Bulletin of the Iranian Mathematical Society
Vol. 28, No. 1, pp 29-42 (2002)

α-VALUATIONS OF SPECIAL CLASSES OF QUADRATIC GRAPHS

"To the memory of Jaromir Abrham (1931-1997)"

Kourosh Eshghi

Department of Industrial Engineering, Sharif University of Technology, Tehran, Iran
eshghi@sharif.edu

Abstract: It is shown that the quadratic graph Q(5,4k) (consisting of 5 cycles of length 4k) has an α-valuation (a stronger form of the graceful valuation) for every positive integer k. Furthermore, additional results are obtained from the main theorem of this paper.

1. BASIC DEFINITIONS

Let $G = (V, E)$ be a graph with $m = |V|$ vertices and $n = |E|$ edges. By the term graph, we mean an undirected finite graph without loops or multiple edges. All parameters in this paper are positive integers. A graceful valuation (or β-valuation) of a graph $G = (V, E)$ is a one-to-one mapping Ψ of the vertex set $V(G)$ into the set $\{0, 1, 2, \ldots, n\}$ with

[0] *MSC (2000):* 05C78

[0] *Keywords:* Graphing labeling, Graceful valuations, α-valuation

[0] *Received:* 26 January, 2001; *Revised:* 6 November, 2001

[0] The author is partially supported by Sharif University of Technology Grant No. 717

this property: If we define, for any edge $e = \{u, v\} \in E(G)$, the value $\Psi^{\bullet}(e) = |\Psi(u) - \Psi(v)|$ then $\Psi^{\bullet}$ is a one-to-one mapping of the set $E(G)$ onto the set $\{1, 2, \ldots, n\}$.

A graph is called graceful if it has a graceful valuation. An α-valuation (or α-labeling) of a graph $G = (V, E)$ is a graceful valuation of G which satisfies the following additional condition: There exists a number $\gamma (0 \leq \gamma \leq |E(G)|)$ such that, for any edge $e \in E(G)$ with end vertices $u, v \in V(G), \min[\Psi(u), \Psi(v)] \leq \gamma < \max[\Psi(u), \Psi(v)]$.

The concept of a graceful valuation and of an α-valuation were introduced by Rosa [7]. Rosa proved that, if G is graceful and if all vertices of G are of even degree, then $|E(G)| \equiv 0$ or 3 (mod 4). This implies that if G has an α-valuation and if all vertices of G are of even degrees, then $|E(G)| \equiv 0$ (mod 4) (G is bipartite). In [7] it is also shown that these conditions are also sufficient if G is a cycle. The symbol Cm will denote a cycle on m vertices. Abrham and Kotzig [2] proved that Rosa's condition is also sufficient for 2-regular graphs with two components.

A snake is a tree with exactly two vertices of degree 1. In [7], it was proved that every snake has an α-valuation. A snake with n edges will be denoted by P_n.

A detailed history of the graph labeling problem and related results appears in Gallian [4,5]. One of the results of Abrham and Kotzig should be mentioned here: If G is a 2-regular graph on n vertices and n edges which has a graceful valuation Ψ then there exists exactly one number $x (0 < x < n)$ such that $\Psi(v) \neq x$ for all $v \in V(G)$; this number x is referred to as the missing value of the graceful valuation[2].
A quadratic graph $Q(r, s)$ is a graph with r components, each of which is an s-cycle.

Here are some of the results published in the references:

1. A $Q(1, s)$-graph (i.e. an s-cycle) is graceful if and only if $s \equiv 0$ or 3 (mod 4). It has an α-valuation if and only if $s \equiv 0$ (mod 4) [7].

2. A $Q(2,s)$-graph has an α-valuation if and only if s is even and $s > 2$ [6].

3. A $Q(3,4k)$-graph has an α-valuation for each $k > 1$. The $Q(3,4)$-graph does not have an α-valuation but it is graceful [6].

4. A $Q(r,3)$-graph is graceful if and only if $r = 1$. A $Q(r,5)$-graph is not graceful for any r [6].

5. A $Q(r,4)$-graph has an α-valuation [6].

2. TRANSFORMATIONS OF LABELING OF A GRAPH

The transformations presented below are used extensively in this paper.

2.1 Transformation Type 1

Lemma 1: (Abrham & Kotzig [1]) *Let r be a non-negative integer and let s be an odd integer, $s = 2k+1$. Then P_s has an α-valuation Ψ with endpoints labeled w and z that satisfies the conditions $z - w = k+1$ and $w = r$.(w.l.o.g., we assume that $w < z$.)*

Suppose that we have two series of vertex labels as follows where $0 \leq n, n+k \leq 2r$ and $2r+1 \leq m, m+k < 4r+1$ and $|4r+1-(m+n+k)| \leq 1$:

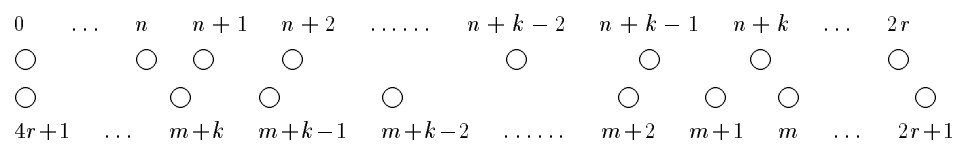

Figure 1: Arrangement of vertex labels in transformation type 1

We apply the transformation type 1 to the vertex labels $(n, n+1, n+2, \ldots, n+k-2, n+k-1, n+k)$ and $(m, m+1, m+2, \ldots, m+k-2, m+k-1, m+k)$ by choosing the vertices w' and z' as end points in the following steps:

Step1: First we modify the vertex labels as follows:

1) From each label $(n, n+1, n+2, \ldots, n+k-2, n+k-1, n+k)$ subtract n;

2) From each label $(m, m+1, m+2, \ldots, m+k-2, m+k-1, m+k)$ subtract $m-(k+1)$

Step2: According to Lemma 1, we construct an α-valuation for the snake P_{2k+1} on the new labels, with end vertices having labels w and z such that $0 \leq w \leq k$ and $k+1 \leq z \leq 2k+1$ and $z - w = k+1$.

Step3: Now if we modify again the new values to the original values in the following way:

1) Add n to each new label $(0, 1, 2, \ldots, k-2, k-1, k)$;

2) Add $m-(k+1)$ to each new value $(k+1, k+2, \ldots, 2k-1, 2k, 2k+1)$;

Then the end vertices of P_{2k+1} will be labeled $w' = w+n$ and $z' = z+m-(k+1)$ and the edge values will be $m-n-k, m-n-k+1, \ldots, m-n+k$.

2.2 Transformation Type 2

Lemma 2: (Abrham & Kotzig [1]) *Let r be a non-negative integer and let s be an even integer, $s = 2k$. Then P_s has an α-valuation Ψ with endpoints labeled w and z that satisfies the conditions $z + w = k$ and $w = r$.(w.l.o.g., we assume that $w < z$.)*

Suppose that we have two series of vertex labels as follows where $0 \leq n, n+k \leq 2r$ and $2r+1 \leq m, m+k-1 < 4r$ and $|4r+1-(m+n+k)| \leq 1$:

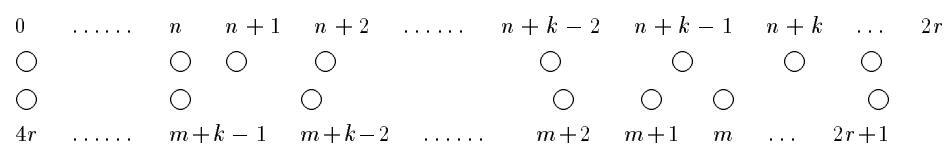

Figure 2: Arrangement of vertex labels in transformation type 2

We apply the transformation type 2 to the vertex labels $(n, n+1, n+2, \ldots, n+k-2, n+k-1, n+k)$ and $(m, m+1, m+2, ..., m+k-2, m+k-1)$ by choosing the vertices w' and z' as end points in the following steps:
Step1: First we modify the vertex labels as follows:

1) From each label $(n, n+1, n+2, \ldots, n+k-2, n+k-1, n+k)$ subtract n;

2) From each label $(m, m+1, m+2, \ldots, m+k-2, m+k-1)$ subtract $m-(k+1)$

Step2: According to Lemma 2, we construct an α-valuation for the snake P_{2k} on the new labels, with end vertices having labels w and z such that $0 \leq w < (k/2) < z \leq k$ and $z + w = k$.

Step3: Now if we transform again the new values to the original values in the following way:

1) Add n to each new label $(0, 1, 2, \ldots, k-2, k-1, k)$;

2) Add $m-(k+1)$ to each new value $(k+1, k+2, \ldots, 2k-1, 2k)$;

Then the end vertices of P_{2k} will be labeled $w' = w + n$ and $z' = z$ and the edge values will be $m-n-k, m-n-k+1, \ldots, m-n+k-1$.

3. BASIC THEOREM

Theorem 1: *The quadratic graph Q(5,4k) has an α-valuation for all $k \geq 1$.*

Proof: The missing value of the α-valuation of this graph is 5k. Now let us assume $k \geq 5$. The vertices of the first C_{4k} will be $[8k, 12k, 8k+1, 12k-1, 8k+2, 12k-2, \ldots, 9k-1, 11k+1, 9k+1, 11k, 9k+2, 11k-1, \ldots, 10k-1, 10k+2, 10k, 10k+1]$; this yields the edge values $4k, 4k-1, 4k-2, 4k-3, \ldots, 2k+2, 2k+1, 2k, ..., 3, 2, 1$. Next we will describe the labeling of the second C_{4k}. The successive vertices will be labeled

as follows:
$[14k+1, 6k, 14k, 6k+1, \ldots, 13k+3, 7k-2, 13k+2, 7k-1, 13k, 7k, \ldots, 12k+2, 8k-2, 12k+1, 8k-1]$. The edge labels of this C_{4k} will be then $8k+1, 8k, 8k-1, \ldots, 6k+5, 6k+4, 6k+3, 6k+2, 6k+1, 6k, \ldots, 4k+4, 4k+3, 4k+2$.
The third C_{4k} will be labeled in three stages as follows:

I. Form the snake $(6k-1, 14k+2, 6k-2, 14k+3, 6k-3, \ldots, 5k+2, 15k-1, 5k+1, 15k, 5k-1, 15k+1, 5k-2, 15k+2, \ldots, 16k-3, 4k+2, 16k-2, 4k+1)$. The values of the edges are then $8k+3, 8k+4, 8k+5, 8k+6, \ldots, 10k-3, 10k-2, 10k-1, 10k+1, \ldots, 12k-5, 12k-4, 12k-3$.

II. Join the vertex labeled $6k-1$ to the vertex labeled $16k-1$ to generate the edge labeled $10k$.

III. Form another snake in such a way that its vertices are labeled as follows: $(16k-1, 4k-1, 16k+2, 4k-2, 16k, 4k+1)$. The resulting values of the edges of this snake are then $12k-1, 12k, 12k+2, 12k+3, 12k+4$.

Now we construct the fourth cycle C_{4k} according to the following stages:

a) The edge labels $12k+1$ and $12k+5$ are generated by joining the following pairs of vertices respectively: $4k$ and $16k+1$; $4k-4$ and $16k+1$.

b) Apply transformation type 2 to the vertex labels $(3k-2, 3k-1, 3k, \ldots, 4k-5, 4k-4, 4k-3)$ and $(16k+3, 16k+4, 16k+5, \ldots, 17k, 17k+1)$ by choosing the two vertices $3k-1$ and $4k-4$ as end vertices. The corresponding edge values of this transformation will be $12k+6, 12k+7, 12k+8, \ldots, 14k+2, 14k+3$.

c) The edges labeled $14k+4$ and $14k+5$ are obtained by joining the following pairs of vertices respectively: $3k-1$ and $17k+3$; $4k$ and $18k+5$.

d) Construct the snake $(17k+3, 3k-3, 17k+4, 3k-4, 17k+5, 3k-5, \ldots, 18k, 2k, 18k+1, 2k-1, 18k+5)$. The corresponding edge labels will then be $14k+6, 14k+7, 14k+8, \ldots, 16k, 16k+1, 16k+2, 16k+6$.

Finally the last cycle C_{4k} will be constructed as follows when $k \geq 14$ and $k \neq 22$:

1. The edges labeled $12k-2, 4k+1, 8k+2$ and $16k+3$ are obtained by connecting the following pairs of vertices respectively: $k+3$ and $13k+1$; $9k$ and $13k+1$; $9k$ and $17k+2$; $k-1$ and $17k+2$.

2. Construct the snake $(18k+2, 2k-2, 18k+3, 2k-4, 18k+4, 2k-5, 18k+6, 2k-6, 18k+7, 2k-3)$. This yields the edge labels $16k+4, 16k+5, 16k+7, ..., 16k+13$.

3. The edge labeled $16k+14$ is obtained by joining the two vertex labels $2k-3$ and $18k+11$ together.

4. Apply transformation type 1 to the vertex labels $(k+4, k+5, \ldots, 2k-8, 2k-7)$ and $(18k+8, 18k+9, 18k+10, 18k+11, \ldots, 19k-4, 19k-3)$ by using the two vertices $k+7$ and $18k+11$ as end points. This transformation generates the edge labels $16k+15, 16k+16, ..., 18k-8, 18k-7$.

5. Connect the following pairs of vertices to each other to obtain the edges labeled $18k-6$ and $18k-5$ respectively: 8 and $18k+2$; $k+7$ and $19k+2$.

6. Construct the snake $(19k-2, k+2, 19k-1, k+1, 19k, k, 19k+1, k-1)$. The resulting values of the edges are then $18k-4, 18k-3, \ldots, 18k+2$.

7. The edge labeled $18k+3$ is obtained by joining the two vertices $k+3$ and $19k+6$.

8. Construct the snake $(19k+2, k-2, 19k+3, k-3, 19k+4, k-4, 19k+5, k-5, 19k+6)$. The edge labels $18k+4, 18k+5, 18k+6, \ldots, 18k+11$ are generated by this snake.

9. Connect the two vertices $k-14$ and $19k-2$ to each other to generate the edge label $18k+12$.

10. Finally apply transformation type 2 to the vertex labels $(0, 1, 2, \ldots, 8, \ldots, k-14, \ldots, k-6)$ and $(19k+7, 19k+8, \ldots, 20k-1, 20k)$ by considering the two vertices 8 and $k-14$ as end vertices. The rest of the edge values will be generated by this transformation and the last C_{4k} will be completed.

The construction of an α-valuation of the last C_{4k} when $5 \leq k \leq 13$ or $k = 22$ has been given in the Table 1 as follows:

k	The construction of the fifth cycle C_{4k} in the graph Q(5,4k)
5	[45, 66, 8, 92, 7, 94, 6, 96, 5, 97, 3, 98, 2, 99, 1, 100, 0, 93, 4, 87]
6	[54, 79, 9, 120, 0, 119, 1, 118, 2, 117, 3, 116, 4, 110, 10, 111, 8, 112, 7,114,6, 115, 5, 104]
7	[63, 92, 10, 129, 12, 128, 8, 130, 5, 135, 4, 136, 3, 137, 2, 138, 1, 139, 0, 140, 11, 132, 9, 133, 7, 134, 6, 121]
8	[72, 105, 11, 160, 0, 159, 1, 158, 2, 157, 3, 156, 4, 155, 5, 151, 9, 152, 8, 153, 6, 154, 13, 150, 10, 148, 12, 147, 14, 146, 7, 138]
9	[81, 118, 12, 176, 4, 175, 5, 178, 2, 177, 3, 180, 0, 179, 1, 170, 10, 171, 9, 172, 7, 173, 6, 174, 15, 169, 11, 168, 13, 166, 14, 165, 16, 164, 8, 155]
10	[90, 131, 13, 197, 3, 196, 4, 199, 2, 198, 0, 200, 1, 192, 17, 187, 14, 186, 15, 184, 16, 183, 18, 182, 8, 193, 7, 194, 6, 195, 5, 188, 12, 189, 11, 190, 10, 191, 9, 172]
11	[99, 144, 14, 218, 2, 219, 1, 220, 0, 215, 5, 216, 4, 217, 3, 212, 7, 213, 6, 214, 15, 206, 13, 207, 12, 208, 11, 209, 19, 205, 16, 204, 17, 202, 18, 201, 20, 200, 8, 211, 9, 210, 10, 189]
12	[108, 157, 15, 230, 6, 234, 7, 233, 8, 218, 22, 219, 20, 220, 19, 222, 18, 223, 21, 227, 13, 226, 14, 225, 16, 224, 17, 237,3, 236, 4, 235, 5, 240, 0, 239, 1, 238, 2, 231, 10, 232, 9, 228, 12, 229, 11, 206]
13	[117, 170, 16, 256, 4, 255, 5, 258, 2, 257, 3, 260, 0, 259, 1, 250, 19, 242, 18, 243, 17, 244, 15, 245, 23, 241, 20, 240, 21, 238, 22, 237, 24, 236, 8, 252, 7, 253, 6, 254, 11, 249, 10, 251, 9, 246, 14, 247, 13, 248, 12, 223]
22	[198, 287, 25, 430, 10, 429, 11, 432, 8, 431, 9, 426, 14, 427, 13, 428, 12, 423, 17, 424, 16, 425, 15, 440, 0, 439, 1, 438, 2, 437, 3, 436, 4, 435, 5, 434, 6, 433, 7, 398, 42, 399, 40, 400, 39, 402, 38, 403, 41, 407, 34, 406, 35, 405, 36, 404, 37, 411, 30, 410, 31, 409, 32, 408, 33, 415, 26, 414, 27, 413, 28, 412, 29, 419, 22, 418, 23, 417, 24, 416, 18, 422, 19, 421, 20, 420, 21, 376]

Table 1: The construction of the fifth cycle C_{4k} in the graph Q(5,4k),
$5 \leq k \leq 13, k = 22$

For $1 \leq k \leq 4$, the successive vertices of each cycle of Q(5,4k) will be labeled according to the following table:

k	The construction of an α-valuation of Q(5,4k)
1	[0, 18, 1, 20] , [2, 16, 4, 17] , [3, 14, 9, 19] , [6, 13, 7, 15] , [8, 11, 10, 12]
2	[0, 37, 4, 33, 8, 39, 1, 40] , [2, 36, 18, 27, 5, 35, 3, 38] , [6, 34, 7, 31, 11, 30, 9, 32] , [12, 29, 15, 25, 14, 26, 13, 28] , [16, 24, 17, 23, 19, 22, 20, 21]
3	[0, 60, 4, 53, 27, 40, 6, 54, 7, 58, 1, 59] , [2, 57, 12, 49, 8, 52, 9, 51, 5, 55, 3, 56] , [10, 50, 11, 47, 17, 44, 16, 45, 14, 46, 13, 48] , [18, 43, 23, 37, 22, 38, 21, 39, 20, 41, 19, 42] , [24, 36, 25, 35, 26, 34, 28, 33, 29, 32, 30, 31]
4	[0, 80, 5, 74, 4, 70, 36, 53, 7, 75, 3, 76, 2, 78, 1, 79] , [6, 77, 16, 65, 12, 68, 13, 67, 10, 69, 11, 71, 9, 72, 8, 73] , [14, 66, 15, 63, 23, 58, 22, 59, 21, 60, 19, 61, 18, 62, 17, 64] , [24, 57, 31, 49, 30, 50, 29, 51, 28, 52, 27, 54, 26, 55, 25, 56] , [32, 48, 33, 47, 34, 46, 35, 45, 37, 44, 38, 43, 39, 42, 40, 41]

Table 2: The construction of an α-valuation of Q(5,4k) for $1 \leq k \leq 4$

4. THE STANDARD VALUATIONS OF C_{4k}

Definition 1: The standard α-valuation of C_{4k} are given by any of the following sequence of values of the consecutive vertices of C_{4k}:

a) $[4k, 0, 4k-1, 1, 4k-2, 2, \ldots, k-2, 3k+1, k-1, 3k, k+1, 3k-1, k+2, 3k-2, \ldots, 2k+2, 2k-1, 2k+1, 2k]$ with missing value $x = k$.

b) $[0, 4k, 1, 4k-1, 2, 4k-2, \ldots, k-2, 3k+2, k-1, 3k-1, k+1, 3k, k+2, 3k-1, \ldots, 2k-2, 2k+2, 2k, 2k+1]$ with missing value $x = k$.

c) $[4k, 0, 4k-1, 1, 4k-2, 2, \ldots, k-2, 3k+1, k-1, 3k-1, k, 3k-2, \ldots, 2k+1, 2k-2, 2k, 2k-1]$ with missing value $x = 3k$.

d) $[0, 4k, 1, 4k-1, 2, 4k-2, \ldots, k-2, 3k+2, k-1, 3k+1, k, 3k-1, k+1, \ldots, 2k-2, 2k+1, 2k-1, 2k]$ with missing value $x = 3k$.

In Figure 3 one of the standard α-valuations of C_{12} has been shown:

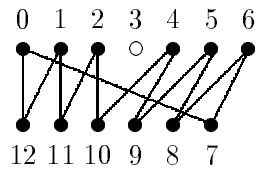

Figure 3: A standard α-valuation of C_{12}

If we suppose that $k_1 + k_2 + \cdots + k_n = k$ and there is an α-valuation for the graph $C_{4k_1} \cup C_{4k_2} \cup \cdots \cup C_{4k_n}$ then in a standard α-valuation of C_{4k}, we can replace C_{4k} by $C_{4k_1} \cup C_{4k_2} \cup \cdots \cup C_{4k_n}$ with its α-valuation and the resulting graph will again have an α-valuation. For example an α-valuation of C_{12} in Figure 3 is replaced by an α-valuation of $2C_6$ in Figure 4:

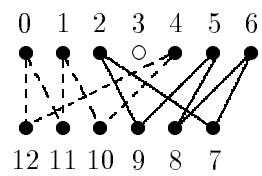

Figure 4: An α-valuation of $2C_6$

Definition 2: The graph C_{4k} has a standard valuation (or standard labeling) if the values of the vertices of C_{4k} can be generated from a standard α-valuation of C_{4k} differ by a constant factor.

For example C_{12} in the α-valuation of $C_{12} \cup C_{20}$ shown in Figure 5 has a standard valuation because it can be generated from a standard α-valuation of C_{12} that differs by a constant factor 10:

Figure 5: An α-valuation of $C_{12} \cup C_{20}$

If a graph C_{4k} has a standard valuation it can be replaced by any α-valuation of $C_{k_1} \cup C_{k_2} \cup \cdots \cup C_{k_n}$ where $k_1 + k_2 + \cdots + k_n = k$ by considering the constant factor. For instance the standard valuation of C_{12} in Figure 5 can be replaced by an α-valuation of $2C_6$ to form an α-valuation of $2C_6 \cup C_{20}$ if we increase the values of the α-valuation $2C_6$ in Figure 4 by constant factor i.e. 10:

Figure 6: An α-valuation of $C_{12} \cup C_{20}$

Theorem 2: *The following graphs have α-valuations:*

a) $\bigcup_{i=1}^{n} C_{4k_i} \bigcup Q(4,4k)$ if $k = \sum_{i=1}^{n} k_i$ and $k_n + k_{n-1} + \cdots + k_{i+2} + k_{i+1} \leq k_i$ for $i = 1, 2, 3, \ldots, n-1$.

b) $\bigcup_{i=1}^{n} 2C_{4k_i} \cup Q(3,4k)$ if $k = \sum_{i=1}^{n} k_i$ and $k_n + k_{n-1} + \cdots + k_{i+2} + k_{i+1} \leq k_i$ for $i = 1, 2, 3, \ldots, n-1$.

c) $\bigcup_{i=1}^{n} C_{4k_i} \cup \bigcup_{j=1}^{t} C_{4p_j} \cup Q(3,4k)$ if $k = \sum_{i=1}^{n} k_i = \sum_{j=1}^{t} p_j$ and $k_n + k_{n-1} + \cdots + k_{i+2} + k_{i+1} \leq k_i$ and $p_t + p_{t-1} + \cdots + p_{j+2} + p_{j+1} \leq p_j$ for $i = 1, 2, 3, \ldots, n-1$ and $j = 1, 2, \ldots, t-1$.

d) $\bigcup_{i=1}^{n} (C_{4k_i} \cup C_{4p_j}) \cup C_{4k_n} \cup Q(4,4k)$ if $k = k_n + \sum_{i=1}^{n} (k_i + p_i)$ and $k_i = 2k_{i+1} + p_{i+1}$ for $i = 1, 2, 3, \ldots, n-1$.

Proof: We know that in construction of α-valuation of Q(5,4k); at least two cycles C_{4k} have standard α-valuation. In order to obtain the different parts of the theorem 2, we replace these two standard valuations with other graphs as follows:

a) Consider one of the standard valuation of C_{4k}. First we replace it by $C_{4k_1} \cup C_{4l_1}$; $l_1 \leq k_1; k = k_1 + l_1$. Then since C_{4l_1} still has a standard

valuation [1], we are able to replace it again by $C_{4k_2} \cup C_{4l_2}$; $l_2 \leq k_2$; $l_1 = k_2 + l_2$. In next stages we continue to replace each C_{4l_i} by $C_{4k_{i+1}} \cup C_{4l_{i+1}}$; $l_{i+1} \leq k_{i+1}$; $l_i = k_{i+1} + l_{i+1}$ for $i = 2, 3, \ldots, n-2$; $k_n = l_{n-1}$.
b) We apply the replacement procedure of part (a) for both C_{4k} which have standard valuations in α-valuation of Q(5,4k).
c) The proof of this part is similar to part (b) except that each standard valuation C_{4k} has been replaced by different disjoint unions of graphs in such a way that their components are not necessarily isomorphic.
d) Consider one of the standard valuation of C_{4k}. First we replace it by $2C_{4k_1} \cup C_{4p_1}$; $k = p_1 + 2k_1$; we know at least one of C_{4k_1} has a standard valuation [3]. Thus we replace C_{4k_1} in the next step by $2C_{4k_2} \cup C_{4p_2}, k_1 = p_2 + 2k_2$. In next stages, we repeat the replacement C_{4k_i} by $2C_{4k_{i+1}} \cup C_{4p_{i+1}}, k_i = p_{i+1} + 2k_{i+1}$; $i = 1, 2, 3, \ldots, n-1$.

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