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ON THE MAXIMAL DEGREE OF THE K-STEP OBRECHKOFF'S METHOD

V.R. Ibrahimov

Department of Mathematics of Computation, Baku State University, Faculty of Mechanics and Mathematics, Baku, Z. Khalilov 23, Azerbaijan Republic
ibvag@yahoo.com

Abstract: In this paper the numerical solution of the Cauchy problem for the ordinary differential equations of arbitrary order is considered. In this regard, the k-step Obrechkoff's method is investigated. In a recent work, the maximal value degree for the k-step Obrechkoff's method was found and the natural conditions on its coeffiecients were defined. Taking this result into account, the convergence of the multi-step method depends on its stability. Here we define maximal value of the degree for stable and nonstable k-step Obrechkoff's method of explicit, implicit and forward-jumping types. These results are developments of some results due to G. Dahlquist, Iserles and Norest.

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Introduction.

As it is known, in solving many applied problems, usually there appears the necessity of finding the solution of the Cauchy problem for the ordinary differential equations. For this aim either one or multi-step methods or their combinations are used. One of the basic questions for their using is in determination of their accuracy. This question is answered in the work of N.S.Bakhvalov, for the explicit stable k-step method with the constant coefficients for $k \leq 10$ (see [1]) and in Dahlquist's work for the implicit stable k-step method with the constant coefficients and for the stable explicit method, when k is arbitrary. As k-step method is applied to the numerical solution of the first order ODE, but for the numerical solution of ODE of second order usually it is used k-step method with the second derivative. The maximal accuracy of the stable k-step method with the second derivative is determined in [3]. This result was obtained in [4] by different ways. Note that the k-step methode with the second derivative, as the numerical method for the solution of the ODE is investigated by many authors (see, for example [5], [6]).

Works, devoted to the numerical solution of ODE of orders more than 2 are considerably few. So the k-step Obrechkoff's method which can be used, as the numerical method for solving any order ODE is investigated here.

The k-step Obrechkoff's method with the constant coefficients may be written as:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{j=1}^{r} h^j \sum_{i=0}^{k} \beta_i^{(j)} y_{n+i}^{(j)}. \tag{1}$$

This method for $r = 2$ was investigated in [7] and maximal value of the degree for the A-stable methods is found there. It is evident that the method (1) can be applied for determination of numerical solution of

the problem:

$$\begin{cases} y^{(j)} = f(x, y, y', \ldots, y^{(j-1)}), & (j = 1, 2, \ldots, r), \\ y(x_0) = y_0, y^{(v)}(x_0) = y_0^{(v)} & (v = 1, \ldots, j-1). \end{cases} \tag{2}$$

It is easy to show, that if $j > 1$, then for determination of the numerical solution of the problem (2), any method of the type (1) can be used and in this case it is necessary to define the solution of the system of the difference equations (hence using some methods for the calculation $y'(x), y''(x), \ldots, y^{(j-1)}(x)$ on the point $x_m (m > 0)$). Convergence and effectiveness of such methods are investigated in [8].

In order to determine the maximal accuracy of the stable method, which is obtained from (1) all over again, one can define the maximal accuracy of the method (1), regardless of its stability.

1. The maximal value of the degree of the k-step Obrechkoff's method.

Usually the concept of the accuracy of a multistep method is concerned with the concept of its order.

Definition 1. The method (1) is said to have the degree p, if for any smooth function $y(x)$,

$$\sum_{i=0}^{k} \left[\alpha_i y(x + ih) - \sum_{j=1}^{r} h^j \beta_i^{(j)} y^{(j)}(x + hi) \right] = O(h^{p+1}), \quad h \to 0. \tag{1.1}$$

It is not difficult to define, that the maximal order of the accuracy for the method (1), coincides with the maximal value of its degree p (see, for exam. [2], [3]). Therefore we shall make busy ourselves with the determination of the maximal value of the degree p, both for stability and for nonstability method, which is received from (1). Consider the next lemma.

Lemma. *Let $y(x)$ be a sufficiently smooth function. Then for implementing relation (1.1), the necessary and sufficient condition is the following:*

$$\begin{cases} \sum_{i=0}^{k} \alpha_i = 0 \ ; \ \sum_{i=0}^{k} \frac{i^v}{v!}\alpha_i = \sum_{j=1}^{v} \sum_{i=0}^{k} \frac{i^{j-1}}{(j-1)!}\beta_i^{(v+1-j)} \ , \ (v = 1, \ldots, r), \\ \sum_{i=0}^{k} \frac{i^{r+l}}{(r+l)!}\alpha_i = \sum_{j=1}^{r} \sum_{i=0}^{k} \frac{i^{r+l-j}}{(r+l-j)!}\beta_i^{(j)} \ , \ (l = 1, \ldots, p-r). \end{cases} \tag{1.2}$$

For the proof of this lemma, it is sufficient to use the following expansions in (1.1)

$$y(x+ih) = y(x) + \sum_{v=1}^{p} \frac{(ih)^v}{v!} y^{(v)}(x) + O(h^{p+1}),$$

$$y^{(j)}(x+ih) = y^{(j)}(x) + \sum_{v=1}^{p-j} \frac{(ih)^v}{v!} y^{(v+j)}(x) + O(h^{p-j+1}), \ \ (j = 1, 2, \ldots, p)$$

and linear independence of the system $1, h, h^2, \ldots, h^p$.

The number of equations in (1.2) is equal to $p+1$, but the number of unknowns is equal to $(r+1)(k+1)$. In order for the system (1.2) to have the non-trivial solution, it must be $p+1 < rk+r+k+1$. Hence, $p \leq r(k+1)+k-1$. From here it follows that $p_{\max} = r(k+1)+k-1$. But the methods with the maximal orders usually are nonstable.

Definition 2. The formula (1) is called stable, if the modulus of roots of the polynomial

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i$$

does not exceed 1, and that the roots of modulus 1 (one) are simple. The method is called stable, if the corresponding formula is stable.

Prior to defining the maximal value of the degree for the stable method, which is received from (1), we shall consider the natural conditions, that we put on coefficients of the formula (1). Suppose that the coefficients of the formula (1) satisfy the following assumptions everywhere.

A. The coefficients $\alpha_i, \beta_i^{(j)}$ $(i = 0, \ldots, k; j = 1, \ldots, r)$ are real and $\alpha_k \neq 0$.

B. The characteristic polynomials $\rho(\lambda)$ and

$$\nu_j(\lambda) \equiv \sum_{i=0}^{k} \beta_i^{(j)} \lambda^i, \quad (j = 1, \ldots, r)$$

have no common factor.

C. The degree of formula (1) satisfies the condition $p \geq r$ and $\nu_r(1) \neq 0$, if $\nu_1(1) = \cdots = \nu_{r-1}(1) = 0$, otherwise $\nu_1(1) \neq 0$ and $p \geq r$.

The necessity of the condition A is evident. Consider, the proof of the necessity of the assumptions B.

Assume that the polynomials $\nu_j(\lambda)$ and $\rho(\lambda)$ have a common factor, degree of which is not less than 1 (one). Then we can write

$$\psi(E)\,(\rho^*(E)y_n - \sum_{j=1}^{r} h^j \nu_j^*(E) y_n^{(j)}) = 0, \tag{1.3}$$

where $\rho(\lambda) \equiv \psi(\lambda)\rho^*(\lambda)$; $\nu_j(\lambda) \equiv \psi(\lambda)\nu_j^*(\lambda)$, $(j = 1, \ldots, r)$, but E is the operator defined by

$$Ey_n = y_{n+1} \quad \text{or} \quad Ey(x) = y(x+h).$$

From (1.3) we have

$$\rho^*(E)y_n = \sum_{j=1}^{r} h^j \nu_j^*(E) y_n^{(j)}. \tag{1.4}$$

It follows from here, that the formula (1) as a difference equation with the order k is equivalent to the difference (1.4) with the order k_1, where $k_1 < k$, i.e. to the difference equation with the lower order. Consequently, for the determination of unique solution of the difference equation (1), it is sufficient to assign the initial values on the first k_1

points. But, as it is known, in this case the solution of difference equation of order $k > k_1$, will be nonunique. The contradiction, which has been obtained, demonstrates the necessity of the condition B.

Now let us prove the necessity of the condition C. Suppose, that the method which is determined by the formula (1) is convergent. Then we can write

$$|y_{n+i} - y(x)| \to 0 \text{ when } h \to 0 \ (x = x_0 + nh), \tag{1.5}$$

here $y(x)$ is exact and y_n is approximate values of the solution of problem (2), calculated by the method, which is determined from the formula (1).

If we substitute (1.5) in (1), then we will have

$$|y(x)| \cdot |\sum_{i=0}^{k} \alpha_i| \leq \varepsilon \sum_{i=0}^{k} |\alpha_i| + O(h). \tag{1.6}$$

Taking this into account $y(x) \not\equiv 0$ and going over to the limit in (1.6) when $h \to 0$, we obtain: $\rho(1) = 0$. From the $\rho(1) = 0$ we can write $\rho(\lambda) = (\lambda - 1)\rho'(\lambda)$. Then from (1) we receive

$$\rho'(E)(y_{i+1} - y_i) - h\nu_1(E)y_i' = O(h^2). \tag{1.7}$$

Summarizing (1.7) over i from 0 to n, we have

$$\rho'(E)(y_{n+1} - y_0) = \nu_1(E)\sum_{i=0}^{n} hy_i' + O(h). \tag{1.8}$$

Put $F_n = \sum\limits_{i=0}^{n} hy_i'$ and consider $y_{n+i} \to y(x)$, $y_i \to y(x_0)$.

$$F_{n+i} \to \int_{x_0}^{x} y'(s)ds \quad (i = 0, 1, \ldots, k).$$

Hence

$$\rho'(1)(y(x) - y_0) = \nu_1(1)\int_{x_0}^{x} y'(s)ds. \tag{1.9}$$

Consequently $\rho'(1) = \nu_1(1)$.

For $\psi(\lambda) \equiv 1$ taking into account in the correlation (1.3)

$$\rho(\lambda) = \rho(1) + \rho'(1)(\lambda - 1) + \frac{1}{2}\rho''(1)(\lambda - 1)^2 + O\left((\lambda - 1)^3\right),$$
$$\nu_1(\lambda) = \nu_1(1) + \nu_1'(1)(\lambda - 1) + O\left((\lambda - 1)^2\right),$$
$$\nu_2(\lambda) = \nu_2(1) + O(\lambda - 1).$$

and $\rho(1) = 0$, $\rho'(1) = \nu_1(1)$, and also $\dfrac{y_{i+1} - y_i}{h} = y_i' + \dfrac{hy_i''}{2} + O(h^2)$ we can write

$$\frac{1}{2}\rho''(1)\left(\frac{y_{i+2} - y_{i+1}}{h} - \frac{y_{i+1} - y_i}{h}\right) - \nu_1'(1)(y_{i+1}' - y_i') - h\left(\nu_2(1) - \frac{h}{2}\rho'(1)\right)y_i'' = O(h^2).$$

Hence

$$\rho''(1)(y_{i+1}' - y_i') - 2\nu_1'(1)(y_{i+1}' - y_i') - h(2\nu_2(1) - \nu_1(1))y_i'' + \frac{h}{2}\rho''(1)(y_{i+1}'' - y_i'') = O(h^2).$$

Summing up the last correlation over i from 0 to n, we obtain

$$(\rho''(1) - 2\nu_1'(1))(y_{i+1}' - y_0') = (2\nu_2(1) - \nu_1(1))\sum_{i=0}^{n} hy_i'' - \frac{h}{2}\rho''(1)(y_{n+1}'' - y_0'') + O(h). \tag{1.10}$$

Going over to the limit in (1.10), when $h \to 0$, we have

$$(\rho''(1) - 2\nu_1'(1))(y'(x) - y_0') = (2\nu_2(1) - \rho'(1))\int_{x_0}^{x} y''(s)ds. \tag{1.11}$$

Hence

$$\rho''(1) = 2\nu_1'(1) + 2\nu_2(1) - \nu_1(1). \tag{1.12}$$

Taking into account that $\rho'(1) = \nu_1(1)$, the correlation (1.12) can be written in the next form:

$$\rho''(1) + \rho'(1) = 2\nu_1'(1) + 2\nu_2(1). \tag{1.13}$$

Using the expansion

$$\rho(\lambda) = \rho(1) + \rho'(1)(\lambda - 1) + \frac{1}{2}\rho''(1)(\lambda - 1)^2 + \frac{1}{6}\rho'''(1)(\lambda - 1)^3 + O\left((\lambda - 1)^4\right),$$
$$\nu_1(\lambda) = \nu_1(1) + \nu_1'(1)(\lambda - 1) + \frac{1}{2}\nu_1''(1)(\lambda - 1)^2 + O\left((\lambda - 1)^3\right),$$
$$\nu_2(\lambda) = \nu_2(1) + \nu_2'(1)(\lambda - 1) + O\left((\lambda - 1)^2\right),$$
$$\nu_3(\lambda) = \nu_3(1) + O(\lambda - 1),$$
$$\frac{y_{i+1}^{(j)} - y_i^{(j)}}{h} = y_i^{(j+1)} + \frac{h}{2}y_i^{(j+2)} + \frac{h^2}{6}y_i^{(j+3)} + O(h^3) \ \ (j = 0, 1)$$

in the next expression

$$\rho(E)y_i - h\nu_1(E)y_i' - h^2\nu_2(E)y_i'' - h^3\nu_3(E)y_i''' = O(h^4) \qquad (1.14)$$

we receive

$$\begin{aligned}
&\rho'(1)(y_{i+1} - y_i) + \frac{h}{2}\rho''(1)\left(\frac{y_{i+2} - y_{i+1}}{h} - \frac{y_{i+1} - y_i}{h}\right) \\
&+ \frac{h^2}{6}\rho'''(1)\left(\frac{y_{i+3} - y_{i+2}}{h} - 2\frac{y_{i+2} - y_{i+1}}{h} + \frac{y_{i+1} - y_i}{h}\right) \\
=&h\nu_1(1)y_i' + h\nu_1'(1)(y_{i+1}' - y_i') + \frac{h^2}{2}\nu_1''(1)\left(\frac{y_{i+2} - y_{i+1}}{h} - \frac{y_{i+1} - y_i}{h}\right) \\
&+ h^2\nu_2(1)y_i'' + h^2\nu_2'(1)(y_{i+1}'' - y_i'') + h^3\nu_3(1)y_i''' + O(h^4).
\end{aligned}$$

If we use $\rho'(1) = \nu_1(1)$ and correlation (1.13), then we can write

$$\rho'''(1) + 3\rho''(1) + \rho'(1) = 3\nu_1''(1) + 3\nu_1'(1) + 6\nu_2'(1) + 6\nu_3(1). \qquad (1.15)$$

Note, that by realizations of the correlations (1.13) and (1.15) we can obtain $p = 2$ and $p = 3$, respectively.

It is obvious, that if we continue this process, before using in (1.14) the expansion of $y_i^{(r)}$, then we will receive correlation similar to (1.13) and (1.15), from which the relation $p = r$ will follow.

Note, that for the receiving of above mentioned correlation we can

use

$$\rho(\lambda)(\ln\lambda)^{-1} = \frac{\rho(\lambda)}{\lambda-1}\sum_{i=0}^{\infty} C_i(\lambda-1)^i,$$

$$\nu_1(\lambda) + \sum_{j=1}^{r-1}\nu_j(\lambda)(\ln\lambda)^j = \frac{\rho(\lambda)}{\lambda-1}\sum_{j=0}^{p-1} C_j(\lambda-1)^j + O\left((\lambda-1)^p\right), \quad \lambda\to 1,$$

here

$$C_m = \sum_{i=1}^{m}(-1)^{i-1}\frac{C_{m-i}}{i+1}.$$

If the last condition is satisfied then the formula (1) will have the degree equal to p.

The second part of the assumption C, is concerned with the fact that if $\nu_1(\lambda)\not\equiv 0$ and $\nu_1(1)=0$, then as it is obvious from the (1.9), method will be divergent, which contradicts to the assumption.

But if we consider the case $\nu_1(\lambda)\equiv 0, \nu_2(\lambda)\not\equiv 0$, and $\nu_2(1)=0$ then as it is obvious from (1.11) convergence of the considered method will be absent, since values of the function $y'(x)$ are involved into the method. Other cases may be explained by analogy.

Note, that when the second part of the assumptions C is not realized, then the method will be nonstable, what proves validity of the above given reasons.

Frequently there arises the necessity to find relation between k and r, i.e. between order of the k-step method (1) and order of the derivatives of the function $y(x)$, used in (1). For the determination of the relation between k and r, the formula can be written in the next form:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = \sum_{j=1}^{r}\delta_j h^j \sum_{i=0}^{k}\beta_i^{(j)} y_{n+i}^{(j)}, \tag{1.16}$$

here $\delta_j (j=1,\ldots,r)$ takes values 0 (zero) or 1 (one).

It is clear that if $\delta_1=0$, then the method, which is determined by the formula (1.16) cannot be stable. Therefore the notion of l-stability introduced by G.Dahlquist will be used (see [3 p.19]), for $\delta_1=\delta_2=\cdots=\delta_{l-1}=0$ and $\delta_1=1$.

Definition 3. Formula (1.16) is said to be l-stable, if the roots of polynomials $\rho(\lambda)$ are located within or on the unit circle and there is not multiple root on the unit circle, except $\lambda = 1$ multiplicity to l.

The method is called l-stable, if the corresponding formula is l-stable. Relation between k and r may be written in the next form:

$$\sum_{j=1}^{r} \delta_j k \geq r \quad \text{or} \quad k \geq \frac{r}{\sum_{j=1}^{r} \delta_j}.$$

If we consider the case $\delta_1 = 0$ and $\delta_2 \neq 0$ then we receive well-known method of Shtermer. In this case $k \geq 2$. Now, we can consider the maximal value of degree for the stable methods, received from the correlation (1).

2. The maximal value of degree for the stable k-step Obrechkoff's method.

For the investigation of the maximal value of degree for the stable k-step Obrechkoff's method, consider in general form, i.e. not taking into account property of explicitly of the considered method, which imposes some limitation on coefficients $\beta_k^{(l)}$ $(l = 1, \ldots, r)$. In general, property of explicitly for formula (1) depends on its application. In particular, if the formula (1) is applied to numerical solution of problem (2), then for $j = r$ formula will be explicit by $\beta_k^{(r)} = 0$ but for $j = 1$ formula will be explicit by $\beta_k^{(l)} = 0$, $(l = 1, \ldots, r)$.

Suppose, that $|\beta_k^{(1)}| + |\beta_k^{(2)}| + \cdots + |\beta_k^{(r)}| \neq 0$ and we shall now prove a theorem, by which relation between p, k, and r can be determined.

Theorem 1. *Suppose, that the formula (1) has the degree p, stable $\alpha_k \neq 0$. Then*

$$p \leq \begin{cases} (k+1)r + 1 & \textit{by even } k \textit{ and odd } r, \\ (k+1)r & \textit{by odd } k \textit{ and even } r. \end{cases}$$

There exists stable formula with the degree $p = (k+1)r + 1$ in the case, when k is even and r is odd, but with the degree $p = (k+1)r$ in the other cases, for arbitrary k.

Proof. Taking into account condition of theorem 1, we can write

$$\rho(E)y_n - \sum_{j=1}^{r} h^j \nu_j(E) y_n^{(j)} \sim C h^{p+1} y_n^{(p+1)} \quad (h \to 0). \tag{2.1}$$

Consider the special case and we put $y(x) = \exp(x)$ (see [2]). Denote by the $\tau = \exp(h)$. Then correlation (2.1) may be written in the next form:

$$\rho(\tau) - \sum_{j=1}^{r} \nu_j(\tau)(\ln \tau)^j \sim C(\tau - 1)^{p+1} \quad (\tau \to 1). \tag{2.2}$$

Replacing by

$$\tau = \frac{(z+1)}{(z-1)} \quad , \quad z = \frac{(\tau+1)}{(\tau-1)}.$$

To use the next notation

$$R(z) = \left(\frac{1}{2}(z-1)\right)^k \rho(\tau) \equiv \sum_{i=0}^{k} a_i z^i,$$

$$S_l(z) = \left(\frac{1}{2}(z-1)\right)^k \nu_1(\tau) \equiv \sum_{i=0}^{k} b_i^{(l)}, z^i \quad (l = 1, \ldots, r),$$

in (2.2) we have, that

$$R(z) - \sum_{j=1}^{r} S_j(z) \left(\ln \frac{z+1}{z-1}\right)^j \sim C \left(\frac{2}{z}\right)^{p-k+1} \ , \ z \to \infty.$$

From here we can write

$$R(z) \left(\ln \frac{z+1}{z-1}\right)^{-1} - S_1(z) - \sum_{j=2}^{r} S_j(z) \left(\ln \frac{z+1}{z-1}\right)^{j-1} \sim C \left(\frac{2}{z}\right)^{p-k} \ , \ z \to \infty. \tag{2.3}$$

Considering the following equalities

$$\left(\ln \frac{z+1}{z-1}\right)^{-1} = \frac{z}{2} - \sum_{i=0}^{\infty} \mu_{2i+1} z^{-(2i+1)} \quad (\mu_{2i+1} > 0), \quad \ln \frac{z+1}{z-1} = 2 \sum_{i=0}^{\infty} \frac{z^{-(2i+1)}}{2i+1},$$

in (2.3), then we have

$$R(z)\left(\frac{z}{2}-\sum_{i=0}^{\infty}\mu_{2i+1}z^{-(2i+1)}\right)-S_1(z)-\sum_{v=1}^{r-1}2^vA_vS_{v+1}(z)\ \sim\ C\left(\frac{2}{z}\right)^{p-k}\ ,\ z\to\infty, \tag{2.4}$$

where

$$A_v=\begin{cases}\sum_{s=l-1}^{\infty}C_{2s+1}^{(l)}z^{-(2s+1)} & \text{for}\quad v=2l-1,\\ \sum_{s=l}^{\infty}C_{2s}^{(l)}z^{-2s} & \text{for}\quad v=2l\ (C_m^{(l)}>0,\ m>0).\end{cases}$$

Let the coefficients of the formula (1) satisfy the condition A, B and C, then we can write $a_k=0$, as $\rho(1)=0$.

If the condition of stability from the polynomial $\rho(\lambda)$ carries over to the polynomial $R(\lambda)$, then we have:

1. R(z) has not roots with the positive real parts.

2. R(z) does not have the multiple roots on the imaginary axis The coefficient $a_{k-1}\neq 0$, as $\rho'(1)\neq 0$.

It is clear, that the left-hand side of the relation (2.4) may be written in the next form:

$$R(z)\left(\frac{z}{2}-\sum_{i=0}^{\infty}\mu_{2i+1}z^{-(2i+1)}\right)-S_1(z)-\sum_{v=1}^{r-1}2^vA_vS_{v+1}(z)=\sum_{i=1}^{\infty}C_iz^{-i}. \tag{*}$$

It is easy to determine, that to prove the theorem 1, it will be nec-

essary to investigate consistency of the next system:

$$
\begin{aligned}
&b_k^{(1)} = \frac{1}{2}a_{k-1},\\
&b_{k-1}^{(1)} + 2C_1^{(1)}b_k^{(2)} = \frac{1}{2}a_{k-2},\\
&b_{k-2}^{(1)} + 2C_1^{(1)}b_{k-1}^{(2)} + 2^2C_2^{(1)}b_k^{(3)} = \frac{1}{2}a_{k-3} - \mu_1 a_{k-1},\\
&b_{k-3}^{(1)} + 2C_1^{(1)}b_{k-2}^{(2)} + 2C_3^{(1)}b_k^{(2)} + 2^2C_2^{(1)}b_{k-1}^{(3)} + 2^3C_3^{(2)}b_k^{(4)} = \frac{1}{2}a_{k-4} - \mu_1 a_{k-2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&b_0^{(1)} + 2\cdot\sum_{v=0}^{[\frac{(k+1)}{2}]-1} C_{2v+1}^{(1)}b_{2v+1}^{(2)} + 2^2\cdot\sum_{v=0}^{[\frac{k}{2}]-1} C_{2v+2}^{(1)}b_{2v+2}^{(3)}\\
&+ 2^3\cdot\sum_{v=1}^{[\frac{(k+1)}{2}]-1} C_{2v+3}^{(2)}b_{2v+3}^{(4)} + \cdots + 2^{r-1}\cdot\sum_{v=[\frac{(r-1)}{2}]}^{\frac{\xi_{k,r}}{2}} C_{2v}^{([\frac{r}{2}])}b_{2v}^{(r)} = -\sum_{v=0}^{[\frac{(k+1)}{2}]-1} a_{2v+1}\mu_{2v+1}
\end{aligned}
\tag{2.5}
$$

with the following system $C_1 = C_2 = \cdots = C_{(r-1)k+r+1} = 0$.

The system (2.5) is received from the $(*)$ by the comparison coefficients of the linearly independent system z^j $(j = 0, \ldots, k)$.

Note, that the system (2.5) may be consistent, since the number of the equations and unknowns are identical. (It is not difficult to prove, that the system (2.5) is consistent). Therefore we will investigate the system $C_1 = C_2 = \cdots = C_{(r-1)k+r+1} = 0$, which can be written in the

next form:

$$2\sum_{v=0}^{[\frac{k}{2}]} C^{(1)}_{2v+1} b^{(2)}_{2v} + 2^2 \sum_{v=0}^{[\frac{(k+1)}{2}]-1} C^{(1)}_{2v+2} b^{(3)}_{2v+1} + 2^3 \sum_{v=1}^{[\frac{k}{2}]} C^{(2)}_{2v+1} b^{(4)}_{2v} + \dots$$

$$\dots + 2^{r-1} \sum_{v=\frac{(r-2)}{2}}^{\frac{\xi_{k,r}}{2}} C^{([\frac{k}{2}])}_{2v+1} b^{(r)}_{2v} = - \sum_{v=0}^{[\frac{k}{2}]-\xi^{(3)}_k} a_{2v}\mu_{2v+1},$$

$$2\sum_{v=0}^{[\frac{k+1}{2}]-1} C^{(1)}_{2v+3} b^{(2)}_{2v+1} + 2^2 \sum_{v=0}^{[\frac{k}{2}]} C^{(1)}_{2v+2} b^{(3)}_{2v} + 2^3 \sum_{v=0}^{[\frac{(k+1)}{2}]-1} C^{(2)}_{2v+3} b^{(4)}_{2v+1} + \dots$$

$$\dots + 2^{r-1} \sum_{v=\frac{(r-3)}{2}}^{\frac{\xi_{k,r}}{2}} C^{([\frac{r}{2}])}_{2v+2} b^{(r)}_{2v} = - \sum_{v=0}^{[\frac{(k+1)}{2}]-1} a_{2v+1}\mu_{2v+3},$$

. . . . . .

$$2\sum_{v=0}^{\frac{(\xi_{k,l}-\xi^{(3)}_l)}{2}} C^{(1)}_{2v+l+\xi^{(3)}_l} b^{(2)}_{2v+\xi^{(3)}_l} + 2^2 \sum_{v=0}^{\frac{(\xi_{k,l-1}+\xi^{(3)}_l)}{2}} C^{(1)}_{2v+l+1-\xi^{(3)}_l} b^{(3)}_{2v+1-\xi^{(3)}_l} + \dots$$

$$\dots + 2^{r-1} \sum_{v=0}^{\frac{(\xi^{(r)}_{k,l}-\xi^{(4)}_l)}{2}} C^{([\frac{r}{2}])}_{2v+l+\xi^{(4)}_l} b^{(r)}_{2v+\xi^{(4)}_l} = - \sum_{v=0}^{[\frac{(k+\xi^{(3)}_l)}{2}]-\xi^{(3)}_l-\xi^{(3)}_k} a_{2v+\xi^{(3)}_l}\mu_{2v+l+\xi^{(3)}_l},$$

$$2\cdot \sum_{v=0}^{\frac{(\xi^{(1)}_{k,l+1}-\xi^{(3)}_{l+1})}{2}} C^{(1)}_{2v+l+1+\xi^{(3)}_{l+1}} b^{(2)}_{2v+\xi^{(3)}_{l+1}} + 2^2\cdot \sum_{v=0}^{\frac{(\xi^{(1)}_{k,l+1}+1-\xi^{(3)}_{l+1})}{2}} C^{(1)}_{2v+l+2-\xi^{(3)}_{l+1}} b^{(3)}_{2v+1-\xi^{(3)}_{l+1}} + \dots$$

$$\dots + 2^{r-1}\cdot \sum_{v=0}^{\frac{(\xi^{(r)}_{k,l+1}+1-\xi^{(4)}_{l+1})}{2}} C^{([\frac{r}{2}])}_{2v+l+2-\xi^{(3)}_{l+1}} b^{(r)}_{2v+l-\xi^{(4)}_{l+1}} = - \sum_{v=0}^{[\frac{(k+\xi^{(3)}_{l+1})}{2}]-\xi^{(3)}_{l+1}-\xi^{(3)}_k} a_{2v+\xi^{(3)}_{l+1}}\mu_{2v+l+1+\xi^{(3)}_{l+1}}.$$

The system which is received for $k = 2i$ may be divided upon two

subsystems. The first of them can be written in the next form:

$$
\begin{gathered}
2\sum_{v=0}^{\frac{k}{2}} C^{(1)}_{2v+1} b^{(2)}_{2v} + 2^2 \cdot \sum_{v=0}^{\frac{(k-2)}{2}} C^{(1)}_{2v+2} b^{(3)}_{2v+1} + \dots \\
\dots + 2^{r-1} \sum_{v=\frac{(r-2)}{2}}^{\frac{\xi_{k,r}}{2}} C^{([\frac{r}{2}])}_{2v+1} b^{(r)}_{2v} = -\sum_{v=0}^{\frac{(k-2)}{2}} a_{2v}\mu_{2v+1}, \qquad (2.6)
\end{gathered}
$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$
\begin{gathered}
2\sum_{v=0}^{\frac{k}{2}} C^{(1)}_{2v+l+\xi_r^{(3)}} b^{(2)}_{2v} + 2^2 \cdot \sum_{v=0}^{\frac{(k-2)}{2}} C^{(1)}_{2v+l+1+\xi_r^{(3)}} b^{(3)}_{2v+1} + \dots \\
\dots + 2^{r-1} \sum_{v=\frac{(r-2)}{2}}^{\frac{\xi_{k,r}}{2}} C^{([\frac{r}{2}])}_{2v+l+\xi_r^{(4)}} b^{(r)}_{2v} = -\sum_{v=0}^{\frac{(k-2)}{2}} a_{2v}\mu_{2v+l+2+\xi_r^{(3)}}.
\end{gathered}
$$

The second subsystem can be written in the next form:

$$
\begin{gathered}
2\sum_{v=0}^{\frac{(k-2)}{2}} C^{(1)}_{2v+3} b^{(2)}_{2v+1} + 2^2 \cdot \sum_{v=0}^{\frac{k}{2}} C^{(1)}_{2v+2} b^{(3)}_{2v} + \dots \\
\dots + 2^{r-1} \sum_{v=\frac{(r-3)}{2}}^{\frac{\xi_{k,r}}{2}} C^{([\frac{r}{2}])}_{2v+2} b^{(r)}_{2v} = -\sum_{v=0}^{\frac{(k-2)}{2}} a_{2v+1}\mu_{2v+3}, \qquad (2.7)
\end{gathered}
$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$
\begin{gathered}
2\sum_{v=0}^{\frac{(k-2)}{2}} C^{(1)}_{2v+l+2-\xi_r^{(3)}} b^{(2)}_{2v+1} + 2^2 \sum_{v=0}^{\frac{k}{2}} C^{(1)}_{2v+l+1-\xi_r^{(3)}} b^{(3)}_{2v} + \dots \\
\dots + 2^{r-1} \sum_{v=\frac{(r-3)}{2}}^{\frac{\xi_{k,r}}{2}} C^{([\frac{r}{2}])}_{2v+l+1-\xi_r^{(4)}} b^{(3)}_{2v} = -\sum_{v=0}^{\frac{(k-2)}{2}} a_{2v+1}\mu_{2v+l+2-\xi_r^{(3)}},
\end{gathered}
$$

where

$$\xi_{k,v} = \begin{cases} \left[\frac{v+1}{2}\right] - \left[\frac{v+2}{2}\right] + 2\left[\frac{k+1}{2}\right], & v = 2j, \\ \left[\frac{v+2}{2}\right] - \left[\frac{v+1}{2}\right] + 2\left[\frac{k}{2}\right], & v = 2j-1, \end{cases}$$

$$\xi^{(1)}_{k,v} = \begin{cases} \left[\frac{v}{2}\right] - \left[\frac{v+1}{2}\right] + 2\left[\frac{k+1}{2}\right], & v = 2j-1, \\ \left[\frac{v+1}{2}\right] - \left[\frac{v}{2}\right] + 2\left[\frac{k}{2}\right], & v = 2j, \end{cases}$$

$$l = rk - k + r, \quad \xi^{(3)}_j = \left[\frac{j+2}{2}\right] - \left[\frac{j+1}{2}\right],$$

$$\xi^{(4)}_{l+j} = \begin{cases} \xi^{(3)}_{l+j}, & r = 2m \ (0 \le j < k), \\ 1 - \xi^{(3)}_{l+j}, & r = 2m-1, \end{cases} \qquad \xi^{(r)}_{k,v} = \begin{cases} \xi_{k,v}, & v = 2n, \\ \xi^{(1)}_{k,v}, & v = 2n-1. \end{cases}$$

If we prove, that the system (2.6) or (2.7) is not consistent, then we shall receive that the above mentioned system is not consistent.

Consider the first subsystem. In the system (2.6) number of the equations will be equal to $2ij + j - i + 1$, if we assume $r = 2j$. It is not difficult to show, that in this case number of the unknowns will be equal to $2ij + j - i$. It is easy to show, that the system for $a_{2n} = 0$ $(n = 0, \ldots, k-1)$, will be consistent and in this case it has the trivial solution.

Now consider the second subsystem. In the considered case number of the equations in the system (2.7) is equal to $2ij + j - i$, but number of the unknowns is equal to $2ij + j - i - 1$. If we consider, that $a_{k-1} = a_{2i-1} \neq 0$, then we shall receive, that the system (2.7) is not consistent. Really, if we solve the system (2.7), then we have:

$$\gamma_3 \sum_{i=0}^{\frac{(k-2)}{2}} a_{2i+1}\mu_{2i+3} + \gamma_5 \sum_{i=0}^{\frac{(k-2)}{2}} a_{2i+1}\mu_{2i+5} + \ldots$$
$$\cdots + \gamma_{l+2-\xi^{(3)}_r} \sum_{i=0}^{\frac{(k-2)}{2}} a_{2i+1}\mu_{2i+l+2-\xi^{(3)}_r} = 0.$$

It should be noted that all the nonzero coefficients a_i $(i = 0, \ldots, k-1)$ have identical sign.

The received relation we write in the next form:

$$\sum_{i=0}^{\frac{(k-2)}{2}} a_{2i+1}\varphi(i) = 0,$$

here

$$\varphi(i) = \int_{-1}^{1} x^{2i+2}\psi_{l-1-\xi_r^{(3)}}(x)\left(\pi^2 + \ln^2\frac{1+x}{1-x}\right)^{-1} dx,$$
$$\psi_{l-1-\xi_r^{(3)}}(x) = \gamma_3 + \gamma_5 x^2 + \cdots + \gamma_{l+2-\xi_r^{(3)}} x^{l-1-\xi_r^{(3)}}.$$

By the following notation

$$F_k(x) = a_1x^2 + a_3x^4 + \cdots + a_{k-1}x^k,$$

it can be written

$$\int_{-1}^{1} F_k(x)\psi_{l-1-\xi_r^{(3)}}(x)\left(\pi^2\ln^2\frac{1+x}{1-x}\right)^{-1} dx = 0.$$

Hence, using parity of the integrant functions and the mean-value theorem we have

$$\int_0^{\xi} F_k(x)\psi_{l-1-\xi_r^{(3)}}(x)dx = 0 \quad \text{or} \quad F_k(\xi)\int_{\xi_1}^{\xi}\psi_{l-1-\xi_r^{(3)}}(x)dx = 0.$$

If we denote by the

$$\Phi_1'(x) = F_k(x)\psi_{l-1-\xi_r^{(3)}}(x); \quad \Phi_2'(x) = \psi_{l-1-\xi_r^{(3)}}(x)$$

and consider $F_k(\xi) \neq 0$, then we shall have

$$\Phi_1(\xi) = 0; \;\; \Phi_2(\xi) = \Phi_2(\xi_1).$$

Then using Rolle's theorem we obtain, that the polynomial $\psi_{l-1-\xi_r^{(3)}}(x)$ by $\Phi_2(\xi) \neq 0$ has $l+1-\xi_r^{(3)}$ roots, what is impossible. If $\Phi_2(\xi) = 0$ then granting $\Phi_2(0) = 0$ we can write

$$\int_0^{\xi}\psi_{l-1-\xi_r^{(3)}}(x)dx = 0.$$

Then

$$\int_0^{\xi} (F_k(x) - F_k(\xi)\psi_{l-1-\xi_r^{(3)}}(x))dx = \int_0^{\xi} F_k'(\eta_k)(x-\xi)\psi_{l-1-\xi_r^{(3)}}(x)dx = 0.$$

Hence, it follows that $\Phi_3(\xi) \neq 0$ $(\Phi_3'(x) = (x-\xi)\psi_{l-1-\xi_r^{(3)}}(x))$ or

$$\int_0^{\xi} (x-\xi)\psi_{l-1-\xi_r^{(3)}}(x)dx = 0.$$

If $\Phi_3(x) \neq 0$, then we have, that the function $(x-\xi)\psi_{l-1-\xi_r^{(3)}}(x)$ has $l-1-\xi_r^{(3)}$ roots and consequently the system (17) is not consistent. But if $\Phi_3(\xi) = 0$, then using the relation $F_k(x) - F_k(\xi) = F_k'(\xi)(x-\xi) + F_k''(\eta_2)\dfrac{(x-\xi)^2}{2}$ and above described procedure, then we can write

$$\Phi_4(\xi) \neq 0 \ \ (\Phi_4'(x) = (x-\xi)^2\psi_{l-1-\xi_r^{(3)}}(x)) \ \text{ or } \ \int_0^{\xi} (x-\xi)^2\psi_{l-1-\xi_r^{(3)}}(x)dx = 0.$$

Carrying on by the above-described scheme, we have

$$\int_0^{\xi} (x-\xi)^v\psi_{l-1-\xi_r^{(3)}}(x)dx = 0 \quad (v = 0, 1, 2, \ldots, k)$$

or $\Phi_k(\xi) \neq 0$. Here $\Phi_k'(x) = (x-\xi)^k\psi_{l-1-\xi_r^{(3)}}(x)$. If $\Phi_k(\xi) \neq 0$ then system (2.7) is not consistent. But if $\Phi_k(\xi) = 0$, then using the last relation we can write

$$\int_0^{\xi} \varphi(x)\psi_{l-1-\xi_r^{(3)}}(x)dx = 0,$$

where $\varphi(x)$ polynomial of the degree which cannot be more than k.

Obviously, that the received correlation was put on any limitation to coefficients γ_j $(j = 3, 5, \ldots, l+2-\xi_r^{(3)})$ which inadmissible, since they are determined by the solving system (2.7). Particularly, if $r = 2$, then we have $l - 1 - \xi_r^{(3)} = k$. Naturally in this connection we may put $\varphi(x) = \psi_k(x)$. It is clear, that the received relation is not correct. Obviously, that the functions $F_k(x)$ and $\varphi(x)$ has the different properties and therefore they can not coincide. Consequently, the system (2.7) is not consistent. Hence we received, that $p-k \leq rk+r-k$ or $p \leq r(k+1)$.

But if we suppose, that a_{2i+1} changes its sign, then maybe $F_k(\xi) = 0$. Naturally in this connection the system (2.7) may be consistent.

Suppose, that $k = 2i$ and $r = 2j - 1$. In this case number of the equations in the systems (2.6) and (2.7) coincides and equals to $2ij + j - 2i$. It is clear, that the system (2.6) may has the trivial solution. Hence one must they consistent it.

Therefore consider the second system, in which the number of the unknowns is equal to $2ij + j - 2i - 1$. Consequently, the system (2.7) is not consistent, since the number of the equation in that system is equal to $2ij + j - 2i$. Thus we received

$$p - k \leq l + 1 = rk + r - k + 1 \text{ or } p \leq r(k+1) + 1.$$

Note, that in the case $k = 2i$ and $r = 2j$ the last equation in the system (2.6) received as the coefficient $z^{-(l+1)}$, since l is even. Therefore the indicated equation can be written in next form:

$$2C_{l+1}^{(1)} b_0^{(2)} + 2C_{l+3}^{(1)} b_2^{(2)} + \dots.$$

In this case the last equation of the system (2.7) can be written in the following form:

$$2C_{l+1}^{(1)} b_1^{(2)} + 2C_{l+3}^{(1)} b_3^{(2)} + \dots.$$

Now consider the case, when $k = 2i - 1$, that is k is odd. Suppose, that $r = 2j$. Then the number of the equations in the system (2.6) will be equal to $2ij - i + 1$. But number of the unknowns is equal to $2ij - i$. Taking into account, that $a_{2r-2} \neq 0$ can be predicated, then the system (2.6) is not consistent. In this case we may show, that the system (2.7) will be consistent. Consequently,

$$p - k \leq l \quad \text{or} \quad p \leq (k+1)r.$$

Using above mentioned scheme we can prove, that also in the case, when $k = 2i - 1$ and $r = 2j - 1$, the system (2.6) is not consistent, but

the system (2.7) is consistent. Consequently,

$$p \leq (k+1)r.$$

After the combination of all the above-mentioned cases, we receive statement of the theorem. ■

Thus, we take for granted the theorem 1. Now consider the case, when $\beta_k^{(j)} = 0 \quad (j = 1(1)r)$, that is investigate the formula, used in the problem (2), which is explicit for all the values of the parameter j. The maximal value for the degree of the stable explicit method may be established by the next theorem.

Theorem 2. *Suppose, that the formula (1) is stable for* $\beta_k^{(j)} = 0 \ (j = 1, \ldots, r)$, *has the degree* p *and* $\alpha_k \neq 0$. *Then* $p \leq rk$. *There exist stable formulas with the degree* $p = rk$ *for the arbitrary* k.

Proof. Taking here exactly the same way, as in theorem 1, we receive the systems similar to the systems (2.6) and (2.7). ■

It should be noted, that these systems can not have trivial solution, since in this connection it is received, that the unknowns $\beta_i^{(j)} \quad (i = 0, \ldots, k-1, \ 1 \leq j \leq r)$ for every fixed j can be determined from the system, which consists of the $k+1$ equations. It may be proved, that in this case these systems will not be consistent. Consider the case $k = 2i$. Then the system similar to the systems (2.6) and (2.7) can be written

in the next form:

$$2\sum_{v=0}^{\frac{k}{2}} C^{(1)}_{2v+1} b^{(2)}_{2v} + 2^2 \sum_{v=0}^{\frac{(k-2)}{2}} C^{(1)}_{2v+2} b^{(3)}_{2v+1} + \dots$$

$$\dots + 2^{r-1} \sum_{v=\frac{(r-2)}{2}}^{\frac{\xi_{k,r}}{2}} C^{([\frac{r}{2}])}_{2v+1} b^{(r)}_{2v} = -\sum_{v=0}^{\frac{(k-2)}{2}} a_{2v}\mu_{2v+1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \tag{2.8}$$

$$2\sum_{v=0}^{\frac{k}{2}} C^{(1)}_{2v+l+1} b^{(2)}_{2v} + 2^2 \sum_{v=0}^{\frac{(k-2)}{2}} C^{(1)}_{2v+l+2} b^{(3)}_{2v+1} + \dots$$

$$\dots + 2^{r-1} \sum_{v=\frac{(r-2)}{2}}^{\frac{(\xi^{(r)}_{k,l+1}-\xi^{(4)}_{l+1})}{2}} C^{([\frac{r}{2}])}_{2v+l+1+\xi^{(4)}_{l+1}} b^{(r)}_{2v+\xi^{(4)}_{l+1}} = -\sum_{v=0}^{\frac{(k-2)}{2}} a_{2v}\mu_{2v+l+1},$$

where $l = (r-1)k$. The second system has the following form:

$$2\cdot\sum_{v=0}^{\frac{(k-2)}{2}} C^{(1)}_{2v+l+1} b^{(2)}_{2v+1} + 2^2\cdot\sum_{v=0}^{\frac{k}{2}} C^{(1)}_{2v+2} b^{(3)}_{2v} + \dots$$

$$\dots + 2^{r-1}\cdot \sum_{v=\frac{(r-3)}{2}}^{\frac{\xi_{k,r}}{2}} C^{([\frac{r}{2}])}_{2v+2} b^{(r)}_{2v} = -\sum_{v=0}^{\frac{(k-2)}{2}} a_{2v+1}\mu_{2v+1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \tag{2.9}$$

$$2\cdot\sum_{v=0}^{\frac{(k-2)}{2}} C^{(1)}_{2v+l+1} b^{(2)}_{2v+1} + 2^2\cdot\sum_{v=0}^{\frac{k}{2}} C^{(1)}_{2v+1} b^{(3)}_{2v} + \dots$$

$$\dots + 2^{r-1}\cdot \sum_{v=\frac{(r-3)}{2}}^{\frac{(\xi^{(r)}_{k,l}-\xi^{(4)}_{l})}{2}} C^{([\frac{r}{2}])}_{2v+l+\xi^{(4)}_{l}} b^{(r)}_{2v+\xi^{(4)}_{l}} = -\sum_{v=0}^{\frac{(k-2)}{2}} a_{2v+1}\mu_{2v+l+1}.$$

In the system (2.8) number of the equations is equal to $(r-1)\,i+1$, but the number of the unknowns is independent of the property of parity of r and equals to $(r-1)i$. Consequently, the system (2.8) is not consistent.

In the system (2.9) number of the equalities is equal to $(r-1)i$, but number of the unknowns equals to $(r-1)i-1$, since one of the unknowns is determined by the coefficient z^0. Hence, it follows that

$$p-k \le l \quad \text{or} \quad p \le kr \quad (l = k(r-1)).$$

This theorem for $k = 2i-1$ is proved analogously to the case $k = 2i$.

If we apply the theorem to formula (1.16), then we shall receive the following theorem.

Theorem 3. *Suppose that the formula (1.16) is stable, has the degree p and $\alpha_k \neq 0$. Then*

$$p \le (k+1)\sum_{j=1}^{r}\delta_j + 1, \quad (p_{\max} = (k+1)\sum_{j=1}^{r}\delta_j + 1).$$

There exists stable formula with the degree $p = p_{\max}$ for $k = 2i$ and $r = 2v-1$, but in other cases there exists stable formula with the degree $p = p_{\max} - 1$ and does not exist stable formula with the degree $p > p_{\max} - 1$.

It is not difficult to determine, that if there exists stable formula with the degree $p > (k+1)r + 1$, then it must be in the class of the forward-jumping formulas.

Really, if we consider forward-jumping formula in the next form:

$$\sum_{i=0}^{k-m}\alpha_i y_{n+i} = \sum_{j=1}^{r} h^j \sum_{i=0}^{k}\beta_i^{(j)} y_{n+i}^{(j)}, \tag{2.10}$$

then the theorem 1 can be formulated in the next form:

Theorem 4. *Suppose, that the formula (2.10) is stable, has the degree p and $\alpha_{k-m} \neq 0$. Then*

$$p \le (k+1)r + m.$$

There exist stable forward-jumping formulas with the degree $p = (k+1)r + m - 1$ for $k = 2i \ge 3m$, $r = 2j$ and $k - m = 2v - 1$.

In other cases there exist stable forward-jumping formulas with the degree $p = (k+1)r + m$ for $k \geq 3m$ if the property parity of k and m is identical and for $k \geq 3m+1$ if the property parity of k and m is not identical.

Proof. Behaving here in exactly the same way, as in theorem 1 and multiplying the polynomial $\rho(\tau)$ and $\nu_l(\tau)$ to $\left(\frac{1}{2}(z-1)\right)^{k-m}$, we receive the next system, consistency of which is questionable

$$\sum_{v=1}^{m} d_v^{(1)} \beta_{k-m+v}^{(1)} + 2 \sum_{v=0}^{[\frac{(k-m)}{2}]} C_{2v+1}^{(1)} b_{2v}^{(2)} + 2^2 \sum_{v=0}^{[\frac{(k-m+1)}{2}]-1} C_{2v+2}^{(1)} b_{2v+1}^{(3)} + \dots$$
$$\dots + 2^{r-1} \sum_{v=\frac{(r-2)}{2}}^{\frac{\xi_{k-m,r}}{2}} C_{2v+1}^{([\frac{r}{2}])} b_{2v}^{(r)} = - \sum_{v=0}^{[\frac{(k-m)}{2}]} a_{2v} \mu_{2v+1},$$

$$\sum_{v=1}^{m} d_v^{(2)} \beta_{k-m+v}^{(1)} + 2 \sum_{v=-1}^{[\frac{(k-m+1)}{2}]-1} C_{2v+3}^{(1)} b_{2v+1}^{(2)} + 2^2 \sum_{v=0}^{[\frac{(k-m)}{2}]} C_{2v+2}^{(1)} b_{2v}^{(3)} + \dots$$
$$\dots + 2^{r-1} \sum_{v=\frac{(r-4)}{2}}^{\frac{\xi_{k-m,r}}{2}} C_{2v+2}^{([\frac{r}{2}])} b_{2v}^{(r)} = - \sum_{v=0}^{[\frac{(k-m+1)}{2}]-1} a_{2v+1} \mu_{2v+3},$$

. . . . . .

$$\sum_{v=1}^{m} d_v^{(l)} \beta_{k-m+v}^{(1)} + 2 \sum_{v=-\frac{(l-1)}{2}}^{\frac{\xi_{k-m,l}}{2}} C_{2v+l}^{(1)} b_{2v}^{(2)} + 2^2 \sum_{v=-\frac{(l-2)}{2}}^{\frac{\xi_{k-m,l}^{(1)}}{2}} C_{2v+l}^{(1)} b_{2v}^{(3)} + \dots$$
$$\dots + 2^{r-1} \sum_{v=\frac{-(l-r+1)}{2}}^{\frac{\xi_{k-m,l}^{(2)}}{2}} C_{2v+l}^{([\frac{r}{2}])} b_{2v}^{(r)} = - \sum_{v=0}^{[\frac{(k-m+\xi_l^{(3)})}{2}]-\xi_l^{(3)}-\xi_{k-m}^{(3)}} a_{2v+\xi_l^{(3)}} \mu_{2v+l+\xi_l^{(3)}},$$

$$\sum_{v=1}^{m} d_v^{(l+1)} \beta_{k-m+v}^{(1)} + 2 \sum_{v=-\frac{l}{2}}^{\frac{\xi_{k-m,l+1}}{2}} C_{2v+l+1}^{(1)} b_{2v}^{(2)} + 2^2 \sum_{v=-\frac{(l-1)}{2}}^{\frac{\xi_{k-m,l+1}^{(1)}}{2}} C_{2v+l+1}^{(1)} b_{2v}^{(3)} + \dots$$
$$\dots + 2^{r-1} \sum_{v=-\frac{(l-r)}{2}}^{\frac{\xi_{k-m,l+1}^{(2)}}{2}} C_{2v+l+1}^{([\frac{r}{2}])} b_{2v}^{(r)} = - \sum_{v=0}^{[\frac{(k-m+\xi_{l+1}^{(3)})}{2}]-\xi_{l+1}^{(3)}-\xi_{k-m}^{(3)}} a_{2v+\xi_{l+1}^{(3)}} \mu_{2v+l+1+\xi_{l+1}^{(3)}},$$

where $l = 2m+kr-k+r-1$. This system is divided into two subsystems. The first of them for $k = 2i$ and $r = 2j$ can be written in the next form:

$$\sum_{v=1}^{m} d_v^{(1)}\beta_{k-m+v}^{(1)} + \sum_{v=0}^{[\frac{(k-m)}{2}]} C_{2v+1}^{(1)} b_{2v}^{(2)} + 2^2 \sum_{v=0}^{[\frac{(k-m+1)}{2}]-1} C_{2v+2}^{(1)} b_{2v+1}^{(3)} + \dots$$
$$\dots + 2^{r-1} \sum_{v=\frac{r}{2}}^{[\frac{(k-m)}{2}]} C_{2v+1}^{([\frac{r}{2}])} b_{2v}^{(r)} = - \sum_{v=0}^{[\frac{(k-m)}{2}]} a_{2v}\mu_{2v+1},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \tag{2.11}$$
$$\sum_{v=1}^{m} d_v^{(1)}\beta_{k-m+v}^{(1)} + 2 \sum_{v=-\frac{(l-1)}{2}}^{[\frac{(k-m)}{2}]} C_{2v+l}^{(1)} b_{2v}^{(2)} + 2^2 \sum_{v=-\frac{(l-1)}{2}}^{[\frac{(k-m+1)}{2}]-1} C_{2v+l+1}^{(1)} b_{2v+1}^{(3)} + \dots$$
$$\dots + 2^{r-1} \sum_{v=-\frac{l-r+1}{2}}^{[\frac{(k-m)}{2}]} C_{2v+l}^{([\frac{r}{2}])} b_{2v}^{(r)} = - \sum_{v=0}^{[\frac{(k-m)}{2}]} a_{2v}\mu_{2v+l}.$$

Number of the equations in the system (2.11) is equal to $2ij-i+j+m$. But number of the unknowns depends on parity $k-m$. If $k-m = 2n$, then number of the unknowns is equal to $2ij - i + j + m$.

Consequently, the system (2.11) may be consistent. But if $k - m = 2n - 1$, then quantity of the unknowns in the system (2.11) is equal to $2ij - i + j + m - 1$. It is obvious, that the system (2.11) can not have the trivial solution, since in this connection $\beta_{k-m+v}^{(1)} = 0$ $(v = 1, \dots, m)$. It may be proved, that the mentioned system has not trivial solution (which identically different from zero). Consequently, the system (2.11) is not consistent. Then we have

$$p - k + m \le 2m + rk - k + r - 1 \quad \text{or} \quad p \le (k+1)r + m - 1.$$

If we consider the case $k = 2i$ and $r = 2j - 1$, then the last equation in the system (2.11) can be written in the next form.

$$\sum_{v=1}^{m} d_v^{(l+1)}\beta_{k-m+v}^{(1)} + 2C_1^{(1)} b_{-(l-1)}^{(2)} + \dots.$$

In this case number of unknowns in the system (2.11) for $k-m=2n$ or for $k-m=2n-1$ is equal to $2ij+j-2i+m-1$, but the number of equation is equal to $2ij+j-2i+m+1$.

As it was proved above, here we can prove, that the system (2.11) is not consistent. Consequently,

$$p-k+m \leq l+1 \quad \text{or} \quad p \leq (k+1)r+m.$$

Now consider the case $k=2i$ and $r=2j$, when the system (2.11) is consistent. In this case the second subsystem is written in the next form:

$$\begin{gathered}\sum_{v=1}^{m} d_v^{(2)}\beta_{k-m+v}^{(1)} + \sum_{v=-1}^{[\frac{(k-m+1)}{2}]-1} C_{2v+3}^{(1)} b_{2v+1}^{(2)} + 2^2 \sum_{v=0}^{[\frac{(k-m)}{2}]} C_{2v+2}^{(1)} b_{2v}^{(3)} + \dots \\ \dots + 2^{r-1} \sum_{v=\frac{(r-4)}{2}}^{[\frac{(k-m+1)}{2}]-1} C_{2v+3}^{([\frac{r}{2}])} b_{2v+1}^{(r)} = -\sum_{v=0}^{[\frac{(k-m+1)}{2}]-1} a_{2v+1}\mu_{2v+3}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\end{gathered} \tag{2.12}$$

$$\begin{gathered}\sum_{v=1}^{m} d_v^{(l+1)}\beta_{k-m+v}^{(1)} + 2\sum_{v=-\frac{(l-1)}{2}}^{[\frac{(k-m+1)}{2}]-1} C_{2v+l+1}^{(1)} b_{2v+1}^{(2)} + 2^2 \sum_{v=-\frac{(l-1)}{2}}^{[\frac{(k-m)}{2}]} C_{2v+l+1}^{(1)} b_{2v}^{(3)} + \dots \\ \dots + 2^{r-1} \sum_{v=-\frac{l-r+1}{2}}^{[\frac{(k-m+1)}{2}]-1} C_{2v+l}^{([\frac{r}{2}])} b_{2v+1}^{(r)} = -\sum_{v=0}^{[\frac{(k-m+1)}{2}]-1} a_{2v+1}\mu_{2v+l+2}.\end{gathered}$$

Number of the equations m the system (2.12) for $k-m=2n$ is equal to $2ij+j-i+m$, but number of the unknowns is equal to $2ij+j-i+m-1$. We can prove, that the system (2.12) in this case is not consistent. Hence, it follows that

$$p \leq (k+1)r+m.$$

Now consider the case, when the systems (2.11) and (2.12) are consistent. In order for the consistency of the systems (2.11) and (2.12) to be followed by consistency of the initial system, the unknowns $\beta_{k-m+v}^{(1)}(v=$

$1, \ldots, m$), found from these systems, must be equal to each other, in general this question depends on parity k and m. If k and m are even or odd simultaneously, then degree of the stable forward-jumping formula has the maximal value for $k \geq 3m$ otherwise for $k \geq 3m+1$. ■

Consider the case, when in the system (1.16) $\delta_1 = 0$. It is clear, that the formula (1.16) can not be stable. Because in this case we use the notation of 2-stability. If to consider the case $\delta_1 = \delta_2 = \cdots = \delta_{l-1} = 0$ and $\delta_l \neq 0$, then we use the notation of l-stability. It is not difficult to prove, that there exists l-stable method determined by the formula (1.16). For this aim consider the next theorem.

Theorem 5. *Let the formula (1.16) has the degree p, is l-stable, $a_k \neq 0$ and $\delta_1 = \cdots = \delta_{l-1} = 0, \delta_1 = 0$. Then there exists l-stable formula with the degree $p = (\delta_l + \cdots + \delta_r)(k+1) + l$ in the case, when $k = 2i$, $r = 2j$, $l = 2v$ or $k = 2i$, $r = 2j-1$, $v = 2i-1$. In the other cases there exist l-stable formulas with the degree $p = (\delta_l + \cdots + \delta_r)(k+1) + l - 1$.*

Under solving some problems, it is useful to determine beforehand the sign of the coefficients $\beta_k^{(j)} (j = 1, \ldots, r)$, and also the relation between them. For example, in using of two sided methods, just as in construction of the new methods having Obrechkoff's type there arises the question on determination of the sign of some coefficients. For this aim consider the next theorem.

Theorem 6. *Suppose, that the formula (1) is stable, has the degree p, which got a maximal value and $\alpha_k > 0$. Then*

$$\beta_k^{(j)} = (-1)^{j-1} l_j (l_j > 0), \ |\beta_k^{(m)}| > |\beta_k^{(m+1)}| \ (m = 1, \ldots, r-1, j = 1, \ldots, r).$$

But if $\beta_k^{(v)} \neq 0, \beta_k^{(v+1)} = \cdots = \beta_k^{(v+s)} = 0$, $\beta_k^{(v+s+1)} \neq 0$, then $\beta_k^{(v)} \beta_k^{(v+s+1)} < 0$ and $|\beta_k^{(v)}| > |\beta_k^{(v+s+1)}|$.

Let $\beta_k^{(1)} = \beta_k^{(2)} = \cdots = \beta_k^{(s-1)} = 0$ and $\beta_k^{(s)} \neq 0$. Then $\beta_k^{(s)} > 0$.

As it is obvious from the formulation of the theorem 6, here the maximal value of the degree for formula (1) is taken, in every considered cases. For example, $p_{\max} = 5$ in the case $r = 2, k = 2$ and $\beta_k^{(1)} = 0$.

Note. Below we reduced some concrete methods constructed by author several times

$$y_{n+1} = \frac{12y_n - h(f_{n+2} - 8f_{n+1} - 5f_n)}{12} \quad (r = 1, k = 2, p = 3)$$

(local trun. err.$h^4 y_n^{(4)}/24 + O(h^5)$),

$$y_{n+2} = \frac{8y_{n+1} + 11y_n}{19} - \frac{h(f_{n+3} - 24f_{n+2} - 57f_{n+1} - 10f_n}{57} \quad (r = 1, k = 3, p = 5)$$

(local trun. err. $-11h^6 y_n^{(6)}/3420 + O(h^7)$),

$$y_{n+2} = \frac{(416y_{n+1} - 103y_n)}{313} + \frac{h(157f_{n+3} + 11232f_{n+2} + 8451f_{n+1} - 2830f_n)}{25353} - \frac{h^2(11g_{n+3} + 630g_{n+2} - 1557g_{n+1} + 92g_n)}{8451} \quad (r = 2, k = 3, p = 9)$$

(local trun. err.$103h^{10} y_n^{(10)}/212965200 + O(h^{11})$),
here $g(x, y) = f'_x(x, y) + f'_y(x, y)f(x, y), y' = f(x, y)$.

It is noted, that there are concrete methods for which theorem 6 is correct in the case, when the value of the degree of the stable methods is less than maximal.

Obrechkoff's method that is the formula (1.16), in more general form was investigated for $r = 2$ and arbitrary k, in [9].

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