EXTENDING FUNCTIONS IN THE MODEL SUBSPACES OF $H^2(\mathbb{R})$ to \mathbb{C}

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Abstract: It is shown that each function f in a model subspace K_{Θ} of $H^2(\mathbb{R})$ can be extended to \mathbb{C} . The extension to the upper half plane is in $H^2(\mathbb{C}_+)$ and the extension to the lower half plane is in Θ $H^2(\mathbb{C}_-)$. We also show that f is analytic at each point of the real line where Θ is analytic. Finally, we completely characterize K_{Θ} for $\Theta(x) = e^{i\sigma x}$ and for Θ being a meromorphic Blaschke product.

1. Introduction

Let f be an analytic function in the upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$. Let $\|f_y\|_p = \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx\right)^{\frac{1}{p}}$ and $\|f\|_p = \sup_{y>0} \|f_y\|_p$ for $0 . The Hardy space <math>H^p(\mathbb{C}_+)$ consists of all f's with

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 $||f||_p < \infty$. The Hardy space $H^{\infty}(\mathbb{C}_+)$ consists of all bounded analytic functions in the upper half plane. In this case $||f||_{\infty} = \sup_{z \in \mathbb{C}_+} |f(z)|$. For $0 , <math>H^p(\mathbb{C}_+)$ with the distance $||f - g||_p^p$ is a complete metric space. For $1 \le p < \infty$, $H^p(\mathbb{C}_+)$, $|| \cdot ||_p$ is a Banach space. In particular, $H^2(\mathbb{C}_+)$, $|| \cdot ||_2$ is a Hilbert space. Finally, $H^{\infty}(\mathbb{C}_+)$ is a Banach algebra [9, pages 70-74].

For each $f \in H^p(\mathbb{C}_+)$, and for almost all $x \in \mathbb{R}$, $\lim_{z \neq \to x} f(z)$ exists. Denoting this limit by f(x), we have $f \in L^p(\mathbb{R})$, and furthermore, $\| f \|_p = \| f \|_{L^p(\mathbb{R})}$. In the preceding limit, z is required to tend to x from within sectors of opening < 180° having vertex at x, and symmetric about the vertical line passing through x. We frequently say that $f(z) \to f(x)$ as z tends to x non-tangentially [4, page 6].



Therefore, there is a canonical correspondence between $H^p(\mathbb{C}_+)$ and a subspace of $L^p(\mathbb{R})$, denoted by $H^p(\mathbb{R})$. The space $H^p(\mathbb{R})$ can also be independently defined as the set of all $f \in L^p(\mathbb{R})$, with (as a distribution) $\hat{f}(x) = 0$ for x < 0. The two definitions are equivalent [7, page 172]. The Hardy spaces $H^p(\mathbb{C}_-)$ are defined similarly. The functions in $H^p(\mathbb{C}_-)$ live in the lower half plane and the family of their boundary values, as functions on \mathbb{R} , is precisely the space $\overline{H^p(\mathbb{R})}$. See also Chapter 11 of [4].

The function $\Theta \in H^{\infty}(\mathbb{R})$ is said to be inner if $|\Theta(x)| = 1$ for almost all $x \in \mathbb{R}$. For each inner function Θ , the set $\Theta H^2(\mathbb{R})$ is a closed subspace of the Hilbert space $H^2(\mathbb{R})$ [9, pages 79-80]. Now we are able to introduce our hero.

Definition The model space K_{Θ} is the orthogonal complement of

 Θ $H^2(\mathbb{R})$ in $H^2(\mathbb{R})$.

In this paper we study the model space K_{Θ} corresponding to the inner function Θ . In Section 2 we briefly discuss the role of model subspaces in operator theory. In Section 3 an analytic description of K_{Θ} is given. This new formulation, which is obtained using Hilbert space characteristic of $H^2(\mathbb{R})$, can be exploited to define K_{Θ} in the Hardy space $H^p(\mathbb{R})$. This representation also enables us to extend each function in K_{Θ} to the whole complex plane. In Section 4 we extend an $f \in K_{\Theta}$ to \mathbb{C} . This extension has three fundamental properties. First, $\lim_{z \neq \to x} f(z) = f(x)$ for almost all $x \in \mathbb{R}$. In these limit, z is allowed to tend to x non-tangentially from *either* half plane. Second, f as a function defined in the upper half plane is in $H^2(\mathbb{C}_+)$. Third, f as a function defined in the lower half plane is in $\Theta H^2(\mathbb{C}_-)$. Therefore, f is at least analytic in the upper half plane and is meromorphic in the lower half plane. Furthermore, in Section 5 we show that f is already analytic wherever Θ is on the real line. Finally in Sections 6 and 7 we completely characterize K_{Θ} corresponding to $\Theta(x) = e^{i\sigma x}$ and for Θ being a meromorphic Blaschke product. In these two cases, and only for them, each $f \in K_{\Theta}$ is analytic on the whole real line.

2. Link to operator theory

In this section we explain the origin of model subspaces of $H^2(\mathbb{R})$. Let $f \in H^2(\mathbb{R})$. By the Fourier-Plancherel theorem, if we write

$$\hat{f}_N(\lambda) = \int_{-N}^N e^{-i\lambda t} f(t) dt,$$

then, as $N \to \infty$, the $\hat{f}_N(\lambda)$ tend in $L^2(\mathbb{R})$ to a function $\hat{f}(\lambda)$, called the *Fourier-Plancherel transform* of f. We can characterize an $f \in H^2(\mathbb{R})$ in terms of its Fourier-Plancherel transform. A function $f \in L^2(\mathbb{R})$ is in $H^2(\mathbb{R})$ if and only if $\hat{f}(\lambda) = 0$ for almost every $\lambda < 0$ [9, page 131]. Therefore, there is a canonical isomorphism between $H^2(\mathbb{R})$ and

 $L^{2}((0,\infty))$. Based on the preceding observation, a function $f \in L^{2}(\mathbb{R})$ is in $\overline{H^{2}(\mathbb{R})}$ if and only if $\hat{f}(\lambda) = 0$ for almost every $\lambda > 0$.

Let $\delta > 0$. Then the map T_{δ}

$$egin{array}{rcl} H^2(\mathbb{R}) &\mapsto & H^2(\mathbb{R}) \ f(t) &\mapsto & \exp(i\delta \ t) \ f(t), \end{array}$$

is called a forward shift operator on $H^2(\mathbb{R})$. Since for each $f \in H^2(\mathbb{R})$

$$\left[\overline{T_{\delta}(f)}\left(\lambda
ight) = \widehat{f}\left(\lambda-\delta
ight), \quad \lambda \in \mathbb{R},$$

 T_{δ} shifts the spectrum of f forward by δ units. Beurling in his classical paper [2] characterized the invariant subspaces of $H^2(\mathbb{R})$ for the forward shift operators.

Beurling's theorem: A closed subspace of $H^2(\mathbb{R})$ is invariant under T_{δ} , for each $\delta > 0$, if and only if it is of the form $\Theta H^2(\mathbb{R})$ for some inner function Θ .

The adjoint of a forward shift operator, T^*_{δ} , is called a *backward shift* operator. By direct verification, one verifies that T^*_{δ} is defined by

$$\widehat{T^*_{\delta}(f)}(\lambda) = \begin{cases} f(\lambda + \delta), & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda < 0 \end{cases}$$

for $f \in H^2(\mathbb{R})$ [6]. Therefore, T^*_{δ} shifts the spectrum of f backward by δ units, and then chops off the negative part of what is thus obtained. In a Hilbert space, a closed subspace M is invariant under a bounded operator T if and only if M^{\perp} is invariant under T^* [5, page 40]. Therefore according to the Beurling's theorem, A closed subspace of $H^2(\mathbb{R})$ is invariant under T^*_{δ} for each $\delta > 0$ if and only if it is the orthogonal complement of $\Theta H^2(\mathbb{R})$ for some inner function Θ . Therefore, the subspaces K_{Θ} are precisely those which are invariant under T^*_{δ} for each $\delta > 0$. That is why some authors call the K_{Θ} a *coinvariant subspace* of $H^2(\mathbb{R})$.

3. Analytic description of K_{Θ}

Let Θ be an inner function for the upper half plane. Then $\Theta H^2(\mathbb{R})$ is a closed subspace of the Hilbert space $H^2(\mathbb{R})$. According to the notation introduced before, the orthogonal complement of $\Theta H^2(\mathbb{R})$ in $H^2(\mathbb{R})$ is denoted by K_{Θ} . The following lemma gives an analytic description of K_{Θ} which can be used as the definition of it in all Hardy spaces $H^p(\mathbb{R})$, 0 .

Theorem 3.1. For each inner function Θ

$$K_{\Theta} = H^2(\mathbb{R}) \cap \Theta \overline{H^2(\mathbb{R})}.$$

Proof. Uses the properties $\Theta \in H^{\infty}$ and $\Theta \overline{\Theta} = 1$. By definition, $f \in K_{\Theta}$ if and only if $f \in H^2(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} f(x) \ \overline{\Theta(x) g(x)} \ dx = 0$$

for each $g \in H^2(\mathbb{R})$. Thus, $f \in K_{\Theta}$ if and only if $f \in H^2(\mathbb{R})$

$$\int_{-\infty}^{\infty} \frac{f(x)}{\Theta(x)} \ \overline{g(x)} \ dx = 0$$

for each $g \in H^2(\mathbb{R})$. This condition is equivalent to $\frac{f}{\Theta} \in \overline{H^2(\mathbb{R})}$. Therefore $f \in K_{\Theta}$ if and only if $f \in H^2(\mathbb{R})$ and also $f \in \Theta$ $\overline{H^2(\mathbb{R})}$.

4. Extension to upper and lower half planes

Let $h \in L^2(\mathbb{R})$. Then the Poisson integral formula

$$P_h(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2} h(t) dt, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

gives an extension of h to the upper and to the lower half planes. It can be shown that $h \in H^2(\mathbb{R})$ if and only if P_h , as a function defined in the upper half plane, is in $H^2(\mathbb{C}_+)$. Similarly, $h \in \overline{H^2(\mathbb{R})}$ if and only if P_h , as a function defined in the lower half plane, is in $H^2(\mathbb{C}_-)$ [12]. An $f \in K_{\Theta}$ belongs in particular to $H^2(\mathbb{R})$. Therefore it has an extension f(z) to the upper half plane, belonging to $H^2(\mathbb{C}_+)$ and given there by the formula

$$f(z) = P_f(z)$$
 for $z \in \mathbb{C}_+$.

An inner function Θ can be (formally) extended to the lower half plane by putting

$$\Theta(z) = \frac{1}{\overline{\Theta(\bar{z})}}$$

for $z \in \mathbb{C}_-$. The extension of an $f \in K_{\Theta}$ to the lower half plane is indirect (depending on Θ). For such an f we have $\overline{\Theta} f \in \overline{H^2(\mathbb{R})}$ by Theorem 3.1, so, by the preceding observation, $\overline{\Theta} f$ has an analytic extension to the lower half plane, equal there to

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2} \,\overline{\Theta(t)} \, f(t) \, dt = P_{\overline{\Theta}_f}(z), \quad z \in \mathbb{C}_-.$$

We then *define* the extension of $f \in K_{\Theta}$ to \mathbb{C}_{-} by putting

$$f(z) = \Theta(z) P_{\overline{\Theta}_f}(z) \text{ for } z \in \mathbb{C}_-,$$

with $\Theta(z)$ defined as above in \mathbb{C}_{-} . This extension is at least meromorphic in the lower half plane.

Remark: We have $\lim_{z \neq \to x} \Theta(z) = \Theta(x)$ and $\lim_{z \neq \to x} f(z) = f(x)$ for almost all $x \in \mathbb{R}$. In these limits, z is allowed to tend to x non-tangentially from *either* half plane.

With above definitions, Theorem 3.1 yields the following characterization of K_{Θ} .

Theorem 4.1. The space K_{Θ} consists precisely of the functions $f \in L^2(\mathbb{R})$ with extension to the upper half plane belonging to $H^2(\mathbb{C}_+)$ and whose extension to the lower half plane makes $\frac{f}{\Theta} \in H^2(\mathbb{C}_-)$.

For further applications of this result see [8].

5. Analytic continuation along \mathbb{R}

A function $f \in K_{\Theta}$ can be continued analytically across intervals of \mathbb{R} on which Θ is analytic. This result has important consequences in characterizing elements of K_B when B is a meromorphic Blaschke product.

Theorem 5.1. If Θ is analytic in a neighborhood of the interval $(a,b) \subset \mathbb{R}$ then any $f \in K_{\Theta}$ is also analytic there.

Proof. By Theorem 4.1, f and $\frac{f}{\Theta}$ are respectively holomorphic in the upper and lower half planes. Without loss of generality, suppose Θ is holomorphic inside the rectangle $\{z : a < \Re z < b, -2 < \Im z < 2\}$. Thus $f = \Theta \cdot \frac{f}{\Theta}$ is also analytic inside that rectangle except possibly on (α, β) , and for almost all $x \in \mathbb{R}$, $\lim_{|y|\to 0} f(x+iy)$ exists. Choose $\alpha, \beta \in (a, b)$ such that this is true for $x = \alpha$ and for $x = \beta$.

With $\varepsilon > 0$, let us take the paths

$$\begin{split} \Gamma &= & [\alpha + i, \alpha - i] \cup [\alpha - i, \beta - i] \cup [\beta - i, \beta + i] \cup [\beta + i, \alpha + i], \\ \Gamma_{\varepsilon} &= & [\alpha + i\varepsilon, \alpha - i\varepsilon] \cup [\alpha - i\varepsilon, \beta - i\varepsilon] \cup [\beta - i\varepsilon, \beta + i\varepsilon] \cup [\beta + i\varepsilon, \alpha + i\varepsilon], \end{split}$$

each oriented counterclockwise. For each point z inside Γ , let $g(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$. Then g is holomorphic inside Γ . By the Cauchy integral formula,

$$g(z) = f(z) + \frac{1}{2\pi i} \int_{\Gamma_{\varepsilon}} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

for $\varepsilon < |\Im z| < 2$. Since f is bounded on the vertical segments through

 α and β ,

$$\lim_{\varepsilon \to 0} \int_{[\alpha + i\varepsilon, \alpha - i\varepsilon]} \frac{f(\zeta)}{\zeta - z} \, d\zeta = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_{[\beta - i\varepsilon, \beta + i\varepsilon]} \frac{f(\zeta)}{\zeta - z} \, d\zeta = 0$$

On the horizontal segments

$$\begin{split} &\int_{[\alpha-i\varepsilon,\beta-i\varepsilon]} \frac{f(\zeta)}{\zeta-z} \ d\zeta + \int_{[\beta+i\varepsilon,\alpha+i\varepsilon]} \frac{f(\zeta)}{\zeta-z} \ d\zeta = \int_{\alpha}^{\beta} \left(\frac{f(t-i\varepsilon)}{t-i\varepsilon-z} - \frac{f(t+i\varepsilon)}{t+i\varepsilon-z} \right) \ dt \\ & \text{Both } \frac{f(t-i\varepsilon)}{t-i\varepsilon-z} \text{ and } \frac{f(t+i\varepsilon)}{t+i\varepsilon-z} \text{ converge in } L^2(dt) \text{ norm to } \frac{f(t)}{t-z}, \text{ as } \\ & \varepsilon \to 0. \text{ Thus} \\ & \lim_{\varepsilon \to 0} \int_{[\alpha-i\varepsilon,\beta-i\varepsilon]} \frac{f(\zeta)}{\zeta-z} \ d\zeta + \int_{[\beta+i\varepsilon,\alpha+i\varepsilon]} \frac{f(\zeta)}{\zeta-z} \ d\zeta = 0. \end{split}$$

Hence $g \equiv f$ in the lower and in the upper part of the interior of Γ . Therefore, f is holomorphic on (α, β) . Since α and β can be taken as close to a and to b as we want, f is holomorphic on (a, b).

6. Paley-Wiener spaces as model subspaces

Let $\sigma > 0$. Then, $\Theta(x) = \exp(i\sigma x)$ is an entire inner function. In this case, the functions $f(x) \in K_{\Theta}$ differ by the factor $e^{i\sigma x/2}$ from those in a Paley-Wiener space.

Theorem 6.1. Let $\sigma > 0$. Then $f \in K_{e^{i\sigma x}}$ if and only if f is an entire function of exponential type, square integrable on the real line, with

$$-\sigma \leq \limsup_{y \to +\infty} \frac{\log |f(iy)|}{y} \leq 0 \quad \text{ and } \quad 0 \leq \limsup_{y \to -\infty} \frac{\log |f(iy)|}{|y|} \leq \sigma.$$

Proof. Since $\Theta(x) = \exp(i\sigma x)$ is analytic across \mathbb{R} , each $f \in K_{e^{i\sigma x}}$ is also analytic there. Furthermore, $f \in H^2(\mathbb{C}_+)$ and $\frac{f}{\Theta} \in H^2(\mathbb{C}_-)$ imply that f is analytic on \mathbb{C}_+ and also on \mathbb{C}_- , that $f \in L^2(\mathbb{R})$, and besides that the support of the Fourier-Plancherel transform of f is a subset of $[0,\sigma]$. Thus $\hat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and for each $z = x \in \mathbb{R}$,

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 $f(z) = \int_0^\sigma \hat{f}(t) e^{izt} dt$. By the uniqueness theorem for analytic functions, equality holds everywhere. Therefore f is an entire function of exponential type with the indicated growth conditions on the imaginary axis. The if part is an easy consequence of the celebrated Paley-Wiener theorem.

The following Corollary is an immediate consequence of the Paley-Wiener representation of entire functions of exponential type and the preceding theorem. The indicated representation, by itself, shows that each $f \in K_{e^{i\sigma x}}$ is an entire function of exponential type which is square integrable on the real line. The representation, moreover, restricts the rate of growth of f along the imaginary axis.

Corollary 6.2. Each $f \in K_{e^{i\sigma x}}$ has the representation

$$f(z) = \int_0^\sigma \hat{f}(t) e^{izt} dt$$

, where $\hat{f} \in L^2(0,\sigma)$.

7. The model space K_B

Let $\{z_k\}_{k\geq 1}$ be a sequence of complex numbers in the upper half plane \mathbb{C}_+ . Let $b_k(z) = e^{i\alpha_k} \cdot \frac{z-z_k}{z-\bar{z}_k}$, where α_k is so chosen that $e^{i\alpha_k} \cdot \frac{i-z_k}{i-\bar{z}_k} \geq 0$. The rational function $B_K = \prod_{k=1}^K b_k$ is called a *finite Blaschke product* for the upper half plane; B_K is analytic at each point of the real line and $|B_K(x)| = 1$ for $x \in \mathbb{R}$. The relation $\sum_{k=1}^{\infty} \frac{\Im z_k}{|z_k+i|^2} < \infty$ is a necessary and sufficient condition for the uniform convergence of B_K on compact sets, disjoint from the closure of $\{\bar{z}_k; k \geq 1\}$, to a non-zero analytic function $B = \prod_{k=1}^{\infty} b_k = \lim_{K \to \infty} B_K$, and we call B an *infinite Blaschke product* for the upper half plane [9, page 120]. Furthermore, |B(z)| < 1 for $z \in \mathbb{C}_+$. Therefore, by Fatou's theorem [9, page 57],

for almost all $x \in \mathbb{R}$, $\lim_{z \neq \to x} B(z)$ exists. Denoting that limit by B(x)(wherever it exists), one has |B(x)| = 1 almost everywhere [9, page 66]. A Blaschke sequence in the upper half plane, $\{z_k\}$, has no accumulation point on the real line if and only if $\lim_{k \to \infty} |z_k| = \infty$. Here, since the z_k stay away from zero, a necessary and sufficient condition for the uniform convergence of B_K to B on compact sets disjoint from $\{\bar{z}_k; k \geq 1\}$ is that $\sum_{k=1}^{\infty} \frac{\Im z_k}{|z_k|^2} < \infty$. In this case, B is a meromorphic function with poles at the \bar{z}_k . For this reason, it is called a *meromorphic Blaschke product*. The function B is analytic at each point of \mathbb{R} , and |B(x)| = 1 for $x \in \mathbb{R}$. Let us multiply B by a constant of modulus one to get B(0) = 1. Then for each z different from all the \bar{z}_k ,

$$B(z) = \prod_{k=1}^{\infty} \left(\frac{\overline{z}_k}{z_k} \cdot \frac{z - z_k}{z - \overline{z}_k} \right).$$

To emphasize legitimacy of repetition, let $\{z_k\}_{k\geq 1}$ be a distinct sequence in the upper half plane with $z_k \to \infty$ and let $\{m_k\}_{k\geq 1}$ be a sequence of positive integers. Suppose that $\sum_{k=1}^{\infty} \frac{m_k \Im z_k}{|z_k|^2} < \infty$. Then B(z) = $\prod_{k=1}^{\infty} \left(\frac{\bar{z}_k}{z_k} \cdot \frac{z-z_k}{z-\bar{z}_k}\right)^{m_k}$ is a meromorphic Blaschke product.

Theorem 7.1. The space K_B consists precisely of the meromorphic functions f with poles of order at most m_k at the \overline{z}_k , such that $f \in$ $H^2(\mathbb{C}_+)$ and also $\frac{f}{B} \in H^2(\mathbb{C}_-)$.

Proof. Let $f \in K_B$. Then by Theorem 4.1, f and $\frac{f}{B}$ are respectively analytic in the upper and lower half planes. Hence $f = B \cdot \frac{f}{B}$ is a meromorphic function in the lower half plane, with poles of order at most m_k at the \bar{z}_k . Finally, by Theorem 5.1, f is analytic at each point of the real line. If, on the other hand, $f \in H^2(\mathbb{C}_+)$ and $\frac{f}{B} \in H^2(\mathbb{C}_-)$, then at least $f \in L^2(\mathbb{R})$. Thus $f \in K_B$ by Theorem 4.1.

The following result is an easy consequence of Theorem 7.1. It can also

be shown that K_B is actually the closed subspace of $H^2(\mathbb{R})$ generated by the elements $\frac{1}{(x-\bar{z}_k)^{\ell_k}}$ with $1 \leq \ell_k \leq m_k$ and $k \geq 1$.

Corollary 7.2. For each ℓ_k , $1 \leq \ell_k \leq m_k$ and $k \geq 1$, we have $\frac{1}{(z - \bar{z}_k)^{\ell_k}} \in K_B$.

The following result gives a complete description of K_B when B is a finite Blaschke product.

Corollary 7.3. Let B be the finite Blaschke product

$$B(z) = \prod_{k=1}^{K} \left(\frac{z-z_k}{z-\bar{z}_k}\right)^{m_k}.$$

Then K_B consists precisely of the linear combinations of the simple fractions $\frac{1}{(z-\bar{z}_k)^{\ell_k}}$, where $1 \leq k \leq K$ and $1 \leq \ell_k \leq m_k$. Thus $f \in K_B$ if and only if

$$f(z) = \frac{P(z)}{\prod_{k=1}^{K} (z - \bar{z}_k)^{m_k}},$$

where P is a polynomial of degree $m_1 + \cdots + m_K - 1$.

Every meromorphic Blaschke product can be represented as

$$B(z) = \frac{\overline{E(\bar{z})}}{E(z)} \quad for \quad z \in \mathbb{C},$$

where E is an entire function with zeros at the \bar{z}_k [13]. The order of \bar{z}_k as a zero of E is the same as its order as a pole of B. In the general case, E is not necessarily of exponential type. In the following we write $E^*(z)$ for $\overline{E(\bar{z})}$. This observation enables us to give another characterization of K_B .

Theorem 7.4 The space K_B consists precisely of functions of the form $\frac{f}{E}$, where f is an entire function with both $\frac{f}{E} \in H^2(\mathbb{C}_+)$ and $\frac{f}{E^*} \in H^2(\mathbb{C}_-)$.

Proof. Let $g \in K_B$. Then by Theorem 7.1, g is a meromorphic function with poles of order at most m_k at the \bar{z}_k . Hence g E is an entire function, where E is the entire function furnished by before. Put f = g E. Then $\frac{f}{E} = g \in H^2(\mathbb{C}_+)$, and $\frac{f}{E^*} = \frac{g}{B} \in H^2(\mathbb{C}_-)$. On the other hand, if f satisfies these conditions, then $\frac{f}{E} \in K_B$ by Theorem 7.1.

The preceding result enables us to characterize the *minimal* majorant for K_B when B is a meromorphic Blaschke product with zeros in a Stoltz domain [8].

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