EXTENDING FUNCTIONS IN THE MODEL SUBSPACES OF $H^2(\mathbb{R})$ TO $\mathbb{C}$

Javad Mashreghi

Département de mathématiques et de statistique Université Laval Québec,
QC Canada G1K 7P4.
Javad.Mashreghi@mat.ulaval.ca

Abstract: It is shown that each function $f$ in a model subspace $K_\Theta$ of $H^2(\mathbb{R})$ can be extended to $\mathbb{C}$. The extension to the upper half plane is in $H^2(\mathbb{C}_+)$ and the extension to the lower half plane is in $\Theta H^2(\mathbb{C}_-)$. We also show that $f$ is analytic at each point of the real line where $\Theta$ is analytic. Finally, we completely characterize $K_\Theta$ for $\Theta(x) = e^{ix}$ and for $\Theta$ being a meromorphic Blaschke product.

1. Introduction

Let $f$ be an analytic function in the upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$. Let $\|f\|_p = \left( \int_{\infty}^{\infty} |f(x+iy)|^p \, dx \right)^{1/p}$ and $\|f\|_p = \sup_{y>0} \|f_y\|_p$ for $0 < p < \infty$. The Hardy space $H^p(\mathbb{C}_+)$ consists of all $f$’s with

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The Hardy space $H^\infty(\mathbb{C}_+)$ consists of all bounded analytic functions in the upper half plane. In this case $\|f\|_\infty = \sup_{z \in \mathbb{C}_+} |f(z)|$. For $0 < p < 1$, $H^p(\mathbb{C}_+)$ with the distance $\|f - g\|_p^p$ is a complete metric space. For $1 \leq p < \infty$, $H^p(\mathbb{C}_+)$, $\| \cdot \|_p$ is a Banach space. In particular, $H^2(\mathbb{C}_+)$, $\| \cdot \|_2$ is a Hilbert space. Finally, $H^\infty(\mathbb{C}_+)$ is a Banach algebra [9, pages 70-74].

For each $f \in H^p(\mathbb{C}_+)$, and for almost all $x \in \mathbb{R}$, $\lim_{z \to x, z \in \mathbb{C}_+} f(z)$ exists. Denoting this limit by $f(x)$, we have $f \in L^p(\mathbb{R})$, and furthermore, $\|f\|_p = \|f\|_{L^p(\mathbb{R})}$. In the preceding limit, $z$ is required to tend to $x$ from within sectors of opening $< \theta$ having vertex at $x$, and symmetric about the vertical line passing through $x$. We frequently say that $f(z) \to f(x)$ as $z$ tends to $x$ non-tangentially [4, page 6].

Therefore, there is a canonical correspondence between $H^p(\mathbb{C}_+)$ and a subspace of $L^p(\mathbb{R})$, denoted by $H^p(\mathbb{R})$. The space $H^p(\mathbb{R})$ can also be independently defined as the set of all $f \in L^p(\mathbb{R})$, with (as a distribution) $\hat{f}(x) = 0$ for $x < 0$. The two definitions are equivalent [7, page 172]. The Hardy spaces $H^p(\mathbb{C}_-)$ are defined similarly. The functions in $H^p(\mathbb{C}_-)$ live in the lower half plane and the family of their boundary values, as functions on $\mathbb{R}$, is precisely the space $\overline{H^p(\mathbb{R})}$. See also Chapter 11 of [4].

The function $\Theta \in H^\infty(\mathbb{R})$ is said to be inner if $|\Theta(x)| = 1$ for almost all $x \in \mathbb{R}$. For each inner function $\Theta$, the set $\Theta H^2(\mathbb{R})$ is a closed subspace of the Hilbert space $H^2(\mathbb{R})$ [9, pages 79-80]. Now we are able to introduce our hero.

**Definition** The model space $K_\Theta$ is the orthogonal complement of
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$\Theta H^2(\mathbb{R})$ in $H^2(\mathbb{R})$.

In this paper we study the model space $K_\Theta$ corresponding to the inner function $\Theta$. In Section 2 we briefly discuss the role of model subspaces in operator theory. In Section 3 an analytic description of $K_\Theta$ is given. This new formulation, which is obtained using Hilbert space characteristic of $H^2(\mathbb{R})$, can be exploited to define $K_\Theta$ in the Hardy space $H^2(\mathbb{R})$. This representation also enables us to extend each function in $K_\Theta$ to the whole complex plane. In Section 4 we extend an $f \in K_\Theta$ to $\mathbb{C}$. This extension has three fundamental properties. First, $\lim_{z \to x} f(z) = f(x)$ for almost all $x \in \mathbb{R}$. In these limit, $z$ is allowed to tend to $x$ non-tangentially from either half plane. Second, $f$ as a function defined in the upper half plane is in $H^2(\mathbb{C}_+)$, Third, $f$ as a function defined in the lower half plane is in $\Theta H^2(\mathbb{C}_-)$. Therefore, $f$ is at least analytic in the upper half plane and is meromorphic in the lower half plane. Furthermore, in Section 5 we show that $f$ is already analytic wherever $\Theta$ is on the real line. Finally in Sections 6 and 7 we completely characterize $K_\Theta$ corresponding to $\Theta(x) = e^{ix}$ and for $\Theta$ being a meromorphic Blaschke product. In these two cases, and only for them, each $f \in K_\Theta$ is analytic on the whole real line.

2. Link to operator theory

In this section we explain the origin of model subspaces of $H^2(\mathbb{R})$. Let $f \in H^2(\mathbb{R})$. By the Fourier-Plancherel theorem, if we write

$$\hat{f}_N(\lambda) = \int_{-N}^{N} e^{-it\lambda} f(t) \, dt,$$

then, as $N \to \infty$, the $\hat{f}_N(\lambda)$ tend in $L^2(\mathbb{R})$ to a function $\hat{f}(\lambda)$, called the Fourier-Plancherel transform of $f$. We can characterize an $f \in H^2(\mathbb{R})$ in terms of its Fourier-Plancherel transform. A function $f \in L^2(\mathbb{R})$ is in $H^2(\mathbb{R})$ if and only if $\hat{f}(\lambda) = 0$ for almost every $\lambda < 0$ [9, page 131]. Therefore, there is a canonical isomorphism between $H^2(\mathbb{R})$ and
$L^2((0, \infty))$. Based on the preceding observation, a function $f \in L^2(\mathbb{R})$ is in $H^2(\mathbb{R})$ if and only if $\hat{f}(\lambda) = 0$ for almost every $\lambda > 0$.

Let $\delta > 0$. Then the map $T_\delta$

$$H^2(\mathbb{R}) \mapsto H^2(\mathbb{R})$$

$$f(t) \mapsto \exp(i\delta t) f(t),$$

is called a forward shift operator on $H^2(\mathbb{R})$. Since for each $f \in H^2(\mathbb{R})$

$$\widehat{T_\delta (f)}(\lambda) = \hat{f}(\lambda - \delta), \quad \lambda \in \mathbb{R},$$

$T_\delta$ shifts the spectrum of $f$ forward by $\delta$ units. Beurling in his classical paper [2] characterized the invariant subspaces of $H^2(\mathbb{R})$ for the forward shift operators.

**Beurling’s theorem:** A closed subspace of $H^2(\mathbb{R})$ is invariant under $T_\delta$, for each $\delta > 0$, if and only if it is of the form $\Theta H^2(\mathbb{R})$ for some inner function $\Theta$.

The adjoint of a forward shift operator, $T_\delta^*$, is called a backward shift operator. By direct verification, one verifies that $T_\delta^*$ is defined by

$$\widehat{T_\delta^* (f)}(\lambda) = \begin{cases} f(\lambda + \delta), & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda < 0 \end{cases}$$

for $f \in H^2(\mathbb{R})$ [6]. Therefore, $T_\delta^*$ shifts the spectrum of $f$ backward by $\delta$ units, and then chops off the negative part of what is thus obtained. In a Hilbert space, a closed subspace $M$ is invariant under a bounded operator $T$ if and only if $M^\perp$ is invariant under $T^*$ [5, page 40]. Therefore according to the Beurling’s theorem, A closed subspace of $H^2(\mathbb{R})$ is invariant under $T_\delta^*$ for each $\delta > 0$ if and only if it is the orthogonal complement of $\Theta H^2(\mathbb{R})$ for some inner function $\Theta$. Therefore, the subspaces $K_\Theta$ are precisely those which are invariant under $T_\delta^*$ for each $\delta > 0$. That is why some authors call the $K_\Theta$ a coinvariant subspace of $H^2(\mathbb{R})$. 
3. Analytic description of $K_\Theta$

Let $\Theta$ be an inner function for the upper half plane. Then $\Theta H^2(\mathbb{R})$ is a closed subspace of the Hilbert space $H^2(\mathbb{R})$. According to the notation introduced before, the orthogonal complement of $\Theta H^2(\mathbb{R})$ in $H^2(\mathbb{R})$ is denoted by $K_\Theta$. The following lemma gives an analytic description of $K_\Theta$ which can be used as the definition of it in all Hardy spaces $H^p(\mathbb{R})$, $0 < p \leq \infty$.

**Theorem 3.1.** For each inner function $\Theta$

\[ K_\Theta = H^2(\mathbb{R}) \cap \Theta \overline{H^2(\mathbb{R})}. \]

**Proof.** Uses the properties $\Theta \in H^\infty$ and $\Theta \overline{\Theta} = 1$. By definition, $f \in K_\Theta$ if and only if $f \in H^2(\mathbb{R})$ and

\[ \int_{-\infty}^{\infty} f(x) \overline{\Theta(x)} g(x) \, dx = 0 \]

for each $g \in H^2(\mathbb{R})$. Thus, $f \in K_\Theta$ if and only if $f \in H^2(\mathbb{R})$

\[ \int_{-\infty}^{\infty} \frac{f(x)}{\Theta(x)} \overline{g(x)} \, dx = 0 \]

for each $g \in H^2(\mathbb{R})$. This condition is equivalent to $\frac{f}{\Theta} \in \overline{H^2(\mathbb{R})}$. Therefore $f \in K_\Theta$ if and only if $f \in H^2(\mathbb{R})$ and also $f \in \Theta \overline{H^2(\mathbb{R})}$. \[ \square \]

4. Extension to upper and lower half planes

Let $h \in L^2(\mathbb{R})$. Then the Poisson integral formula

\[ P_h(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|z-t|^2}{|z-t|^2} h(t) \, dt, \quad z \in \mathbb{C} \setminus \mathbb{R}, \]

gives an extension of $h$ to the upper and to the lower half planes. It can be shown that $h \in H^2(\mathbb{R})$ if and only if $P_h$, as a function defined in the upper half plane, is in $H^2(\mathbb{C}_+)$ Similarly, $h \in \overline{H^2(\mathbb{R})}$ if and only if
$P_b$, as a function defined in the lower half plane, is in $H^2(\mathbb{C}_-)$ [12]. An $f \in K_\Theta$ belongs in particular to $H^2(\mathbb{R})$. Therefore it has an extension $f(z)$ to the upper half plane, belonging to $H^2(\mathbb{C}_+)$ and given there by the formula

$$f(z) = P_f(z) \text{ for } z \in \mathbb{C}_+.$$  

An inner function $\Theta$ can be (formally) extended to the lower half plane by putting

$$\Theta(z) = \frac{1}{\Theta(\overline{z})}$$

for $z \in \mathbb{C}_-$. The extension of an $f \in K_\Theta$ to the lower half plane is indirect (depending on $\Theta$). For such an $f$ we have $\overline{\Theta} f \in H^2(\mathbb{R})$ by Theorem 3.1, so, by the preceding observation, $\overline{\Theta} f$ has an analytic extension to the lower half plane, equal there to

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|3z|}{|z-t|^2} \overline{\Theta(t)} f(t) \, dt = P_{\overline{\Theta} f}(z), \quad z \in \mathbb{C}_-.$$  

We then define the extension of $f \in K_\Theta$ to $\mathbb{C}_-$ by putting

$$f(z) = \Theta(z) P_{\overline{\Theta} f}(z) \text{ for } z \in \mathbb{C}_-,$$

with $\Theta(z)$ defined as above in $\mathbb{C}_-$. This extension is at least meromorphic in the lower half plane.

**Remark:** We have $\lim_{z \to x} \Theta(z) = \Theta(x)$ and $\lim_{z \to x} f(z) = f(x)$ for almost all $x \in \mathbb{R}$. In these limits, $z$ is allowed to tend to $x$ non-tangentially from either half plane.

With above definitions, Theorem 3.1 yields the following characterization of $K_\Theta$.

**Theorem 4.1.** The space $K_\Theta$ consists precisely of the functions $f \in L^2(\mathbb{R})$ with extension to the upper half plane belonging to $H^2(\mathbb{C}_+)$ and whose extension to the lower half plane makes $\frac{f}{\Theta} \in H^2(\mathbb{C}_-)$.

For further applications of this result see [8].
5. Analytic continuation along $\mathbb{R}$

A function $f \in K_\Theta$ can be continued analytically across intervals of $\mathbb{R}$ on which $\Theta$ is analytic. This result has important consequences in characterizing elements of $K_B$ when $B$ is a meromorphic Blaschke product.

**Theorem 5.1.** If $\Theta$ is analytic in a neighborhood of the interval $(a, b) \subset \mathbb{R}$ then any $f \in K_\Theta$ is also analytic there.

**Proof.** By Theorem 4.1, $f$ and $\frac{f}{\Theta}$ are respectively holomorphic in the upper and lower half planes. Without loss of generality, suppose $\Theta$ is holomorphic inside the rectangle $\{ z; a < Rz < b, -2 < \Im z < 2 \}$. Thus $f = \Theta \cdot \frac{f}{\Theta}$ is also analytic inside that rectangle except possibly on $(\alpha, \beta)$, and for almost all $x \in \mathbb{R}$, $\lim_{|y| \to 0} f(x + iy)$ exists. Choose $\alpha, \beta \in (a, b)$ such that this is true for $x = \alpha$ and for $x = \beta$.

With $\varepsilon > 0$, let us take the paths

$\Gamma = [\alpha + i, \alpha - i] \cup [\alpha - i, \beta - i] \cup [\beta - i, \beta + i] \cup [\beta + i, \alpha + i],$

$\Gamma_z = [\alpha + i\varepsilon, \alpha - i\varepsilon] \cup [\alpha - i\varepsilon, \beta - i\varepsilon] \cup [\beta - i\varepsilon, \beta + i\varepsilon] \cup [\beta + i\varepsilon, \alpha + i\varepsilon],$

each oriented counterclockwise. For each point $z$ inside $\Gamma$, let $g(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$. Then $g$ is holomorphic inside $\Gamma$. By the Cauchy integral formula,

$g(z) = f(z) + \frac{1}{2\pi i} \int_{\Gamma_z} \frac{f(\zeta)}{\zeta - z} d\zeta$

for $\varepsilon < |\Re z| < 2$. Since $f$ is bounded on the vertical segments through
\[ \alpha \text{ and } \beta, \]
\[ \lim_{\varepsilon \to 0} \int_{[\alpha + i\varepsilon, \alpha - i\varepsilon]} \frac{f(\zeta)}{\zeta - z} \, d\zeta = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_{[\beta - i\varepsilon, \beta + i\varepsilon]} \frac{f(\zeta)}{\zeta - z} \, d\zeta = 0. \]

On the horizontal segments
\[ \int_{[\alpha - i\varepsilon, \beta - i\varepsilon]} \frac{f(\zeta)}{\zeta - z} \, d\zeta + \int_{[\beta + i\varepsilon, \alpha + i\varepsilon]} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \int_{\alpha}^{\beta} \left( \frac{f(t - i\varepsilon)}{t - i\varepsilon - z} - \frac{f(t + i\varepsilon)}{t + i\varepsilon - z} \right) \, dt. \]

Both \( \frac{f(t - i\varepsilon)}{t - i\varepsilon - z} \) and \( \frac{f(t + i\varepsilon)}{t + i\varepsilon - z} \) converge in \( L^2(dt) \) norm to \( \frac{f(t)}{t - z} \), as \( \varepsilon \to 0 \). Thus
\[ \lim_{\varepsilon \to 0} \int_{[\alpha - i\varepsilon, \beta - i\varepsilon]} \frac{f(\zeta)}{\zeta - z} \, d\zeta + \int_{[\beta + i\varepsilon, \alpha + i\varepsilon]} \frac{f(\zeta)}{\zeta - z} \, d\zeta = 0. \]

Hence \( g \equiv f \) in the lower and in the upper part of the interior of \( \Gamma \). Therefore, \( f \) is holomorphic on \( (\alpha, \beta) \). Since \( \alpha \) and \( \beta \) can be taken as close to \( a \) and to \( b \) as we want, \( f \) is holomorphic on \( (a, b) \). \( \blacksquare \)

6. Paley-Wiener spaces as model subspaces

Let \( \sigma > 0 \). Then, \( \Theta(x) = \exp(i \sigma x) \) is an entire inner function. In this case, the functions \( f(x) \in K_\Theta \) differ by the factor \( e^{i\sigma x^2/2} \) from those in a Paley-Wiener space.

**Theorem 6.1.** Let \( \sigma > 0 \). Then \( f \in K_{e^{i\sigma x}} \) if and only if \( f \) is an entire function of exponential type, square integrable on the real line, with

\[ -\sigma \leq \limsup_{y \to +\infty} \log \frac{|f(iy)|}{y} \leq 0 \quad \text{and} \quad 0 \leq \limsup_{y \to -\infty} \frac{\log |f(iy)|}{|y|} \leq \sigma. \]

**Proof.** Since \( \Theta(x) = \exp(i \sigma x) \) is analytic across \( \mathbb{R} \), each \( f \in K_{e^{i\sigma x}} \) is also analytic there. Furthermore, \( f \in H^2(\mathbb{C}_+) \) and \( \frac{f}{\Theta} \in H^2(\mathbb{C}_-) \) imply that \( f \) is analytic on \( \mathbb{C}_+ \) and also on \( \mathbb{C}_- \), that \( f \in L^2(\mathbb{R}) \), and besides that the support of the Fourier-Plancherel transform of \( f \) is a subset of \([0, \sigma]\). Thus \( \hat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \), and for each \( z = x \in \mathbb{R} \),
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\[ f(z) = \int_0^\sigma \hat{f}(t) e^{itz} \, dt. \]  

By the uniqueness theorem for analytic functions, equality holds everywhere. Therefore \( f \) is an entire function of exponential type with the indicated growth conditions on the imaginary axis. The if part is an easy consequence of the celebrated Paley-Wiener theorem.

The following Corollary is an immediate consequence of the Paley-Wiener representation of entire functions of exponential type and the preceding theorem. The indicated representation, by itself, shows that each \( f \in K_{\sigma, \infty} \) is an entire function of exponential type which is square integrable on the real line. The representation, moreover, restricts the rate of growth of \( f \) along the imaginary axis.

**Corollary 6.2.** Each \( f \in K_{\sigma, \infty} \) has the representation

\[ f(z) = \int_0^\sigma \hat{f}(t) e^{itz} \, dt \]

, where \( \hat{f} \in L^2(0, \sigma) \).

7. The model space \( K_B \)

Let \( \{z_k\}_{k \geq 1} \) be a sequence of complex numbers in the upper half plane \( \mathbb{C}_+ \). Let \( b_k(z) = e^{i\alpha_k} \cdot \frac{z - z_k}{z - \overline{z}_k} \), where \( \alpha_k \) is so chosen that \( e^{i\alpha_k} \cdot \frac{i - z_k}{i - \overline{z}_k} \geq 0 \). The rational function \( B_K = \prod_{k=1}^K b_k \) is called a *finite Blaschke product* for the upper half plane; \( B_K \) is analytic at each point of the real line and \( |B_K(x)| = 1 \) for \( x \in \mathbb{R} \). The relation \( \sum_{k=1}^{\infty} \frac{3z_k}{|z_k + i|^2} < \infty \) is a necessary and sufficient condition for the uniform convergence of \( B_K \) on compact sets, disjoint from the closure of \( \{z_k; k \geq 1\} \), to a non-zero analytic function \( B = \prod_{k=1}^{\infty} b_k = \lim_{K \to \infty} B_K \), and we call \( B \) an *infinite Blaschke product* for the upper half plane [9, page 120]. Furthermore, \( |B(z)| < 1 \) for \( z \in \mathbb{C}_+ \). Therefore, by Fatou's theorem [9, page 57],
for almost all $x \in \mathbb{R}$, $\lim_{z \to x} B(z)$ exists. Denoting that limit by $B(x)$ (wherever it exists), one has $|B(x)| = 1$ almost everywhere [9, page 66]. A Blaschke sequence in the upper half plane, $\{z_k\}$, has no accumulation point on the real line if and only if $\lim_{k \to \infty} |z_k| = \infty$. Here, since the $z_k$ stay away from zero, a necessary and sufficient condition for the uniform convergence of $B_K$ to $B$ on compact sets disjoint from $\{z_k; k \geq 1\}$ is that $\sum_{k=1}^{\infty} \frac{|\Im z_k|}{|z_k|^2} < \infty$. In this case, $B$ is a meromorphic function with poles at the $\bar{z}_k$. For this reason, it is called a *meromorphic Blaschke product*. The function $B$ is analytic at each point of $\mathbb{R}$, and $|B(x)| = 1$ for $x \in \mathbb{R}$. Let us multiply $B$ by a constant of modulus one to get $B(0) = 1$. Then for each $z$ different from all the $\bar{z}_k$,

$$B(z) = \prod_{k=1}^{\infty} \left( \frac{\bar{z}_k}{z_k} \cdot \frac{z - z_k}{\overline{z - z_k}} \right).$$

To emphasize legitimacy of repetition, let $\{z_k\}_{k \geq 1}$ be a distinct sequence in the upper half plane with $z_k \to \infty$ and let $\{m_k\}_{k \geq 1}$ be a sequence of positive integers. Suppose that $\sum_{k=1}^{\infty} \frac{m_k \Im z_k}{|z_k|^2} < \infty$. Then $B(z) = \prod_{k=1}^{\infty} \left( \frac{\bar{z}_k}{z_k} \cdot \frac{z - z_k}{\overline{z - z_k}} \right)^{m_k}$ is a meromorphic Blaschke product.

**Theorem 7.1.** The space $K_B$ consists precisely of the meromorphic functions $f$ with poles of order at most $m_k$ at the $\bar{z}_k$, such that $f \in H^2(\mathbb{C}_+)$ and also $\frac{f}{B} \in H^2(\mathbb{C}_-)$. 

**Proof.** Let $f \in K_B$. Then by Theorem 4.1, $f$ and $\frac{f}{B}$ are respectively analytic in the upper and lower half planes. Hence $f = B \cdot \frac{f}{B}$ is a meromorphic function in the lower half plane, with poles of order at most $m_k$ at the $\bar{z}_k$. Finally, by Theorem 5.1, $f$ is analytic at each point of the real line. If, on the other hand, $f \in H^2(\mathbb{C}_+)$ and $\frac{f}{B} \in H^2(\mathbb{C}_-)$, then at least $f \in L^2(\mathbb{R})$. Thus $f \in K_B$ by Theorem 4.1. 

The following result is an easy consequence of Theorem 7.1. It can also
be shown that $K_B$ is actually the closed subspace of $H^2(\mathbb{R})$ generated by the elements $\frac{1}{(x - z_k)^{\ell_k}}$ with $1 \leq \ell_k \leq m_k$ and $k \geq 1$.

**Corollary 7.2.** For each $\ell_k$, $1 \leq \ell_k \leq m_k$ and $k \geq 1$, we have

$$\frac{1}{(z - z_k)^{\ell_k}} \in K_B.$$ 

The following result gives a complete description of $K_B$ when $B$ is a finite Blaschke product.

**Corollary 7.3.** Let $B$ be the finite Blaschke product

$$B(z) = \prod_{k=1}^{K} \left( \frac{z - z_k}{z - \bar{z}_k} \right)^{m_k}.$$ 

Then $K_B$ consists precisely of the linear combinations of the simple fractions $\frac{1}{(z - z_k)^{\ell_k}}$, where $1 \leq k \leq K$ and $1 \leq \ell_k \leq m_k$. Thus $f \in K_B$ if and only if

$$f(z) = \frac{P(z)}{\prod_{k=1}^{K} (z - z_k)^{m_k}},$$

where $P$ is a polynomial of degree $m_1 + \cdots + m_K - 1$.

Every meromorphic Blaschke product can be represented as

$$B(z) \approx \frac{\overline{E(z)}}{\overline{E(\overline{z})}} \text{ for } z \in \mathbb{C},$$

where $E$ is an entire function with zeros at the $z_k$ [13]. The order of $z_k$ as a zero of $E$ is the same as its order as a pole of $B$. In the general case, $E$ is not necessarily of exponential type. In the following we write $E^*(z)$ for $\overline{E(\overline{z})}$. This observation enables us to give another characterization of $K_B$.

**Theorem 7.4** The space $K_B$ consists precisely of functions of the form $\frac{f}{E^*}$, where $f$ is an entire function with both $\frac{f}{E} \in H^2(\mathbb{C}_+)$ and $\frac{f}{E^*} \in H^2(\mathbb{C}_-)$. 

Proof. Let $g \in K_B$. Then by Theorem 7.1, $g$ is a meromorphic function with poles of order at most $m_k$ at the $z_k$. Hence $gE$ is an entire function, where $E$ is the entire function furnished by before. Put $f = gE$. Then $\frac{f}{E} = g \in H^2(\mathbb{C}_+)$, and $\frac{f}{B} = g \in H^2(\mathbb{C}_-)$. On the other hand, if $f$ satisfies these conditions, then $\frac{f}{E} \in K_B$ by Theorem 7.1. 

The preceding result enables us to characterize the minimal majorant for $K_B$ when $B$ is a meromorphic Blaschke product with zeros in a Stoltz domain [8].

References


