a(T_b - I)(x) = (T_{ba} - I)(x) - (T_a - I)(x) \in X_{\rho}.$$

This shows that $T_a(X_{\rho}) \subseteq X_{\rho}$ whence $T_a(X_{\sigma}) \subseteq X_{\sigma}$.

Fix $x \in X$. To prove $C_x \subseteq X_{\sigma}$, let $y \in C_x$. By definition there is a net (a_{δ}) in $P_1(\mathcal{A})$ such that $T_{a_{\delta}}(x) \to y$. Write $x = x_{\kappa} + x_{\rho}$, where $x_{\kappa} \in X_{\kappa}$ and $x_{\rho} \in X_{\rho}$. Then since $T_{a_{\delta}}(X_{\rho}) \subseteq X_{\rho}$ and X_{ρ} is closed, we have

$$y - x_{\kappa} = \lim_{\delta} T_{a_{\delta}}(x - x_{\kappa}) = \lim_{\delta} T_{a_{\delta}}(x_{\rho}) \in X_{\rho}$$

So we have shown that $C_x \subseteq X_{\sigma}$ for all $x \in X_{\sigma}$.

Now the last inclusion $E_{\gamma}(X_{\sigma}) \subseteq X_{\sigma}$ follows from (\mathcal{E}_2) .

(d). The first assertion follows from (b). For the second, fix $x \in X_{\sigma}$ and note that $P(x) \in X_{\kappa} \cap C_x$ by (b). To prove the converse inclusion, let $y \in X_{\kappa} \cap C_x$. Then $T_{a_{\delta}}(x) \to y$ for some net (a_{δ}) in $P_1(\mathcal{A})$, and therefore

$$y - x = \lim_{\delta} (T_{a_{\delta}} - I)(x) \in X_{\rho}.$$

Consequently, P(y - x) = 0 and so y = P(y) = P(x) as required. \Box

Theorem above is due to Day [1] and Eberlein [2] for $\mathcal{A} = L^1(\Sigma)$ of a locally compact group or discrete semigroup Σ (cf. [11], Chapter 5).

The second dual \mathcal{A}^{**} of \mathcal{A} is a Lau algebra with the first Arens product defined by the equations

$$\langle F \odot H, f \rangle = \langle F, Hf \rangle, \quad \langle Hf, a \rangle = \langle H, fa \rangle, \quad \langle fa, b \rangle = \langle f, ab \rangle$$

for all $F, H \in \mathcal{A}^{**}, f \in \mathcal{A}^{*}$ [4], Proposition 3.2. An element F of \mathcal{A}^{**} (resp. $P_1(\mathcal{A}^{**})$) is called a *topological left invariant functional* (resp. *mean*) on \mathcal{A}^{*} if $a \odot F = F$ for all $a \in P_1(\mathcal{A})$. The set of all topological left invariant functional (resp. mean) is denoted by $\text{TLIF}(\mathcal{A}^{*})$ (resp. TLIM (\mathcal{A}^{*})).

Lemma 2.2. The vector space $TLIF(\mathcal{A}^*)$ is spanned by $TLIM(\mathcal{A}^*)$.

Proof. Suppose that F is a nonzero element of $\text{TLIF}(\mathcal{A}^*)$. Then $F^* \in \text{TLIF}(\mathcal{A}^*)$, where F^* denotes the adjoint of F. We therefore may assume that F is self-adjoint. So, there exist unique positive functionals F^+ and F^- on \mathcal{A}^* such that

$$F = F^{+} - F^{-}$$
 and $||F|| = ||F^{+}|| + ||F^{-}||$

for all $a \in P_1(\mathcal{A})$ [14], Theorem 1.14.3. Thus $a \odot F^+$ and $a \odot F^-$ are positive functionals on \mathcal{A}^* , and

$$\| F \| = \| F^+ \| + \| F^- \| = \left\langle F^+, u \right\rangle + \left\langle F^-, u \right\rangle$$
$$= \left\langle a \odot F^+, u \right\rangle + \left\langle a \odot F^-, u \right\rangle = \| a \odot F^+ \| + \| a \odot F^- \| .$$

This together with $a \odot F = a \odot F^+ - a \odot F^-$ imply that $a \odot F^+ = F^+$ and $a \odot F^- = F^-$ [14], Theorem 1.14.3. In particular, TLIM(\mathcal{A}^*) is nonempty because $F \neq 0$. So, putting $M^+ = \parallel F^+ \parallel^{-1} F^+$ (resp. $M^- = \parallel F^- \parallel^{-1} F^-$) if $F^+ \neq 0$ (resp. $F^- \neq 0$) and choosing M^+ in TLIM(\mathcal{A}^*) (resp. M^- in TLIM(\mathcal{A}^*)) arbitrarily if $F^+ = 0$ (resp. $F^- = 0$), we see that $M^+, M^- \in \text{TLIM}(\mathcal{A}^*)$ and $F = \parallel F^+ \parallel M^+ - \parallel$ $F^- \parallel M^-$. \Box

We are now prepared to present the main result of this paper.

Theorem 2.3. The following assertions are equivalent. (a) \mathcal{A} is left amenable. A characterization of Lau algebras

(b) Each antirepresentation of \mathcal{A} on a Banach space is ergodic.

(c) The antirepresentation T of \mathcal{A} on \mathcal{A}^* defined by $T_a(f) = fa$ for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, is ergodic.

Proof. (a) \Rightarrow (b). Let *T* be an antirepresentation of \mathcal{A} on a Banach space *X*. Since \mathcal{A} is left amenable, there is a net $(a_{\gamma})_{\gamma \in \Gamma}$ in $P_1(\mathcal{A})$ such that for each $a \in P_1(\mathcal{A})$, $|| aa_{\gamma} - a_{\gamma} || \rightarrow 0$ [4], Theorem 4.6. So, if we put $E_{\gamma} = T_{a_{\gamma}}$ for all $\gamma \in \Gamma$, then

$$\parallel E_{\gamma}(T_a - I) \parallel = \parallel T_{aa_{\gamma} - a_{\gamma}} \parallel \leq \parallel T \parallel \parallel aa_{\gamma} - a_{\gamma} \parallel \to 0$$

for all $a \in P_1(\mathcal{A})$; that is $(E_{\gamma})_{\gamma \in \Gamma}$ satisfies (\mathcal{E}_1) . The condition (\mathcal{E}_2) is also satisfied because $E_{\gamma}(x) = T_{a_{\gamma}}(x) \in C_x$ for all $x \in X$ and $\gamma \in \Gamma$. (b) \Rightarrow (c). Clear.

(c) \Rightarrow (a). Let *T* be as in (c). Then $T_a u = ua = u$ for all $a \in P_1(\mathcal{A})$. That is $u \in (\mathcal{A}^*)_{\kappa}$, and hence $u \notin (\mathcal{A}^*)_{\rho}$ by (c) of Theorem 2.1. Using the Hahn-Banach Theorem, we may find a nonzero element *F* of \mathcal{A}^{**} such that $\langle F, f \rangle = 0$ for all $f \in (\mathcal{A}^*)_{\rho}$. Then $F \in \text{TLIF}(\mathcal{A}^*)$, and hence there is a topological left invariant mean on \mathcal{A}^* by Lemma 2.2. Now invoke [4, Theorem 4.1] to conclude that \mathcal{A} is left amenable. \Box

Suppose that \mathcal{A} is two-sided amenable; i.e. there is an element $M \in P_1(\mathcal{A}^{**})$ with $a \odot M = M \odot a = M$ for all $a \in P_1(\mathcal{A})$. Then there is a net $(a_{\gamma})_{\gamma \in \Gamma}$ in $P_1(\mathcal{A})$ such that

$$|| aa_{\gamma} - a_{\gamma} || + || a_{\gamma}a - a_{\gamma} || \to 0 \text{ for all } a \in P_1(\mathcal{A}).$$

So, if we put $E_{\gamma} = T_{a_{\gamma}}$ for all $\gamma \in \Gamma$, then as in the above proof E_{γ} satisfies the conditions (\mathcal{E}_1) , (\mathcal{E}_2) , and

 (\mathcal{E}_3) $(T_a-I)E_{\gamma} \to 0$ in the strong operator topology for all $a \in P_1(\mathcal{A})$.

Recall from Example 1 of [4] that any commutative Lau algebras is left (and hence two-sided) amenable; in particular, the Fourier algebra A(G) and Fourier-Stieltjes algebra B(G) of a locally compact group Gare always two-sided amenable. Also the group algebra $L^1(G)$ (resp. the measure algebra M(G)) is two-sided amenable if and only if it is left amenable; see 4.1 and 4.2 of [11] or 4.19 of [12]. Therefore, twosided amenable Lau algebras \mathcal{A} form a large class of Lau algebras such that for any antirepresentation T of such algebras on a Banach space X, there is a net $(E_{\gamma})_{\gamma \in \Gamma}$ satisfying $(\mathcal{E}_1), (\mathcal{E}_2)$, and (\mathcal{E}_3) . Theorem 2.1 does not give the closedness of X_{σ} in X. The following result shows that X_{σ} is closed in X if in addition (\mathcal{E}_3) holds.

Proposition 2.4. Let T be an ergodic antirepresentation of \mathcal{A} on a Banach space X with $(E_{\gamma})_{\gamma \in \Gamma}$ satisfying (\mathcal{E}_1) , (\mathcal{E}_2) , and (\mathcal{E}_3) . Then X_{σ} is the closed subspace of X consisting of all $x \in X$ such that $(E_{\delta}(x))$ is weakly convergent in X for some subnet (E_{δ}) of (E_{γ}) .

Proof. Using (a) of Theorem 2.1, $(E_{\gamma}(x))$ is norm (and hence weakly) convergent in $X_{\kappa} \cap C_x \subseteq X$ for all $x \in X_{\sigma}$.

Now, let $x \in X$ and suppose that $(E_{\delta}(x))$ is weakly convergent to $y \in X$ for some subnet (E_{δ}) of (E_{γ}) . We must show that $x \in X_{\sigma}$. Then $y \in C_x$ because C_x is closed in X and $E_{\delta}(x) \in C_x$ for all δ . Also, by (\mathcal{E}_3) , for each $a \in P_1(\mathcal{A})$ we have

$$T_a(y) = \lim_{\delta} T_a(E_{\delta}(x)) = \lim_{\delta} \left[(T_a - I)E_{\delta}(x) + E_{\delta}(x) \right] = y$$

whence $y \in X_{\kappa} \cap C_x$. It follows that $x = y + (x - y) \in X_{\kappa} + X_{\rho} = X_{\sigma}$. \Box

As a consequent of the above proposition, we have the following result.

Corollary 2.5. Let T, X, and (E_{γ}) be as in the above proposition. Then $X_{\sigma} = X$ if C_x is weakly compact for all $x \in X$. In particular, $X_{\sigma} = X$ if X is reflexive.

In conclusion, Proposition 2.4 leads us to ask:

Question. Is there a Lau algebra with an ergodic antirepresentation on a Banach space X such that X_{σ} is not closed?

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