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ON THE RESTRICTION OF CHARACTERS OF SPECIAL LINEAR GROUPS OF DIMENSION THREE

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ABSTRACT. Let G be the special linear group $\mathrm{SL}(3,q)$, where q is power of a prime p. Here, we show that χ_P has a linear constituent with multiplicity one for each irreducible character χ and Sylow p-subgroup P of G. Furthermore, if $\mathrm{cf}(G)$ is the vector space of class functions of G, we show that the restriction of a subset of irreducible characters of G on P is a basis for the vector space of class functions defined on P spanned by $\{\phi_P \mid \phi \in \mathrm{cf}(G)\}$.

1. Introduction

Steinberg asserts, in particular, that for any finite Chevalley group G, each nonprincipal linear character of a maximal unipotent subgroup H (a Sylow p-subgroup where p is the characteristic of G) of G is a constituent of χ_H with multiplicity at most 1 for every irreducible character χ of G [8, Theorem 49]. Moreover, in an earlier work, Gel'fand and Graev [2] showed the same results for groups $\mathrm{SL}(n,q)$ for arbitrary n with a particular attention to the case $n = 3$. If q is a power of a prime p, by constructing the primitive central idempotents of the complex group algebra $\mathbb{C}G = \mathbb{C}\mathrm{SL}(3,q)$, Guzel [3] shows that the restriction of χ to a Sylow p-subgroup of G has a linear constituents with multiplicity 1, for

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each irreducible character χ of G. This result has been referred by the author of this manuscript in [1] without any details.

Here, we provide another explicit proof of this result by using values of the irreducible characters of G on p-elements and writing them down as *integral* linear combinations of some specific characters.

Theorem 1.1. *Let $G = \mathrm{SL}(3,\, q)$, where $q > 2$ is a power of a prime p. Let P be a Sylow p-subgroup of G. Then, for all irreducible characters χ of G, there exists a linear character φ of P such that $\langle \chi_P, \varphi \rangle = 1$.*

In the following section, we describe the structure of conjugacy classes and irreducible characters of G and their restrictions to the Sylow p-subgroup P. Section 3 contains the proof of Theorem 1.1. Finally, in section 4 we conclude that a subset of restricted irreducible characters of G on P is a basis for the vector space of class functions defined on P and spanned by $\{\phi_P \mid \phi \in \mathrm{cf}(G)\}$, where $\mathrm{cf}(G)$ is the vector space of class functions of G.

2. **Structure of characters**

The special linear group $G = \mathrm{SL}(3, q)$, where q is a power of a prime p, of dimension 3 over the finite field $\mathbb{F}_q = \mathrm{GF}(q)$, is the set of all nonsingular 3×3 matrices with determinant 1.

Let LT (a, b, c) denote a 3×3 lower triangular matrix with diagonal entries being 1 and the entries at the positions $(2, 1)$, $(3, 1)$ and $(3, 2)$ being a, b and c, respectively. The set P of all matrices LT (a, b, c) with $a, b, c \in \mathbb{F}_q$ is a Sylow p-subgroup of G of order q^3. We use the character values of G restricted to P to show that for each irreducible character χ of G there exists a linear character φ of P such that $\langle \chi_P, \varphi \rangle = 1$.

The conjugacy classes and the character table of G are given in [7]. We use notations defined in [7]. We shall use that table to get the values of characters on the different conjugacy classes of G which contain the elements of P.

Table 1 is a part of Table 1a of [7] that shows the structure of conjugacy classes of G which contain some elements of the Sylow p-subgroup P. Let $d = \gcd(3, q-1)$, ω be a cube root of unity and $\epsilon^3 \neq 1$, for $\epsilon \in \mathrm{GF}(q)$.

Based on the structure of the elements of P and the fact that ω is a cube root of unity, the elements of P are contained only in the conjugacy classes $\mathcal{C}_1^{(0)}$, $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,l)}$ of G. The centre $Z(P) = \{\mathrm{LT}\ (0, z, 0) \mid z \in$

$\mathbb{F}_q\}$ is an elementary abelian p-group of order q. Using the canonical representative elements of conjugacy classes $\mathcal{C}_1^{(0)}$, $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,l)}$, we see that the minimal polynomials of elements of these conjugacy classes have degrees 1,2 and 3, respectively. The minimal polynomials of nontrivial elements of $Z(P)$ have degree 2, and so nontrivial elements of $Z(P)$ are contained in the conjugacy class $\mathcal{C}_2^{(0)}$.

TABLE 1: Conjugacy classes of SL(3, q) which contain elements of the Sylow p-subgroup P for $d = 1, 3$.

Conjugacy class	Canonical representative	Parameters
$\mathcal{C}_1^{(k)}$	$\begin{pmatrix} \omega^k & 0 & 0 \\ 0 & \omega^k & 0 \\ 0 & 0 & \omega^k \end{pmatrix}$	$0 \leqslant k \leqslant (d-1)$
$\mathcal{C}_2^{(k)}$	$\begin{pmatrix} \omega^k & 0 & 0 \\ 1 & \omega^k & 0 \\ 0 & 0 & \omega^k \end{pmatrix}$	$0 \leqslant k \leqslant (d-1)$
$\mathcal{C}_3^{(k,l)}$	$\begin{pmatrix} \omega^k & 0 & 0 \\ \epsilon^l & \omega^k & 0 \\ 0 & \epsilon^l & \omega^k \end{pmatrix}$	$0 \leqslant k, l \leqslant (d-1)$

The following lemma gives some properties of P.

Lemma 2.1. *Suppose $G = \mathrm{SL}(3, q)$, where q is a power of a prime p. If P is a Sylow p-subgroup of G, then we have:*

(1) *P has $q^2 + q - 1$ conjugacy classes.*

(2) *P has q^2 linear characters and $q - 1$ non-linear characters of degree q such that their values on nontrivial elements of $Z(P)$ are 1 and $q\omega^i$, for some $1 \leqslant i \leqslant p$, respectively, where ω is a primitive p^{th} root of unity.*

(3) *If τ is an irreducible character of degree q of P, then $\tau(x) = 0$, for $x \notin Z(P)$, and $\sum_{1 \neq z \in Z(P)} \tau(z) = -q$.*

Proof. First of all we show $P/Z(P)$ is abelian. Let $x, y \in P$. It is enough to show $x^{-1}y^{-1}xy \in Z(P)$. Let $x = \mathrm{LT}\ (a, b, c)$ and $y =$

LT (d, e, f). Then, $x^{-1}y^{-1}xy = \text{LT }(0, af - dc, 0)$. Hence, $P/Z(P)$ is abelian and $P' \subseteq Z(P)$, where P' is the derived subgroup of P. Conversely, if $z = \text{LT }(0, t, 0) \in Z(P)$, then $z = x^{-1}y^{-1}xy \in P'$, where $x = \text{LT}(t, b, c)$ and $y = \text{LT}(0, 1, e)$, for $b, c, e \in \mathbb{F}_q$. Therefore, $P' = Z(P)$.

Now, suppose $h = \text{LT }(h_1, h_2, h_3) \in P \backslash Z(P)$ so that at least one of h_1, h_3 is not 0. Then $x^{-1}hx = h^x = \text{LT }(h_1, h_1c - ah_3 - h_2, h_3)$.

As x runs over P, $h_1c - ah_3 - h_2$ runs over $\mathbb{F}_q$. Thus, the conjugacy class $\{h^x \mid x \in P\}$ has order q. Therefore, each conjugacy class of P has order 1 or q and P has q single element conjugacy classes, since $|Z(P)| = q$. If n is the number of conjugacy classes of order q, then $|P| = (q \times 1) + (n \times q)$ and so $n = q^2 - 1$. Thus, P has $q^2 + q - 1$ conjugacy classes.

Since $|P : P'| = q^2$, then P has q^2 linear characters and since the number of conjugacy classes of P is $q^2 + q - 1$, then P has $q - 1$ non-linear characters. Let τ be a non-linear irreducible character of P. Since $Z(P) \subseteq Z(\tau)$ and by [5, Corollary 2.30] ,

$$\tau^2(1) \leqslant |P : Z(\tau)| \leqslant |P : Z(P)| = q^2, \tag{2.1}$$

then $\tau(1) \leqslant q$. On the other hand, the number of conjugacy classes of P is $q^2 + q - 1$ and the order of P is q^3 ,and thus

$$q^3 = |P| = \sum_{i=1}^{q^2} \varphi_i(1)^2 + \sum_{j=1}^{q-1} \tau_j(1)^2,$$

where φ_i and τ_j are linear and non-linear irreducible characters of P, respectively. Since $\tau_j(1) \leqslant q$, then $\tau_j(1) = q$ and (2.1) implies $Z(P) = Z(\tau)$. Since $P' = Z(P)$, then the value of all linear characters of P on $Z(P)$ is 1. Also, for an irreducible character τ of degree q, if ρ is the representation which affords τ, then $\rho(z)$ is a scalar for all $1 \neq z \in Z(P)$ and thus $\tau(z) = q\omega^j$, for some $1 \leqslant j \leqslant p$, where ω is a primitive p^{th}-root of unity.

Since $\tau^2(1) = q^2 = |P : Z(P)|$, [5, Corollary 2.30] shows that $\tau(x) = 0$, for all $x \notin Z(P)$. Using the first orthogonality relation, we get

$$\frac{1}{|P|} \sum_{x \in P} \tau(x) 1(x^{-1}) = \frac{1}{|P|} \sum_{x \in P} \tau(x) = \frac{1}{|P|} \sum_{z \in Z(P)} \tau(z) = 0.$$

Therefore, $\tau(1) = q$ implies

$$\sum_{1 \neq z \in Z(P)} \tau(z) = -q, \tag{2.2}$$

and this completes the proof. □

The following lemmas are simple consequences of Clifford's Theorem [5, Theorem 6.2] and Frattini's argument [6, Lemma 1.13].

Lemma 2.2. *Let H be a subgroup of any group G, $x \in N_G(H)$ and ϑ and ψ be characters of H. Then, $\langle \vartheta^x, \psi^x \rangle = \langle \vartheta, \psi \rangle$. In particular, taking $\psi = \vartheta$, ϑ^x is irreducible if and only if ϑ is irreducible.*

Lemma 2.3. *Let G be a normal subgroup of a group L and H be a Sylow p-subgroup of G. Let χ and ϑ be irreducible characters of G and H, respectively. Let $l \in L$. Then,*

$$\langle \chi_H, \vartheta \rangle = \langle \chi^l_H, \vartheta^x \rangle \text{ for some } x \in N_L(H).$$

In particular, $\langle \chi_H, \mathbf{1} \rangle = \langle \chi^l_H, \mathbf{1} \rangle$.

Tables 2 and 3 show the values of the restriction of the irreducible characters of the groups SL(3, q) on the elements of Sylow p-subgroup P when $d = 1$ and $d = 3$, respectively (see [7] Table 1b).

Lemma 2.4. *Let $G = \mathrm{SL}(3,\ q)$, where $q > 2$ is a power of a prime p and let P be the Sylow p-subgroup of G and ψ be the irreducible character of degree $q^2 + q$ of G. Then,*

1. *$\langle \psi_P, \mathbf{1} \rangle = 2$.*
2. *$\langle \psi_P, \tau \rangle = 1$, for each irreducible character τ of degree q of P.*
3. *There exist some non-principal linear characters φ and ϕ of P such that $\langle \psi_P, \varphi \rangle = 0$ and $\langle \psi_P, \phi \rangle = 1$.*

Proof. Suppose $x = \mathrm{LT}\,(a, b, c) \in P$ is contained in the conjugacy class $\mathcal{C}_2^{(0)}$ of G. Since each element in $\mathcal{C}_2^{(0)}$ has a minimal polynomial of degree 2, $(x-1)^2 = \mathrm{LT}\,(0, ac, 0) = 0$. This, together with $x \notin Z(P)$, implies $a = 0$ or $c = 0$ but not both. Therefore the number of possibilities for the elements x with the above properties is $2q(q-1)$. The elements of $Z(P)$ are also contained in $\mathcal{C}_2^{(0)}$ and the values of ψ on $\mathcal{C}_1^{(0)}$, $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,0)}$ are $q^2 + q$, q and 0, respectively. Thus, we have

$$\langle \psi_P, \mathbf{1} \rangle = \frac{1}{|P|} \sum_{x \in P} \psi_P(x) 1(x)$$
$$= \frac{1}{q^3}(\psi_P(1) + \sum_{1 \neq z \in Z(P)} \psi_P(z) + \sum_{z \notin Z(P)} \psi_P(z))$$

$$= \frac{1}{q^3}((q^2+q)+(q-1)q+2q(q-1)q) = 2.$$

This proves the first assertion.

Now, suppose τ is an irreducible character of degree q of P. By using Table 2 for the value of ψ on the conjugacy class $\mathcal{C}_2^{(0)}$ of G which contains the elements of $Z(P)$ and Lemma 2.1, we have

$$\langle \psi_P, \tau \rangle = \frac{1}{|P|}\sum_{x\in P} \psi_P(x)\overline{\tau(x)}$$
$$= \frac{1}{q^3}(\psi_P(1)\tau(1) + \sum_{1\neq z\in Z(P)} \psi_P(z)\overline{\tau(z)} + \sum_{z\notin Z(P)} \psi_P(z)\overline{\tau(z)})$$
$$= \frac{1}{q^3}((q^2+q)q - q^2 + 0) = 1,$$

where $\overline{\tau(x)}$ is the complex conjugate of the value $\tau(x)$. Therefore, for each irreducible character τ of degree q of P,

$$\langle \psi_P, \tau \rangle = 1,$$

as claimed.

Now, since $\langle \psi_P, \tau \rangle = 1$ for each irreducible character τ of degree q of P, then $\psi_P = \sum_{i=1}^{q-1} \tau_i + \sum_{j=1}^{t} m_j\phi_j$, where the ϕ_j are linear characters of P with the multiplicities m_j. Since $\psi(1) = q^2+q$ and $\sum_{i=1}^{q-1} \tau_i(1) = q^2-q$, we have $\sum_{j=1}^{t} m_j\phi_j(1) = 2q$. Since P possesses q^2-1 non-principal linear characters, there exists at least one non-principal linear character φ such that $\langle \psi_P, \varphi \rangle = 0$.

By the first assertion, $\langle \psi_P, \mathbf{1} \rangle = 2$. Hence, $\sum_{j=1}^{\prime t} m_j\phi_j(1) = 2q-2 > 1$, where $\sum'$ runs over $\phi_j \neq \mathbf{1}$. This means there exists some non-principal linear character ϕ of P such that $\langle \psi_P, \phi \rangle \neq 0$. Note that $(\nu_t)_P = \rho_P - \psi_P + \mathbf{1}$ is a character of P and that ρ_P is the regular character of P. It follows that any nonprincipal linear constituent ϕ of ψ_P has multiplicity 1. This completes the proof. □

By the values of characters ω_m and γ_n on the conjugacy classes $\mathcal{C}_1^{(0)}$, $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,l)}$ in the Table 1b of [7], we have

$$\{(\omega_1)_P, (\omega_2)_P, (\omega_3)_P\} = \{(\gamma_1)_P, (\gamma_2)_P, (\gamma_3)_P\}. \tag{2.3}$$

Table 2: Values of characters of SL(3, q) on elements of P when $d = 1$, where $1 \leqslant i, j \leqslant q-2$, $1 \leqslant r \leqslant (q^2-5q+6)/6$, $1 \leqslant s \leqslant (q^2-q)/2$ and $1 \leqslant t \leqslant (q^2+q)/3$.

	$\mathcal{C}_1^{(0)}$	$\mathcal{C}_2^{(0)}$	$\mathcal{C}_3^{(0,0)}$
1	1	1	1
ψ	q^2+q	q	0
ρ	q^3	0	0
ζ_i	q^2+q+1	$q+1$	1
η_j	q^3+q^2+q	q	0
ε_r	q^3+2q^2+2q+1	$2q+1$	1
μ_s	q^3-1	-1	-1
ν_t	q^3-q^2-q+1	$1-q$	1

Table 1: Values of characters of SL(3, q) on elements of P when $d = 3$, where $1 \leqslant i, j \leqslant q-2$, $1 \leqslant r \leqslant (q^2-5q+4)/6$, $1 \leqslant s \leqslant (q^2-q)/2$, $1 \leqslant t \leqslant (q^2+q-2)/3$ and $1 \leqslant k, m, n \leqslant 3$.

	$\mathcal{C}_1^{(0)}$	$\mathcal{C}_2^{(0)}$	$\mathcal{C}_3^{(0,l)}$
1	1	1	1
ψ	q^2+q	q	0
ρ	q^3	0	0
ζ_i	q^2+q+1	$q+1$	1
η_j	q^3+q^2+q	q	0
θ_k	$(q^3+2q^2+2q+1)/3$	$(2q+1)/3$ or $(1-q)/3$	$(2q+1)/3$ or $(1-q)/3$
ε_r	q^3+2q^2+2q+1	$2q+1$	1
μ_s	q^3-1	-1	-1
ν_t	q^3-q^2-q+1	$1-q$	1
ω_m	$(q^3-q^2-q+1)/3$	$(1-q)/3$ or $(2q+1)/3$	$(1-q)/3$ or $(2q+1)/3$
γ_n	$(q^3-q^2-q+1)/3$	$(1-q)/3$ or $(2q+1)/3$	$(1-q)/3$ or $(2q+1)/3$

3. Proof

Proof of Theorem 1. By Table 2, the characters ρ and ψ have degrees q^3 and q^2+q, respectively. Now, if we restrict them to P we see that for all nontrivial $x \in P$, we have $\rho_P(x) = 0$ and $\psi_P(x) = q$ or 0. Thus, from the values of the other characters of G on P, we get

$$(\zeta_i)_P = \psi_P + \mathbf{1} \tag{3.1}$$

$$(\eta_j)_P = \rho_P + \psi_P \tag{3.2}$$

$$(\varepsilon_r)_P = \rho_P + 2\psi_P + \mathbf{1} \tag{3.3}$$

$$(\mu_s)_P = \rho_P - \mathbf{1} \tag{3.4}$$

and

$$(\nu_t)_P = \rho_P - \psi_P + \mathbf{0} \tag{3.5}$$

Since $\rho(1) = q^3$ is the order of P and $\rho_P(x) = 0$, for all $x \neq 1$ in P, thus ρ_P is the regular character of P and $\rho_P = \sum_{\upsilon \in \mathrm{Irr}(P)} \upsilon(1)\upsilon$. On the other hand, by Lemma 2.4 there exists a non-principal linear character φ of P such that $\langle \psi_P, \varphi \rangle = 0$. Then, since $\langle \rho_P, \varphi \rangle = 1$, we have

$$\begin{aligned} \langle (\eta_j)_P, \varphi \rangle = &\ \langle \rho_P + \psi_P, \varphi \rangle = 1, \\ \langle (\varepsilon_r)_P, \varphi \rangle = &\ \langle \rho_P + 2\psi_P + \mathbf{1}, \varphi \rangle = 1, \\ \langle (\mu_s)_P, \varphi \rangle = &\ \langle \rho_P - \mathbf{1}, \varphi \rangle = 1 \end{aligned}$$

and

$$\langle (\nu_t)_P, \varphi \rangle = \ \langle \rho_P - \psi_P + \mathbf{1}, \varphi \rangle = 1.$$

Also, by Lemma 2.4 there exists a non-principal linear character ϕ of P such that $\langle \psi_P, \phi \rangle = 1$. Thus,

$$\langle (\zeta_i)_P, \phi \rangle = \ \langle \psi_P + \mathbf{1}, \phi \rangle = 1.$$

For the case that $d = 3$, the only remaining characters to consider are θ_k, ω_m and γ_n, for $1 \leqslant k, m, n \leqslant 3$.

Using the Frobenius reciprocity, we have

$$\langle \eta_j, \varphi^G \rangle = \langle \varepsilon_r, \varphi^G \rangle = \langle \mu_s, \varphi^G \rangle = \langle \nu_t, \varphi^G \rangle = 1,$$

and $\langle \zeta_i, \varphi^G \rangle = 0$. Also, if

$$\langle (\theta_k)_P, \varphi \rangle = K_k, \langle (\omega_m)_P, \varphi \rangle = M_m \text{ and } \langle (\gamma_n)_P, \varphi \rangle = N_n,$$

then

$$\langle \theta_k, \varphi^G \rangle = K_k, \langle \omega_m, \varphi^G \rangle = M_m \text{ and } \langle \gamma_n, \varphi^G \rangle = N_n,$$

for $1 \leqslant k, m, n \leqslant 3$. Therefore, if we induce φ to G, we get

$$\varphi^G = \rho + (q-2)\eta_j + ((q^2 - 5q + 4)/6)\varepsilon_r + ((q^2 - q)/2)\mu_s$$

$$+((q^2+q-2)/3)\nu_t+\sum_{k=1}^{3}K_k\theta_k+\sum_{m=1}^{3}M_m\omega_m+\sum_{n=1}^{3}N_n\gamma_n.$$

Using the fact that $\varphi^G(1)=|G:P|\varphi(1)$, we calculate the value at 1 and simplifing the above equation, we have

$$|G:P|=-q^2-2q^3+q^5+\sum_{k=1}^{3}K_k\theta_k(1)+\sum_{m=1}^{3}M_m\omega_m(1)+\sum_{n=1}^{3}N_n\gamma_n(1).$$

Since $|G:P|=q^5-q^3-q^2+1$, we get

$$\sum_{k=1}^{3}K_k\theta_k(1)+\sum_{m=1}^{3}M_m\omega_m(1)+\sum_{n=1}^{3}N_n\gamma_n(1)=q^3+1.$$

Since

$$\theta_k(1)=(q^3+2q^2+2q+1)/3$$

and

$$\omega_m(1)=\gamma_n(1)=(q^3-q^2-q+1)/3,$$

we have

$$(\sum_{k=1}^{3}K_k)((q^3+2q^2+2q+1)/3)+(\sum_{m=1}^{3}M_m+\sum_{n=1}^{3}N_n)((q^3-q^2-q+1)/3)=q^3+1.$$

Hence, by considering $K=\sum_{k=1}^{3}K_k$, $M=\sum_{m=1}^{3}M_m$ and $N=\sum_{n=1}^{3}N_n$, we get

$$K((q^3+2q^2+2q+1)/3)+(M+N)((q^3-q^2-q+1)/3)=q^3+1,$$

and so

$$(K+M+N)q^3+(2K-(M+N))q^2+((2K-(M+N))q+(K+M+N)=3(q^3+1).$$

Thus,

$$(A-3)(q^3+1)=-B(q^2+q), \tag{3.6}$$

where $A=K+M+N$ and $B=2K-(M+N)$. Since K, M and N are non negative integers and are not all equal to 0, then A is a positive integer. Since $q\mid -B(q^2+q)$, then $q\mid A-3$ and this means that $A-3=tq$, for some integer t. Hence, simplifying equation (3.6) implies $-B=t(q^2-q+1)$. Thus,

$$0\leqslant 3K=A+B=3-t(q-1)^2.$$

Since $d=\gcd(3,q-1)=3$, then we can consider $q>3$, which in this case $A=3+tq>0$ implies $t\geqslant 0$ and $A+B=3-t(q-1)^2\geqslant 0$

shows $t \leqslant 0$. Thus, $t = 0$, $A = 3$ and $B = 0$, which yield $K = 1$ and $M + N = 2$. Therefore, $\sum_{k=1}^{3} K_k = 1$ and $\sum_{m=1}^{3} M_m + \sum_{n=1}^{3} N_n = 2$. So, for some k, $K_k = 1$ and $\langle (\theta_k)_P, \varphi \rangle = 1$.

Let $\langle (\theta_1)_P, \varphi \rangle = 1$. Then, the characters θ_1, θ_2 and θ_3 are conjugate in $L = \mathrm{GL}(3, q)$ (see [7, Sec. 4]). Hence, by Lemma 2.3, we have

$$\langle (\theta_1)_P, \varphi \rangle = \langle (\theta_2)_P, \varphi^x \rangle = \langle (\theta_3)_P, \varphi^y \rangle = 1,$$

for some $x, y \in N_L(P)$. On the other hand, by Lemma 2.2, φ^x and φ^y are linear characters of P and so the restriction of characters θ_1, θ_2 and θ_3 to P have at least a constituent of degree one with multiplicity one.

Also, by (2.3),

$$\{(\omega_1)_P, (\omega_2)_P, (\omega_3)_P\} = \{(\gamma_1)_P, (\gamma_2)_P, (\gamma_3)_P\},$$

and so $\sum_{m=1}^{3} M_m = 1$ and $\sum_{n=1}^{3} N_n = 1$. Therefore, for some m and n we have $N_n = 1$ and $M_m = 1$, which means $\langle (\omega_m)_P, \varphi \rangle = \langle (\gamma_n)_P, \varphi \rangle = 1$. Without any ambiguity, we can suppose $\langle (\omega_1)_P, \varphi \rangle = \langle (\gamma_1)_P, \varphi \rangle = 1$. Since the elements of each set of characters $\{\omega_1, \omega_2, \omega_3\}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ are conjugate in $L = \mathrm{GL}(3, q)$ (see [7, Sec. 4]), then by Lemma 2.3 and Lemma 2.2, there exist $r, s, t, u \in N_L(G)$ such that $\varphi^r, \varphi^s, \varphi^t$ and φ^u are linear characters of P and

$$\langle (\omega_2)_P, \varphi^r \rangle = \langle (\omega_3)_P, \varphi^s \rangle = \langle (\gamma_2)_P, \varphi^t \rangle = \langle (\gamma_3)_P, \varphi^u \rangle = 1.$$

Hence, for $1 \leqslant m, n \leqslant 3$, the characters $(\omega_m)_P$ and $(\gamma_n)_P$ have a linear constituent with multiplicity 1. This completes the proof. □

4. **Basis for class functions**

If G is a finite group with n conjugacy classes such that t of these classes are p-elements, then the $n \times n$ matrix X constructed from the character table of G has an $n \times t$ submatrix M whose columns correspond to the p-elements. Since X is invertible, the columns of M are linearly independent and so M has rank t. Thus, there exist t irreducible characters of G such that for every irreducible character χ of G, the restriction of χ on p-elements is a linear combination of these t characters. Now, suppose $G = \mathrm{SL}(3, q)$. If $d = 1$, then equations (3.1) to (3.5) show that the class function ϕ_P of P is an integral linear combination of characters ρ_P, ψ_P and $\mathbf{1}$, for each generalized character ϕ of G. If $d = 3$, then using the values of the characters θ_k, ω_m and γ_n on p-elements, we have

$$\rho_P = (\theta_k)_P + (\omega_m)_P + (\gamma_n)_P - \mathbf{1}.$$

Now, considering equations (3.1) to (3.5), we have

$$\begin{aligned}(\zeta_i)_P &= \psi_P + \mathbf{1},\\ (\eta_j)_P &= (\theta_k)_P + (\omega_m)_P + (\gamma_n)_P + \psi_P - \mathbf{1},\\ (\varepsilon_r)_P &= (\theta_k)_P + (\omega_m)_P + (\gamma_n)_P + 2\psi_P,\\ (\mu_s)_P &= (\theta_k)_P + (\omega_m)_P + (\gamma_n)_P - 2\cdot\mathbf{1},\end{aligned}$$

and

$$(\nu_t)_P = (\theta_k)_P + (\omega_m)_P + (\gamma_n)_P - \psi_P.$$

Thus, $\{(\theta_k)_P, (\omega_m)_P, (\gamma_n)_P, \psi_P, \mathbf{1}_P\}$ is a basis for the vector space of class functions defined on P with integer coefficients. This is analogous to the theory of π-partial characters of solvable groups, developed by Isaacs, for the case $\pi = \{p\}$ (see [4]).

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