A NEW CLASS OF BANACH SEQUENCE SPACES

PARVIZ AZIMI

ABSTRACT. We modify the definition of the norm in the class of Banach spaces constructed by Hagler and this author, as presented in "Examples of hereditarily l_1 Banach spaces failing the Schur property" by mimicking the l_p norm rather than l_1 norm and obtain a new class of separable Banach spaces. We prove that if $1 , any member of this class (i) is hereditarily complementably, <math>l_p$ (ii) is a dual space, and (iii) the predual contains complemented subspace isomorphic to l_q where $\frac{1}{p} + \frac{1}{q} = 1$. Other properties of these spaces are investigated.

1. Introduction

In [2] we introduced a class of Banach sequence spaces which among the other interesting properties, each of the spaces is hereditarily l_1 and yet fails the Schur property. In this notes, we modify the definition of the norm in [2] by mimicking the l_p norm rather than l_1 norm and obtain a new class of separable Banach sequence spaces, the $X_{\alpha,p}$ spaces. We prove that, if 1 , any member of this $class contains <math>l_p$ hereditarily complementably, is a dual space, and it's predual contains complemented subspaces isomorphic to l_q where $\frac{1}{p} + \frac{1}{q} = 1$.

Our main result is the following.

MSC(2000): Primary 46B04; Secondary 46B20

Keywords: Banach spaces, Hereditarily complementably l_p , Complemented subspaces isomorphic to l_p .

Received: 5 January 2002, Revised: 21 November 2002

^{© 2002} Iranian Mathematical Society.

⁵⁷

Theorem 1.1. Let $X_{\alpha,p}$ denote a specific space of the class, we have the following:

(1) $X_{\alpha,p}$ is hereditarily complementably l_p .

(2) The sequence (e_i) is a normalized boundedly complete bases for $X_{\alpha,p}$. Thus, $X_{\alpha,p}$ is a dual space.

(3) The predual of $X_{\alpha,p}$ contains complemented subspaces isomorphic to l_q where $\frac{1}{p} + \frac{1}{q} = 1$.

 $X_{\alpha,p}$ spaces have some other properties similar to [1] and [2] Banach spaces, which we state some of them here.

(i) Each complemented non weakly sequentially complete subspace of $X_{\alpha,p}$ contains a complemented isomorph of $X_{\alpha,p}$.

(ii) $X_{\alpha,p}$ and $X_{\beta,p}$ are isomorphic if and only if they are equal as sets.

(iii) The sequence (x_n) with $x_n = e_{2n-1} - e_{2n}$ is weakly null sequence in $X_{\alpha,p}$ but not in norm.

Since $X_{\alpha,p}$ contains l_p hereditarily complementably thus, (iv) $X_{\alpha,p}$ spaces are not prime.

Since for $p > 1, X_{\alpha,p}$ does not contain l_1 and is not reflexive, by (theorem 1.c.12(a)/5)

(v) $X_{\alpha,p}$ is a Banach space without unconditional basis.

2. Definition and Notation.

Difinitions and notation are standard. But we list the most important of these here. The dual space of a Banach space X is denoted by X^* . Let Y be a subspace of X then we say that X contains Y hereditarily if every infinite dimensional subspace of X contain an isomorphic copy of Y. A subspace Y of X is complemented in X if there is a bounded projection $P: X \to Y$ such that P(X) = Y.

Let $l_p = \left\{ x = (x_1, x_2, ...) : || x || = \left[\sum_{i=1}^{\infty} |x_i|^p \right]^{\frac{1}{p}} \right\}$ where x_i are scalars and $\sum_{i=1}^{\infty} |x_i|^p < \infty$. An infinite-dimensional Banach space X is said to be *prime* if every infinite-dimensional complemented subspace of X is isomorphic to X.

Now we go through the definition of the $X_{\alpha,p}$ spaces. First, by a block we mean an interval (finite or infinite) of integers. For any block F, and $x = (t_1, t_2, ...)$ a finitely non-zero sequence of scalars, we let $\langle x, F \rangle = \sum_{j \in F} t_j$. A sequence of blocks $F_1, F_2, ...$ is admissible if

max $F_i < \min F_{i+1}$ for each *i*. Finally, let $1 = \alpha_1 \ge \alpha_2 \ge \alpha_3 \ge \dots$ be a sequence of real numbers with $\lim_{i\to\infty} \alpha_i = 0$ and $\sum_{i=1}^{\infty} \alpha_i = \infty$.

We now define a norm which uses the $\alpha_i^{i} s$ and admissible sequences of blocks in its definition. Let $x = (t_1, t_2, ...)$ be finitely non-zero sequence of reals. Define

$$|| x || = \max[\sum_{i=1}^{n} \alpha_i | \langle x, F_i \rangle|^p]^{\frac{1}{p}}$$

where the max is taken over all n, and admissible sequences F_1, F_2, \ldots . The Banach space $X_{\alpha,p}$ is the completion of the finitely non-zero sequences of scalars in this norm, and 1

Remark. If we consider sequences (α_i) which satisfy $\alpha_i = 1$ for all i and p = 2, then $X_{\alpha,p}$ is the James space [4], and if we require $\sum_{i=1}^{\infty} \alpha_i < \infty$ then the spaces $X_{\alpha,p}$ are all isomorphic to c_0 .

For $x \in X_{\alpha,p}$, put $s(x) = \max |\langle x, G \rangle|$ where the max is taken over all blocks G.

3. The results.

For the proof of theorem 1.1 we first collect several results which are crucial to the proof of theorem 1.1.

Lemma 3.1. There is a constant M such that every normalized block basis of the unit vectors bases (u_i) of $X_{\alpha,p}$ with $\lim_{i\to\infty} s(u_i) = 0$ has a subsequence (v_i) satisfying

$$\|\sum_{i=1}^{n} t_i v_i\|^p \le M \sum_{i=1}^{n} |t_i|^p$$

for all finite scalar sequences (t_i) .

Proof. We select a subsequence (v_i) of (u_i) so that $\lim_{i\to\infty} \langle u_i, N \rangle$ exists. Put $v_i = u_{2i-1} - u_{2i}$, then $||v_i|| \leq 2$ and $\lim_{i\to\infty} \langle v_i, N \rangle = 0$. By passing to a subsequence of (v_i) (not renaming) we may assume that

(A)
$$\sum_{i=1}^{n} |\langle v_i, N \rangle|^q \le 1$$
.

Azimi

By induction, we show that for any n, and admissible blocks $F_1, F_2, ..., F_m$, we have

(B)
$$\sum_{j=1}^{m} \alpha_j \mid < \sum_{i=1}^{n} t_i v_i, F_j > \mid^p$$
$$\leq 2K \sum_{i=1}^{n-1} \mid t_i \mid^p + K \mid t_n \mid^p$$

for $K = 3^{p-1}$. Once this is done, the lemma follows with M = 2K. We now assume (B) to be true for all $k \leq n-1$, and note that it holds for k = 1. We now let l be the largest integer for which

$$\operatorname{support}(v_{n-1}) \cap F_l \neq \phi$$

and suppose that for i = k, ..., n - 1

 $\operatorname{support}(v_i) \cap F_l \neq \phi$

yet

$$support(v_{k-1}) \cap F_l = \phi.$$

Thus $v_{k+1}, ..., v_{n-1}$ are entirely supported in F_l . Next

$$\sum_{j=1}^{m} \alpha_j \mid < \sum_{i=1}^{n} t_i v_i, F_j > \mid^p = \sum_{j=1}^{l-1} \alpha_j \mid < \sum_{i=1}^{k} t_i v_i, F_j > \mid^p$$
(C)
$$+\alpha_l \mid < \sum_{i=k}^{n} t_i v_i, F_l > \mid^p + \sum_{j=l+1}^{m} \alpha_j \mid < t_n v_n, F_j > \mid^p$$

$$= \sum_1 + \sum_2 + \sum_3.$$

We will use the induction hypothesis on \sum_1 , we will leave \sum_3 basically as is, and we now estimate the middle term in \sum_2 .

(D)

$$\begin{split} \sum_{2} &= \alpha_{l} \mid t_{k} < v_{k}, F_{l} > + \sum_{i=k+1}^{n-1} < t_{i}v_{i}, F_{l} > + t_{n} < v_{n}, F_{l} > |^{p} \\ &\leq \alpha_{l} 3^{p-1} [\mid t_{k} < v_{k}, F_{l} > |^{p} + \mid \sum_{i=k+1}^{n-1} < t_{i}v_{i}, F_{l} > |^{p} \\ &+ \mid t_{n} < v_{n}, F_{l} > |^{p}]. \end{split}$$

We estimate the middle term in (D) by

60

$$\begin{aligned} |\sum_{i=k+1}^{n-1} \langle t_i v_i, F_l \rangle|^p &= |\sum_{i=k+1}^{n-1} t_i \langle v_i, F_l \rangle|^p \\ &\leq \left(\sum_{i=k+1}^{n-1} |t_i|^p\right) \left(\sum_{i=k+1}^{n-1} |\langle v_i, F_l \rangle|^q\right)^{\frac{p}{q}} \\ &= \left(\sum_{i=k+1}^{n-1} |t_i|^p\right) \left(\sum_{i=k+1}^{n-1} |\langle v_i, N \rangle|^q\right)^{\frac{p}{q}} \\ &\leq \sum_{i=k+1}^{n-1} |t_i|^p \end{aligned}$$

by (A). Returning to (C) we obtain

$$\begin{split} \sum_{j=1}^{m} \alpha_j &|< \sum_{i=1}^{n} t_i v_i, F_j > |^p \le \left[2K \sum_{i=1}^{k-1} | t_i |^p + K | t_k |^p \right] \\ &+ \left[K | t_k < v_k, F_l > |^p + K \sum_{i=k+1}^{n-1} | t_i |^p + \alpha_l K | t_n < v_n, F_l > |^p \right] \\ &+ \sum_{j=l+1}^{m} \alpha_j | < t_n v_n, F_j > |^p \\ &\le 2K \sum_{i=1}^{n-1} | t_i |^p + K \sum_{j=l}^{m} \alpha_j | < t_n v_n, F_j > |^p \\ &\le 2K \sum_{i=1}^{n-1} | t_i |^p + K | t_n |^p \end{split}$$

as desired. \Box

We use the next lemma (lemma 4 of [2]) to find a lower estimate for a linear combination of vectors in $X_{\alpha,p}$.

Lemma 3.2. Let the sequence (α_i) be as above, let N be an integer and let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that if $0 \le b_i < \delta$ for all i, and $\sum_{i=1}^n \alpha_i b_i = 1$ then $\sum_{i=1}^n \alpha_{i+N} b_i \ge 1 - \varepsilon$.

Lemma 3.3. Let (u_i) be a sequence of norm one vectors in $X_{\alpha,p}$ and (G_i) an admissible sequence of blocks such that $\{j : u_i(j) \neq 0\} \subset G_i$. For each i, put $s_i = s(u_i)$. If $\lim s_i = 0$, then for a subsequence (v_k) of (u_k) and for a given sequence $t_1, t_2, ..., t_k$ of scalars we have

$$\|\sum_{i=1}^{k} t_i v_i \|^p \ge \frac{1}{2} \sum_{i=1}^{k} |t_i|^p.$$

Proof. An argument similar to the proof of lemma 5 in [2] shows that we may assume the following.

There exists subsequence (v_i) of (u_i) and sequence (n_i) of integers and $\delta_i > 0$ satisfying :

(i) $|| v_i || = 1$ for all *i*.

(ii) For integer $n_i (> n_{i-1})$ put $N_i = n_1 + n_2 + \ldots + n_{i-1}, i > 1$ and $N_1 = 0$. Then δ_i satisfies lemma 3.2 for $\varepsilon = \frac{1}{2}$ and $N = N_i$.

Azimi

(iii) For each block F and each i, $|\langle v_i, F \rangle|^p \leq \delta_i$.

(iv) For each *i*, there is a sequence of admissible blocks $F_{n_{i-1}+1}, F_{n_{i-1}+2}, \dots, F_{n_i}$ with,

- (a) $\max F_{n_i} < \min F_{n_i+1}$
- (b) $\sum_{j=1}^{n_i n_{i-1}} \alpha_j | \langle v_i, F_{n_{i-1}+j} \rangle|^p = ||v_i||^p = 1$
- (c) $\langle v_k, F_{n_{i-1}+j} \rangle = 0$ if $i \neq k$, and by lemma 3.2, we have
- (d) $\sum_{j=n_{i-1}+1}^{n_i} \alpha_j \mid < v_i, F_j > \mid^p > \frac{1}{2}.$

Since the sequence $F_1, F_2, ..., F_{n_1}, ..., F_{n_2}, ..., F_{n_k}, ...$ is admissible, it follows from (i)-(iv) above that for scalars $t_1, ..., t_k$ and admissible blocks $F_1, F_2, ..., F_{n_k}$,

$$\|\sum_{i=1}^{k} t_{i} v_{i} \|^{p} \ge \sum_{i=1}^{n_{k}} \alpha_{i} | < \sum_{j=1}^{k} t_{j} v_{j}, F_{i} > |^{p}$$

$$= \sum_{j=1}^{k} |t_j|^p \sum_{i=n_{j-1}+1}^{n_j} \alpha_i | \langle v_j, F_i \rangle|^p \ge \frac{1}{2} \sum_{j=1}^{k} |t_j|^p \square$$

Lemmas 3.1 and 3.3 imply the following lemma

Lemma 3.4. Let (u_i) be a sequence of norm one vectors in $X_{\alpha,p}$ and (G_i) an admissible sequence of blocks such that $\{j : u_i(j) \neq 0\} \subset G_i$. For each *i* put $s_i = s(u_i)$. If $\lim_{i\to\infty} s_i = 0$, then a subsequence (v_k) of (u_k) is equivalent to the usual basis of l_p . In fact there exist K > 0 such that for a given sequence $t_1, t_2, ..., t_n$ of scalars

 $\frac{1}{2}\sum_{i=1}^{n} |t_i|^p \leq ||\sum_{i=1}^{n} t_i v_i||^p \leq K \sum_{i=1}^{n} |t_i|^p.$ The following lemma is the key result.

Lemma 3.5. Let X be a Banach space, and (x_i, f_i) a biorthogonal sequence $(f_i(x_j) = \delta_{ij})$ in $X \times X^*$ such that (i) (x_i) is equivalent to the usual basis of l_p , $1 and (ii) <math>(f_i)$ is equivalent to the usual basis of l_q with $\frac{1}{p} + \frac{1}{q} = 1$. Then $[x_i]$ is complemented in X.

Proof. Define $P: X \to X$ by $Px = \sum_{i=1}^{\infty} f_i(x) x_i$. We show that P is a bounded projection onto $[x_i]$. Fix n and $x \in X$, since (x_i) is equivalent to usual basis of l_p , and (f_i) is equivalent to usual basis of l_q , there exist positive numbers λ , K and L such that

62

$$\lambda \left(\sum_{i=1}^{n} | f_{i}(x) |^{p} \right)^{\frac{1}{p}} \leq \| \sum_{i=1}^{n} f_{i}(x) x_{i} \| \leq K \left(\sum_{i=1}^{n} | f_{i}(x) |^{p} \right)^{\frac{1}{p}}$$

and

$$\|\sum_{i=1}^{n} t_i f_i \| \le L \left(\sum_{i=1}^{n} |t_i|^q \right)^{\frac{1}{q}}$$

where (t_i) is a sequence of reals. Let $\varepsilon_i = sgnf_i(x)$ then

$$(\sum_{i=1}^{n} |f_{i}(x)|^{p})^{\frac{1}{p}} = (\sum_{i=1}^{n} |f_{i}(x)|^{p-1} \varepsilon_{i}f_{i}(x))^{\frac{1}{p}} = [\sum_{i=1}^{n} (\varepsilon_{i} |f_{i}(x)|^{p-1} f_{i})(x)]^{\frac{1}{p}}$$

$$\leq [\|\sum_{i=1}^{n} \varepsilon_{i} |f_{i}(x)|^{p-1} f_{i} \|\|x\|]^{\frac{1}{p}} \leq L \left(\sum_{i=1}^{n} |f_{i}(x)|^{(p-1)q}\right)^{\frac{1}{pq}} |x\|^{\frac{1}{p}} .$$

Therefore

$$\left(\sum_{i=1}^{n} \mid f_{i}(x) \mid^{p}\right)^{\frac{1}{p}} \leq L\left(\sum_{i=1}^{n} \mid f_{i}(x) \mid^{p}\right)^{\frac{1}{pq}} \parallel x \parallel^{\frac{1}{p}}$$

which implies that

$$(\sum_{i=1}^{n} | f_i(x) |^p)^{\frac{1}{p}} \le L^p || x ||$$

and hence

$$\|Px\| = \|\sum_{i=1}^{n} f_i(x) x_i\| \leq KL^p \|x\| . \Box$$

Remark. This result is wrong if (e_i) is equivalent to usual basis of c_0 and (f_i) equivalent to usual basis of l_1 . To see this take $X = l_{\infty} = c_0^{**}$ and $l_1 \subset l_{\infty}^* = l_1^{**}$. Then we have the conditions of lemma 3.7 but we know that c_0 is not complemented in l_{∞} .

Let V be an infinite dimensional subspace of $X_{\alpha,p}$ then V contains a sequence of norm one vectors equivalant to the usual basis of l_p . In fact if (v_i) is a sequence in V with $\lim_{i\to\infty} s(v_i) = 0$, then (v_i) has a subsequence equivalent to the usual basis of l_p .

Let (φ_i) in $X^*_{\alpha,p}$ be defined by

Azimi

$$\varphi_i\left(x\right) = \sum_{j=1}^{n_i} \alpha_{j+N_i} \mid < v_j, F_j^i > \mid^{p-1} \varepsilon_j^i < x, F_j^i >$$

where v_i is normed by $F_1^i, ..., F_{n_i}^i, i = 1, 2, ..., N_i = n_1 + n_2 + ... + n_{i-1}, i > 1, N_1 = 0$ and $\varepsilon_j^i = sgn < v_i, F_j^i > .$ Let $g_i = \frac{\varphi_i}{\varphi_i(v_i)}$ clearly (g_i, v_i) is a biorthogonal sequence. Let Y be a subspace of $X_{\alpha,p}^*$ generated by (g_i) . Theorem 1.1(2) and well known result [3] show that $Y^* = X_{\alpha,p}$. Clearly, each $\varphi_i \in Y$.

Lemma 3.6. Let (φ_i) be as above, taking the sequence (s_i) of scalars, then we have

$$\left\|\sum_{i=1}^{k} s_{i} \frac{\varphi_{i}}{\varphi_{i}(v_{i})}\right\| \leq 2 \left(\sum_{i=1}^{k} |s_{i}|^{q}\right)^{\frac{1}{q}}.$$

Proof. Observe that

$$\varphi_i(v_i) = \sum_{j=1}^{n_i} \alpha_{j+N_i} \mid < v_i, F_j^i > \mid^p \ge \frac{1}{2} \sum_{j=1}^{n_i} \alpha_j \mid < v_i, F_j^i > \mid^p = \frac{1}{2}.$$

Since $\varphi_i(v_i) \leq \sum_{j=1}^{n_i} \alpha_j | \langle v_i, F_j^i \rangle|^p = ||v_i||^p = 1$. This implies that $||\varphi_i|| \leq 1$. Now we go through the calculation of the norm. By Holder inequality and the fact that q(p-1) = p, we have

$$\begin{split} |\sum_{i=1}^{k} s_{i} \frac{\varphi_{i}(x)}{\varphi_{i}(v_{i})}| &\leq 2\sum_{i=1}^{k} |s_{i}| |\varphi_{i}(x)| \\ &\leq 2\sum_{i=1}^{k} |s_{i}| \left(\sum_{j=1}^{n_{i}} \alpha_{j+N_{i}} |< v_{i}, F_{j}^{i} > |^{p-1}| < x, F_{j}^{i} > |\right) \\ &= 2\sum_{i=1}^{k} \sum_{j=1}^{n_{i}} |s_{i}| \alpha_{j+N_{i}} |< v_{i}, F_{j}^{i} > |^{p-1}| < x, F_{j}^{i} > | \\ &= 2\sum_{i=1}^{k} \left(\sum_{j=N_{i}+1}^{N_{i+1}} |s_{i}| \alpha_{j}^{\frac{1}{q}}| < v_{i}, F_{j-N_{i}}^{i} > |^{p-1} \alpha_{j}^{\frac{1}{p}}| < x, F_{j-N_{i}}^{i} > |\right) \\ &\leq 2 \left[\sum_{i=1}^{k} \left(\sum_{j=N_{i}+1}^{N_{i+1}} |s_{i}|^{q} \alpha_{j}| < v_{i}, F_{j-N_{i}}^{i} > |^{q(p-1)}\right)\right]^{\frac{1}{q}} \\ &\times \left[\sum_{i=1}^{k} \left(\sum_{j=N_{i}+1}^{N_{i+1}} \alpha_{j}| < x, F_{j-N_{i}}^{i} > |^{p}\right)\right]^{\frac{1}{p}} \end{split}$$

64

$$\leq 2\left(\sum_{i=1}^{k} \mid s_i \mid^q\right)^{\frac{1}{q}} \parallel x \parallel.$$

Therefore

$$\left\|\sum_{i=1}^{k} s_{i} \frac{\varphi_{i}}{\varphi_{i}(v_{i})}\right\| \leq 2 \left(\sum_{i=1}^{k} |s_{i}|^{q}\right)^{\frac{1}{q}}.$$

Lemma 3.7. By assumptions of lemma 3.6 we have

$$\left\|\sum_{i=1}^{n} s_{i} \frac{\varphi_{i}}{\varphi_{i}(v_{i})}\right\| \ge \delta \left(\sum_{i=1}^{n} |s_{i}|^{q}\right)^{\frac{1}{q}}$$

for some $\delta > 0$.

Proof. Suppose V is a subspace of $X_{\alpha,p}$, then V contains a sequence (v_i) of norm one vectors in V, such that (v_i) is equivalent to the usual basis of l_p . Also $\frac{\varphi_i}{\varphi_i(v_i)}$ is biorthogonal to (v_i) . Let

$$x = \sum_{i=1}^{n} \varepsilon_i \mid s_i \mid^{q-1} v_i, \, \varepsilon_i = sgn(s_i) \,.$$

Since (v_i) is equivalent to usual basis of l_p , there are real numbers λ and K such that

$$\lambda \left(\sum_{i=1}^{n} |s_i|^{p(q-1)} \right)^{\frac{1}{p}} \le \|x\| \le K \left(\sum_{i=1}^{n} |s_i|^{p(q-1)} \right)^{\frac{1}{p}}.$$

Since p(q-1) = q, we have

$$\lambda \left(\sum_{i=1}^{n} |s_i|^q \right)^{\frac{1}{p}} \le \|x\| \le \left(\sum_{i=1}^{n} |s_i|^q \right)^{\frac{1}{p}}.$$

This implies that

$$\begin{split} \| \sum_{i=1}^{n} s_{i} \frac{\varphi_{i}}{\varphi_{i}(v_{i})} \| \geq |\sum_{i=1}^{n} s_{i} \frac{1}{\varphi_{i}(v_{i})} \varphi_{i} \left(\frac{x}{\|x\|}\right)| &= \frac{1}{\|x\|} |\\ \sum_{i=1}^{n} s_{i} \frac{1}{\varphi_{i}(v_{i})} \varphi_{i} \left(\varepsilon_{i} \mid s_{i} \mid^{q-1} v_{i}\right)| \\ &= \frac{1}{\|x\|} \sum_{i=1}^{n} |s_{i}|^{q} \geq \frac{1}{K\left(\sum_{i=1}^{n} |s_{i}|^{q}\right)^{\frac{1}{p}}} \sum_{i=1}^{n} |s_{i}|^{q} = \frac{1}{K} \left(\sum_{i=1}^{n} |s_{i}|^{q}\right)^{\frac{1}{q}}. \end{split}$$

This implies that for $\delta = \frac{1}{K}$

$$\left\|\sum_{i=1}^{n} s_{i} \frac{\varphi_{i}}{\varphi_{i}(v_{i})}\right\| \geq \delta\left(\sum_{i=1}^{n} |s_{i}|^{q}\right)^{\frac{1}{q}}.$$

Lemmas 3.6 and 3.7 prove the following Lemma.

Lemma 3.8. Let V be an infinite dimensional subspace of $X_{\alpha,p}$ and (v_i) a sequence of norm one vectors in V and (v_i) is normed by $F_1^i, F_2^i, \dots F_{n_i}^i, i = 1, 2, \dots$ Let $N_i = n_1 + n_2 + \dots + n_{i-1}, i > 1, N_1 = 0.$ Define

$$\varphi_i\left(x\right) = \sum_{j=1}^{n_i} \alpha_{j+N_i} \mid < v_i, F_j^i > \mid^{p-1} \varepsilon_j^i < x, F_j^i >$$

where $\varepsilon_j^i = sgn < v_i, F_j^i > .$ Then the sequence $\left(\frac{\varphi_i}{\varphi_i(v_i)}\right)$ is equivalent to the usual basis of l_a .

We are now ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1.(1) : Lemma 3.4 and an argument similar to the proof of Theorem 1.(1) of [2], shows that $X_{\alpha,p}$ contains l_p hereditarily. To complete the proof, let V be the subspace of $X_{\alpha,p}$ generated by (v_i) . We show that V is complemented in $X_{\alpha,p}$. Let $\varphi_i \in X_{\alpha,p}^*$ be defined as above. Take $g_i = \frac{\varphi_i}{\varphi_{i(v_i)}}$, then (v_i, g_i) is biorthogonal sequence, with (g_i) equivalent to usual basis of l_q and (v_i) equivalent to usual basis of l_p . Lemma 3.5 implies that the projection $P: X_{\alpha,p} \to [v_i]$ defined by $P(x) = \sum_{i=1}^{\infty} g_i(x) v_i$ is a bounded projection onto $[v_i]$.

Proof of Theorem 1.1.(2). The proof of this part may be found in [2, Theorem1.(2)] (although the spaces are different)

Proof of 1.1.(3). Lemma 3.8. shows that the predual of $X_{\alpha,p}$ con-

tains l_q where $\frac{1}{p} + \frac{1}{q} = 1$. To see that $[g_i]$ with $g_i = \frac{\varphi_i}{\varphi_i(v_i)}$ is complemented in Y the predual of $X_{a,p}$, let (f_i) in $X^*_{\alpha,p}$ be biorthogonal to (e_i) in $X_{\alpha,p}$, and let Y be

the subspace of $X_{\alpha,p}^*$ generated by (f_i) . It is known that $Y^* = X_{\alpha,p}$. Clearly $g_i \in Y$ and (g_i) is equivalent to usual basis of l_q . For $\varphi \in [f_i]$ in $X_{\alpha,p}^*$ the projection $Q: X_{a,p}^* \to [g_i]$ defined by $Q(\varphi) = \sum_{i=1}^{\infty} \varphi(e_i) g_i$ is a bounded projection onto $[g_i]$. This completes the proof.

Proof of 1.1.(i) and 1.1.(*ii*) are similar to the proof of Theorems 3 and 4 of [1].

proof of 1.1.(iii). Clearly $||x_n|| = (1 + \alpha_2)^{\frac{1}{p}}$ for all n. It remains to show that (x_n) tends weakly to 0. This follows from the fact that , for every increasing sequence (n_k) of integers $\lim_{k\to\infty} \frac{||x_{n_1}+x_{n_2}+\ldots+x_{n_k}||}{k} = 0$. Indeed, since $\alpha_i \to 0$

$$\lim_{k \to \infty} \frac{\|x_{n_1} + x_{n_2} + \dots + x_{n_k}\|}{k} = \lim_{k \to \infty} \frac{\left(\sum_{i=1}^{2k} \alpha_i\right)^{\frac{1}{p}}}{k} = 0. \quad \Box$$

Remark. There is still a further question concerning the subspace structure of Y the predual of $X_{\alpha,p}$:

Is Y hereditarily complementably l_q ?

Acknowledgment

The author would like to thank J.Hagler for helpful comments and suggestions. He also provided Lemma 3.5. I want to thank department of mathematics of University of Denver for their hospitality during my sabbatical, when this research was done.

References

[1] A. Andrew, On the Azimi-Hagler Banach spaces, Rocky Mountain

J.Math.17(1987), 51-53.

- [2] P. Azimi, J. Hagler, Examples of hereditarily l₁ Banach spaces failin the Schur property, *Pacific J. Math.***122**(1986),287-297.
- [3] M. M. Day, Normed Linear Spaces, Springer Verlag, Berlin.
- [4] R.C. James. A none-reflexive Banach Spaces isometric with the second conjugate, Proc. Nat. Acd. Sci. U.S.A. 37(1951),174-177.

[5] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Vol. I sequence spaces, Springer Verlag, Berlin.

Department of Mathematics., University of Sistan and Baluchestan., Zahedan, Iran e-mail:azimi@hamoon.usb.ac.ir

68

Azimi