α-VALUATIONS OF SPECIAL CLASSES OF QUADRATIC GRAPHS

“To the memory of Jaromir Abrham (1931-1997)”

Kourosh Eshghi

Department of Industrial Engineering, Sharif University of Technology,

Tehran, Iran

eshghi@sharif.edu

Abstract: It is shown that the quadratic graph \(Q(5,4k)\) (consisting of 5 cycles of length 4k) has an α-valuation (a stronger form of the graceful valuation) for every positive integer \(k\). Furthermore, additional results are obtained from the main theorem of this paper.

1. BASIC DEFINITIONS

Let \(G = (V, E)\) be a graph with \(m = |V|\) vertices and \(n = |E|\) edges. By the term graph, we mean an undirected finite graph without loops or multiple edges. All parameters in this paper are positive integers. A graceful valuation (or \(β\)-valuation) of a graph \(G = (V, E)\) is a one-to-one mapping \(ψ\) of the vertex set \(V(G)\) into the set \(\{0, 1, 2, \ldots, n\}\) with

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this property: If we define, for any edge $e = \{u, v\} \in E(G)$, the value $\Psi^*(e) = |\Psi(u) - \Psi(v)|$ then $\Psi^*$ is a one-to-one mapping of the set $E(G)$ onto the set $\{1, 2, \ldots, n\}$.

A graph is called graceful if it has a graceful valuation. An $\alpha$-valuation (or $\alpha$-labeling) of a graph $G = (V, E)$ is a graceful valuation of $G$ which satisfies the following additional condition: There exists a number $\gamma(0 \leq \gamma \leq |E(G)|)$ such that, for any edge $e \in E(G)$ with end vertices $u, v \in V(G)$, $\min[\Psi(u), \Psi(v)] \leq \gamma < \max[\Psi(u), \Psi(v)]$.

The concept of a graceful valuation and of an $\alpha$-valuation were introduced by Rosa [7]. Rosa proved that, if $G$ is graceful and if all vertices of $G$ are of even degree, then $|E(G)| \equiv 0$ or $3$ (mod $4$). This implies that if $G$ has an $\alpha$-valuation and if all vertices of $G$ are of even degrees, then $|E(G)| \equiv 0$ (mod $4$) ($G$ is bipartite). In [7] it is also shown that these conditions are also sufficient if $G$ is a cycle. The symbol $C_m$ will denote a cycle on $m$ vertices. Abrham and Kotzig [2] proved that Rosa’s condition is also sufficient for 2-regular graphs with two components.

A snake is a tree with exactly two vertices of degree 1. In [7], it was proved that every snake has an $\alpha$-valuation. A snake with $n$ edges will be denoted by $P_n$.

A detailed history of the graph labeling problem and related results appears in Gallian [4,5]. One of the results of Abrham and Kotzig should be mentioned here: If $G$ is a 2-regular graph on $n$ vertices and $n$ edges which has a graceful valuation $\Psi$ then there exists exactly one number $x(0 < x < n)$ such that $\Psi(v) \neq x$ for all $v \in V(G)$; this number $x$ is referred to as the missing value of the graceful valuation[2].

A quadratic graph $Q(r, s)$ is a graph with $r$ components, each of which is an $s$-cycle.

Here are some of the results published in the references:

1. A $Q(1, s)$-graph (i.e. an $s$-cycle) is graceful if and only if $s \equiv 0$ or $3$ (mod $4$). It has an $\alpha$-valuation if and only if $s \equiv 0$ (mod $4$) [7].
2. A $Q(2, s)$-graph has an $\alpha$-valuation if and only if $s$ is even and $s > 2$ [6].

3. A $Q(3, 4k)$-graph has an $\alpha$-valuation for each $k > 1$. The $Q(3, 4)$-graph does not have an $\alpha$-valuation but it is graceful [6].

4. A $Q(r, 3)$-graph is graceful if and only if $r = 1$. A $Q(r, 5)$-graph is not graceful for any $r$ [6].

5. A $Q(r, 4)$-graph has an $\alpha$-valuation [6].

2. TRANSFORMATIONS OF LABELING OF A GRAPH

The transformations presented below are used extensively in this paper.

2.1 Transformation Type 1

Lemma 1: (Abrham & Kotzig [1]) Let $r$ be a non-negative integer and let $s$ be an odd integer, $s = 2k + 1$. Then $P_s$ has an $\alpha$-valuation $\Psi$ with endpoints labeled $w$ and $z$ that satisfies the conditions $z - w = k + 1$ and $w = r.(w.l.o.g., we assume that $w < z$.)

Suppose that we have two series of vertex labels as follows where $0 \leq n, n + k \leq 2r$ and $2r + 1 \leq m, m + k < 4r + 1$ and $|4r + 1 - (m + n + k)| \leq 1$:

\begin{align*}
0 & \quad \ldots \quad n \quad n + 1 \quad n + 2 \quad \ldots \quad n + k - 2 \quad n + k - 1 \quad n + k \quad \ldots \quad 2r \\
\circ & \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \\
4r + 1 & \quad \ldots \quad m + k \quad m + k - 1 \quad m + k - 2 \quad \ldots \quad m + 2 \quad m + 1 \quad m \quad \ldots \quad 2r + 1
\end{align*}

Figure 1: Arrangement of vertex labels in transformation type 1

We apply the transformation type 1 to the vertex labels ($n, n + 1, n + 2, \ldots, n + k - 2, n + k - 1, n + k$) and ($m, m + 1, m + 2, \ldots, m + k - 2, m + k - 1, m + k$) by choosing the vertices $w'$ and $z'$ as end points in the following steps:

Step1: First we modify the vertex labels as follows:
1) From each label \((n, n + 1, n + 2, \ldots, n + k - 2, n + k - 1, n + k)\)
subtract \(n\);

2) From each label \((m, m + 1, m + 2, \ldots, m + k - 2, m + k - 1, m + k)\)
subtract \(m - (k + 1)\)

Step 2: According to Lemma 1, we construct an \(\alpha\)-valuation for the snake
\(P_{2k+1}\) on the new labels, with end vertices having labels \(w\) and \(z\) such
that \(0 \leq w \leq k\) and \(k + 1 \leq z \leq 2k + 1\) and \(z - w = k + 1\).

Step 3: Now if we modify again the new values to the original values
in the following way:

1) Add \(n\) to each new label \((0, 1, 2, \ldots, k - 2, k - 1, k)\);

2) Add \(m - (k + 1)\) to each new value \((k + 1, k + 2, \ldots, 2k - 1, 2k, 2k + 1)\);

Then the end vertices of \(P_{2k+1}\) will be labeled \(w' = w + n\) and \(z' = z + m - (k + 1)\) and the edge values will be
\(m - n - k, m - n - k + 1, \ldots, m - n + k\).

2.2 Transformation Type 2

Lemma 2: (Abrham & Kotzig [1]) Let \(r\) be a non-negative integer and
let \(s\) be an even integer, \(s = 2k\). Then \(P_s\) has an \(\alpha\)-valuation \(\Psi\) with
endpoints labeled \(w\) and \(z\) that satisfies the conditions \(z + w = k\) and
\(w = r\). (w.l.o.g., we assume that \(w < z\).)

Suppose that we have two series of vertex labels as follows where \(0 \leq n, n + k \leq 2r\) and \(2r + 1 \leq m, m + k - 1 < 4r\) and \(|4r + 1 - (m + n + k)| \leq 1:\)

\[
\begin{array}{cccccccccccc}
0 & \ldots & n & n + 1 & n + 2 & \ldots & n + k - 2 & n + k - 1 & n + k & \ldots & 2r \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
4r & \ldots & m + k - 1 & m + k - 2 & \ldots & m + 2 & m + 1 & m & \ldots & 2r + 1
\end{array}
\]

Figure 2: Arrangement of vertex labels in transformation type 2
We apply the transformation type 2 to the vertex labels \((n, n + 1, n + 2, \ldots, n + k - 2, n + k - 1, n + k)\) and \((m, m + 1, m + 2, \ldots, m + k - 2, m + k - 1)\) by choosing the vertices \(w'\) and \(z'\) as end points in the following steps:

**Step 1:** First we modify the vertex labels as follows:

1) From each label \((n, n + 1, n + 2, \ldots, n + k - 2, n + k - 1, n + k)\) subtract \(n\);

2) From each label \((m, m + 1, m + 2, \ldots, m + k - 2, m + k - 1)\) subtract \(m - (k + 1)\)

**Step 2:** According to Lemma 2, we construct an \(\alpha\)-valuation for the snake \(P_{2k}\) on the new labels, with end vertices having labels \(w\) and \(z\) such that \(0 \leq w < (k/2) < z \leq k\) and \(z + w = k\).

**Step 3:** Now if we transform again the new values to the original values in the following way:

1) Add \(n\) to each new label \((0, 1, 2, \ldots, k - 2, k - 1, k)\);

2) Add \(m - (k + 1)\) to each new value \((k + 1, k + 2, \ldots, 2k - 1, 2k)\);

Then the end vertices of \(P_{2k}\) will be labeled \(w' = w + n\) and \(z' = z\) and the edge values will be \(m - n - k, m - n - k + 1, \ldots, m - n + k - 1\).

### 3. BASIC THEOREM

**Theorem 1:** The quadratic graph \(Q(5, 4k)\) has an \(\alpha\)-valuation for all \(k \geq 1\).

**Proof:** The missing value of the \(\alpha\)-valuation of this graph is \(5k\). Now let us assume \(k \geq 5\). The vertices of the first \(C_{4k}\) will be \([8k, 12k, 8k + 1, 12k - 1, 8k + 2, 12k - 2, \ldots, 9k - 1, 11k + 1, 9k + 1, 11k, 9k + 2, 11k - 1, \ldots, 10k - 1, 10k + 2, 10k, 10k + 1]\); this yields the edge values \(4k, 4k - 1, 4k - 2, 4k - 3, \ldots, 2k + 2, 2k + 1, 2k, \ldots, 3, 2, 1\). Next we will describe the labeling of the second \(C_{4k}\). The successive vertices will be labeled
as follows:

\[14k+1, 6k, 14k, 6k+1, \ldots, 13k+3, 7k-2, 13k+2, 7k-1, 13k, 7k, \ldots, 12k+2, 8k-2, 12k+1, 8k-1].\] The edge labels of this \(C_{4k}\) will be then \(8k+1, 8k, 8k-1, \ldots, 6k+5, 6k+4, 6k+3, 6k+2, 6k+1, 6k, \ldots, 4k+4, 4k+3, 4k+2.\)

The third \(C_{4k}\) will be labeled in three stages as follows:

1. Form the snake \((6k-1, 14k+2, 6k-2, 14k+3, 6k-3, \ldots, 5k+2, 15k-1, 5k+1, 15k, 5k-1, 15k+1, 5k-2, 15k+2, \ldots, 16k-3, 4k+2, 16k-2, 4k+1)\). The values of the edges are then \(8k+3, 8k+4, 8k+5, 8k+6, \ldots, 10k-3, 10k-2, 10k-1, 10k+1, \ldots, 12k-5, 12k-4, 12k-3.\)

2. Join the vertex labeled \(6k-1\) to the vertex labeled \(16k-1\) to generate the edge labeled \(10k.\)

3. Form another snake in such a way that its vertices are labeled as follows: \((16k-1, 4k-1, 16k+2, 4k-2, 16k, 4k+1)\). The resulting values of the edges of this snake are then \(12k-1, 12k, 12k+2, 12k+3, 12k+4.\)

Now we construct the fourth cycle \(C_{4k}\) according to the following stages:

a) The edge labels \(12k+1\) and \(12k+5\) are generated by joining the following pairs of vertices respectively: \(4k\) and \(16k+1; 4k-4\) and \(16k+1.\)

b) Apply transformation type 2 to the vertex labels \((3k-2, 3k-1, 3k, \ldots, 4k-5, 4k-4, 4k-3)\) and \((16k+3, 16k+4, 16k+5, \ldots, 17k, 17k+1)\) by choosing the two vertices \(3k-1\) and \(4k-4\) as end vertices. The corresponding edge values of this transformation will be \(12k+6, 12k+7, 12k+8, \ldots, 14k+2, 14k+3.\)
c) The edges labeled $14k + 4$ and $14k + 5$ are obtained by joining the following pairs of vertices respectively: $3k - 1$ and $17k + 3$; $4k$ and $18k + 5$.

d) Construct the snake $(17k + 3, 3k - 3, 17k + 4, 3k - 4, 17k + 5, 3k - 5, \ldots, 18k, 2k, 18k + 1, 2k - 1, 18k + 5)$. The corresponding edge labels will then be $14k + 6, 14k + 7, 14k + 8, \ldots, 16k, 16k + 1, 16k + 2, 16k + 6$.

Finally the last cycle $C_{4k}$ will be constructed as follows when $k \geq 14$ and $k \neq 22$:

1. The edges labeled $12k - 2, 4k + 1, 8k + 2$ and $16k + 3$ are obtained by connecting the following pairs of vertices respectively: $k + 3$ and $13k + 1; 9k$ and $13k + 1; 9k$ and $17k + 2; k - 1$ and $17k + 2$.

2. Construct the snake $(18k + 2, 2k - 2, 18k + 3, 2k - 4, 18k + 4, 2k - 5, 18k + 6, 2k - 6, 18k + 7, 2k - 3)$. This yields the edge labels $16k + 4, 16k + 5, 16k + 7, \ldots, 16k + 13$.

3. The edge labeled $16k + 14$ is obtained by joining the two vertex labels $2k - 3$ and $18k + 11$ together.

4. Apply transformation type 1 to the vertex labels $(k + 4, k + 5, \ldots, 2k - 8, 2k - 7)$ and $(18k + 8, 18k + 9, 18k + 10, 18k + 11, \ldots, 19k - 4, 19k - 3)$ by using the two vertices $k + 7$ and $18k + 11$ as endpoints. This transformation generates the edge labels $16k + 15, 16k + 16, \ldots, 18k - 8, 18k - 7$.

5. Connect the following pairs of vertices to each other to obtain the edges labeled $18k - 6$ and $18k - 5$ respectively: $8$ and $18k + 2; k + 7$ and $19k + 2$.

6. Construct the snake $(19k - 2, k + 2, 19k - 1, k + 1, 19k, k, 19k + 1, k - 1)$. The resulting values of the edges are then $18k - 4, 18k - 3, \ldots, 18k + 2$. 
7. The edge labeled $18k + 3$ is obtained by joining the two vertices $k + 3$ and $19k + 6$.

8. Construct the snake $(19k + 2, k - 2, 19k + 3, k - 3, 19k + 4, k - 4, 19k + 5, k - 5, 19k + 6)$. The edge labels $18k + 4, 18k + 5, 18k + 6, \ldots, 18k + 11$ are generated by this snake.

9. Connect the two vertices $k - 14$ and $19k - 2$ to each other to generate the edge label $18k + 12$.

10. Finally apply transformation type 2 to the vertex labels $(0, 1, 2, \ldots, 8, \ldots, k - 14, \ldots, k - 6)$ and $(19k + 7, 19k + 8, \ldots, 20k - 1, 20k)$ by considering the two vertices 8 and $k - 14$ as end vertices. The rest of the edge values will be generated by this transformation and the last $C_{4k}$ will be completed.

The construction of an $\alpha$-valuation of the last $C_{4k}$ when $5 \leq k \leq 13$ or $k = 22$ has been given in the Table 1 as follows:
\( \alpha \)-valuations of special classes of ...

<table>
<thead>
<tr>
<th>( k )</th>
<th>The construction of the fifth cycle ( C_{4k} ) in the graph ( Q(5,4k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>([45, 66, 8, 92, 7, 94, 6, 96, 5, 97, 3, 98, 2, 99, 1, 100, 0, 93, 4, 87])</td>
</tr>
<tr>
<td>6</td>
<td>([54, 79, 9, 120, 0, 119, 1, 118, 2, 117, 3, 116, 4, 110, 10, 111, 8, 112, 7, 114, 6, 115, 5, 104])</td>
</tr>
<tr>
<td>7</td>
<td>([63, 92, 10, 129, 12, 128, 8, 130, 5, 135, 4, 136, 3, 137, 2, 138, 1, 139, 0, 140, 11, 132, 9, 133, 7, 134, 6, 121])</td>
</tr>
<tr>
<td>8</td>
<td>([72, 105, 11, 160, 0, 159, 1, 158, 2, 157, 3, 156, 4, 155, 5, 151, 9, 152, 8, 153, 6, 154, 13, 150, 10, 148, 12, 147, 14, 146, 7, 138])</td>
</tr>
<tr>
<td>9</td>
<td>([81, 118, 12, 176, 4, 175, 5, 178, 2, 177, 3, 180, 0, 179, 1, 170, 10, 171, 9, 172, 7, 173, 6, 174, 15, 169, 11, 168, 13, 166, 14, 165, 16, 164, 8, 155])</td>
</tr>
<tr>
<td>10</td>
<td>([90, 131, 13, 197, 3, 196, 4, 199, 2, 198, 0, 200, 1, 192, 17, 187, 14, 186, 15, 184, 16, 183, 18, 182, 8, 193, 7, 194, 6, 195, 5, 188, 12, 189, 11, 190, 10, 191, 9, 172])</td>
</tr>
<tr>
<td>11</td>
<td>([99, 144, 14, 218, 2, 219, 1, 220, 0, 215, 5, 216, 4, 217, 3, 212, 7, 213, 6, 214, 15, 206, 13, 207, 12, 208, 11, 209, 19, 205, 16, 204, 17, 202, 18, 201, 20, 200, 8, 211, 9, 210, 10, 189])</td>
</tr>
<tr>
<td>12</td>
<td>([108, 157, 15, 230, 6, 234, 7, 233, 8, 218, 22, 219, 20, 220, 19, 222, 18, 223, 21, 227, 13, 226, 14, 225, 16, 224, 17, 237, 3, 236, 4, 235, 5, 240, 0, 239, 1, 238, 2, 231, 10, 232, 9, 228, 12, 229, 11, 206])</td>
</tr>
<tr>
<td>13</td>
<td>([117, 170, 16, 256, 4, 255, 5, 258, 2, 257, 3, 260, 0, 259, 1, 250, 19, 242, 18, 243, 17, 244, 15, 245, 23, 241, 20, 240, 21, 238, 22, 237, 24, 236, 8, 252, 7, 253, 6, 254, 11, 249, 10, 251, 9, 246, 14, 247, 13, 248, 12, 223])</td>
</tr>
</tbody>
</table>

**Table 1:** The construction of the fifth cycle \( C_{4k} \) in the graph \( Q(5,4k) \), 

\[ 5 \leq k \leq 13, k = 22 \]
For $1 \leq k \leq 4$, the successive vertices of each cycle of $Q(5,4k)$ will be labeled according to the following table:

<table>
<thead>
<tr>
<th>k</th>
<th>The construction of an $\alpha$-valuation of $Q(5,4k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[0, 18, 1, 20], [2, 16, 4, 17], [3, 14, 9, 19], [6, 13, 7, 15], [8, 11, 10, 12]</td>
</tr>
<tr>
<td>2</td>
<td>[0, 37, 4, 33, 8, 39, 1, 40], [2, 36, 18, 27, 5, 35, 3, 38], [6, 34, 7, 31, 11, 30, 9, 32], [12, 29, 15, 25, 14, 26, 13, 28], [16, 24, 17, 23, 19, 22, 20, 21]</td>
</tr>
<tr>
<td>3</td>
<td>[0, 60, 4, 53, 27, 40, 6, 54, 7, 58, 1, 59], [2, 57, 12, 49, 8, 52, 9, 51, 5, 55, 3, 56], [10, 50, 11, 47, 17, 44, 16, 45, 14, 46, 13, 48], [18, 43, 23, 37, 22, 38, 21, 39, 20, 41, 19, 42], [24, 36, 25, 35, 26, 34, 28, 33, 29, 32, 30, 31]</td>
</tr>
<tr>
<td>4</td>
<td>[0, 80, 5, 74, 4, 70, 36, 53, 7, 75, 3, 76, 2, 78, 1, 79], [6, 77, 16, 65, 12, 68, 13, 67, 10, 69, 11, 71, 9, 72, 8, 73], [14, 66, 15, 63, 23, 58, 22, 59, 21, 60, 19, 61, 18, 62, 17, 64], [24, 57, 31, 49, 30, 50, 29, 51, 28, 52, 27, 54, 26, 55, 25, 56], [32, 48, 33, 47, 34, 46, 35, 45, 37, 44, 38, 43, 39, 42, 40, 41]</td>
</tr>
</tbody>
</table>

Table 2: The construction of an $\alpha$-valuation of $Q(5,4k)$ for $1 \leq k \leq 4$

4. THE STANDARD VALUATIONS OF $C_{4k}$

Definition 1: The standard $\alpha$-valuation of $C_{4k}$ are given by any of the following sequence of values of the consecutive vertices of $C_{4k}$:

a) $[4k, 0, 4k - 1, 1, 4k - 2, 2, \ldots, k - 2, 3k + 1, k - 1, 3k, k + 1, 3k - 1, k + 2, 3k - 2, \ldots, 2k + 2, 2k - 1, 2k + 1, 2k]$ with missing value $x = k$.

b) $[0, 4k, 1, 4k - 1, 2, 4k - 2, \ldots, k - 2, 3k + 2, k - 1, 3k - 1, k + 1, 3k, k + 2, 3k - 1, \ldots, 2k - 2, 2k + 2, 2k, 2k + 1]$ with missing value $x = k$.

c) $[4k, 0, 4k - 1, 1, 4k - 2, 2, \ldots, k - 2, 3k + 1, k - 1, 3k - 1, k, 3k - 2, \ldots, 2k + 1, 2k - 2, 2k, 2k - 1]$ with missing value $x = 3k$.

d) $[0, 4k, 1, 4k - 1, 2, 4k - 2, \ldots, k - 2, 3k + 2, k - 1, 3k + 1, k, 3k - 1, k + 1, \ldots, 2k - 2, 2k + 1, 2k - 1, 2k]$ with missing value $x = 3k$. 
In Figure 3 one of the standard $\alpha$-valuations of $C_{12}$ has been shown:

![Diagram of $C_{12}$ with a specific valuation]

**Figure 3:** A standard $\alpha$-valuation of $C_{12}$

If we suppose that $k_1 + k_2 + \cdots + k_n = k$ and there is an $\alpha$-valuation for the graph $C_{4k_1} \cup C_{4k_2} \cup \cdots \cup C_{4k_n}$ then in a standard $\alpha$-valuation of $C_{4k}$, we can replace $C_{4k}$ by $C_{4k_1} \cup C_{4k_2} \cup \cdots \cup C_{4k_n}$ with its $\alpha$-valuation and the resulting graph will again have an $\alpha$-valuation. For example an $\alpha$-valuation of $C_{12}$ in Figure 3 is replaced by an $\alpha$-valuation of $2C_6$ in Figure 4:

![Diagram of $2C_6$ with a specific valuation]

**Figure 4:** An $\alpha$-valuation of $2C_6$

**Definition 2:** The graph $C_{4k}$ has a standard valuation (or standard labeling) if the values of the vertices of $C_{4k}$ can be generated from a standard $\alpha$-valuation of $C_{4k}$ differ by a constant factor.

For example $C_{12}$ in the $\alpha$-valuation of $C_{12} \cup C_{28}$ shown in Figure 5 has a standard valuation because it can be generated from a standard $\alpha$-valuation of $C_{12}$ that differs by a constant factor 10:

![Diagram of $C_{12} \cup C_{28}$ with a specific valuation]

**Figure 5:** An $\alpha$-valuation of $C_{12} \cup C_{28}$
If a graph $C_{4k}$ has a standard valuation it can be replaced by any $\alpha$-valuation of $C_{k_1} \cup C_{k_2} \cup \cdots \cup C_{k_n}$ where $k_1 + k_2 + \cdots + k_n = k$ by considering the constant factor. For instance the standard valuation of $C_{12}$ in Figure 5 can be replaced by an $\alpha$-valuation of $2C_6$ to form an $\alpha$-valuation of $2C_6 \cup C_{28}$ if we increase the values of the $\alpha$-valuation $2C_6$ in Figure 4 by constant factor i.e. 10:

\textbf{Theorem 2:} The following graphs have $\alpha$-valuations:

a) $\bigcup_{i=1}^{n} C_{4k_i} \cup Q(4,4k)$ if $k = \sum_{i=1}^{n} k_i$ and $k_n + k_{n-1} + \cdots + k_{i+2} + k_{i+1} \leq k_i$ for $i = 1, 2, 3, \ldots, n - 1$.

b) $\bigcup_{i=1}^{n} 2C_{4k_i} \cup Q(3,4k)$ if $k = \sum_{i=1}^{n} k_i$ and $k_n + k_{n-1} + \cdots + k_{i+2} + k_{i+1} \leq k_i$ for $i = 1, 2, 3, \ldots, n - 1$.

c) $\bigcup_{i=1}^{n} C_{4k_i} \cup \bigcup_{j=1}^{t} C_{4p_j} \cup Q(3,4k)$ if $k = \sum_{i=1}^{n} k_i = \sum_{j=1}^{t} p_j$ and $k_n + k_{n-1} + \cdots + k_{i+2} + k_{i+1} \leq k_i$ and $p_{i} + p_{i-1} + \cdots + p_{j+3} + p_{j+1} \leq p_j$ for $i = 1, 2, 3, \ldots, n - 1$ and $j = 1, 2, \ldots, t - 1$.

d) $\bigcup_{i=1}^{n} (C_{4k_i} \cup C_{4p_j}) \cup C_{4k_n} \cup Q(4,4k)$ if $k = k_n + \sum_{i=1}^{n} (k_i + p_i)$ and $k_i = 2k_{i+1} + p_{i+1}$ for $i = 1, 2, 3, \ldots, n - 1$.

\textbf{Proof:} We know that in construction of $\alpha$-valuation of $Q(5,4k)$; at least two cycles $C_{4k}$ have standard $\alpha$-valuation. In order to obtain the different parts of the theorem 2, we replace these two standard valuations with other graphs as follows:

a) Consider one of the standard valuation of $C_{4k}$. First we replace it by $C_{4k_i} \cup C_{4k_i}; l_1 \leq k_1; k = k_1 + l_1$. Then since $C_{4k_i}$ still has a standard
valuation [1], we are able to replace it again by $C_{4k_2} \cup C_{4l_2}; \ l_2 \leq k_2; \ l_1 = k_2 + l_2$. In next stages we continue to replace each $C_{4l_i}$ by $C_{4k_{i+1}} \cup C_{4l_{i+1}}; \ k_{i+1} \leq l_{i+1}; \ l_i = k_{i+1} + l_{i+1}$ for $i = 2, 3, \ldots, n - 2; \ k_n = l_{n-1}$.

b) We apply the replacement procedure of part (a) for both $C_{4k}$ which have standard valuations in $\alpha$-valuation of $Q(5,4k)$.

c) The proof of this part is similar to part (b) except that each standard valuation $C_{4k}$ has been replaced by different disjoint unions of graphs in such a way that their components are not necessarily isomorphic.

d) Consider one of the standard valuation of $C_{4k}$. First we replace it by $2C_{4k_1} \cup C_{4p_1}; \ k_1 = p_1 + 2k_1$; we know at least one of $C_{4k_1}$ has a standard valuation [3]. Thus we replace $C_{4k_1}$ in the next step by $2C_{4k_2} \cup C_{4p_2}, k_1 = p_2 + 2k_2$. In next stages, we repeat the replacement $C_{4k_i}$ by $2C_{4k_{i+1}} \cup C_{4p_{i+1}}, k_i = p_{i+1} + 2k_{i+1}$; $i = 1, 2, 3, \ldots, n - 1$.

References


