CATEGORY OF POLYGROUP OBJECTS

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Abstract: In this manuscript we generalize of the notion of polygroup in an arbitrary category \mathcal{E} , which is also a generalization of the notion of group object. We then define the category, $PGrp(\mathcal{E})$, of polygroup objects in \mathcal{E} , and we investigate some of its properties such as having limits and colimits. We also show that $PGrp(\mathcal{E})$ is a concrete category over the category $Mon(\mathcal{E})$ of monoid objects in \mathcal{E} , and that it has free objects and is geometric and essentially algebraic as such. Finally the preservation and reflection of epimorphism and monomorphism by the forgetful functor from $PGrp(\mathcal{E})$ to $Mon(\mathcal{E})$ is investigated.

1. Introduction

The hyperstructure theory was introduced by F. Marty in 1934 [6] at the 8th Congress of Scandinavian Mathematicians.

^oMSC (2000): 20N20, 18A30

⁰Keywords: Polygroup, Polygroup object, Essentially algebraic

^oReceived: 28 December, 2001; Revised: 14 March, 2002.

⁰Supported by a Grant from Ministry of Science, Research and Technology.

Bonansinga and Corsini used this structure and introduced the notion of quasicanonical hypergroup [2], called polygroup by Comer [3], which is a generalization of the notion of a group.

The theory has found applications in many branches of mathematics such as analysis, algebra, geometry, automata and fuzzy set. In this paper we give a generalization of the above notion by categorical methods, having two goals in mind. One is to embed the category of polygroups in a complete category, as limits have not been computed in the category of polygroups yet. Another is to give the hyperstructure theory a categorical organization. Having established these goals, we have come up with several other results, such as those mentioned in the abstract. First we give some notions that are needed in the sequel.

Definition 1.1 Let \mathcal{E} be a category with finite products. We call the triple $(P, *, \overline{E})$ a monoid object in \mathcal{E} if:

(a) $*: P \times P \to P$ is a morphism in \mathcal{E} such that the following diagram commutes.

$$\begin{array}{ccc} P^3 & \stackrel{ia_P \wedge *}{\longrightarrow} & P^2 \\ * \times id_P \downarrow & & \downarrow * \\ P^2 & \stackrel{*}{\longrightarrow} & P \end{array}$$

(b) $\overline{E} : 1 \to P$ is a morphism in \mathcal{E} such that the following diagram commutes.

$$P \xrightarrow{(id_P, E!_P)} P^2$$

$$\langle \overline{E}!_P, id_P \rangle \downarrow \qquad \searrow id_P \qquad \downarrow *$$

$$P^2 \xrightarrow{*} P$$

i.e., $id_P * \overline{E}!_P = \overline{E}!_P * id_P = id_P$, where $!_P : P \to 1$ is the unique morphism from P to the terminal object 1. We call \overline{E} the identity.

If $(P, *, \overline{E})$ and $(P', *', \overline{E}')$ are two monoid objects in \mathcal{E} , a morphism of monoid objects $f : (P, *, \overline{E}) \longrightarrow (P', *', \overline{E}')$ is a morphism $f : P \longrightarrow P'$ in \mathcal{E} such that $f * = *'f^2$, and $f\overline{E} = \overline{E}'$.

Proposition 1.2 Let $(P, *, \overline{E})$ and $(P', *', \overline{E}')$ be monoid objects in

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 \mathcal{E} and $f: (P, *, \overline{E}) \to (P', *', \overline{E}')$ be a morphism. If $E = \overline{E}!_P$ and $E' = \overline{E}'!_{P'}$, then fE = E'f.

Proof: Straightforward.

Theorem 1.3 The collection of monoid objects in \mathcal{E} together with morphisms forms a category, which is denoted by $Mon(\mathcal{E})$.

Proof: The proof is obvious. ■

Notation In the special case, where $\mathcal{E} = Set$ is the category of sets and

functions, we denote the category Mon(Set) by Mon.

Theorem 1.4 Let $\{(P_{\alpha}, *_{\alpha}, \overline{E}_{\alpha})\}_{\alpha \in I}$ be a collection of objects in $Mon(\mathcal{E})$. If \mathcal{E} has products then $(\prod P_{\alpha}, \prod *_{\alpha}, \prod \overline{E}_{\alpha})$ is a product of $\{(P_{\alpha}, *_{\alpha}, \overline{E}_{\alpha})\}$ in $Mon(\mathcal{E})$. In particular $Mon(\mathcal{E})$ has finite products.

Proof: Straightforward.

Definition 1.5 (see [7], Page 98) Let \mathcal{E} be a category with finite products. We call the quadruple (H, *, e, i) a group object in \mathcal{E} if: (a) $*: H \times H \to H$ is a morphism in \mathcal{E} such that the following diagram commutes.

$$\begin{array}{cccc} H^3 & \xrightarrow{1 \times *} & H^2 \\ * & \times & 1 & \downarrow & & \downarrow & * \\ & & H^2 & \xrightarrow{*} & H \end{array}$$

(b) $e : 1 \to H$ is a morphism in \mathcal{E} such that the following diagram commutes.

$$egin{array}{ccc} H & \stackrel{(1,e!_{H})}{\longrightarrow} & H^{2} \ & & & & & \\ \langle e!_{H},1
angle & \downarrow & \searrow & \downarrow & * \ & & H^{2} & \stackrel{-}{\longrightarrow} & H \end{array}$$

(c) $i: H \to H$ is a morphism in $\mathcal E$ such that the following diagram

commutes.

$$\begin{array}{cccc} H & \stackrel{\langle 1,i\rangle}{\longrightarrow} & H^2 \\ \langle i,1\rangle & \downarrow & \stackrel{e^!_H}{\searrow} & \downarrow & * \\ H^2 & \stackrel{*}{\longrightarrow} & H \end{array}$$

We denote the quadruple (H, *, e, i) by \hat{H} .

If \hat{H} and \hat{H}' are two group objects in \mathcal{E} . A morphism of group objects $\hat{f} : \hat{H} \longrightarrow \hat{H}'$ is defined by a morphism $f_H : H \rightarrow H'$ in \mathcal{E} such that $f_H * = *'f_H$.

Theorem 1.6 The collection of group objects in \mathcal{E} together with morphisms forms a category, which is denoted by $Grp(\mathcal{E})$.

Proof: Straightforward. ■

Notation In the special case, where $\mathcal{E} = Set$, we denote the category Grp(Set) by Grp.

Theorem 1.7 (see [5], pp 237-238) If G is a group object in \mathcal{E} , then each hom-set $Hom_{\mathcal{E}}(X,G)$ has the structure of a group, natural in X. Conversely, a group structure on $Hom_{\mathcal{E}}(X,G)$ for each object X of \mathcal{E} , natural in X, gives G the structure of an internal group. Equivalently G is a group object in \mathcal{E} if and only if $Hom_{\mathcal{E}}(-,G)$ is a group object in Set^{$\mathcal{E}^{\circ p}$}.

Definition 1.8 A hyperstructure is a nonempty set H together with a map

$$*: H \times H \longrightarrow P^*(H)$$

which is called hyperoperation, where $P^*(H)$ denotes the set of all nonempty subsets of H.

Remark 1.9 A hyperoperation $*: H \times H \longrightarrow P^*(H)$ yields an operation $\otimes: P^*(H) \times P^*(H) \longrightarrow P^*(H)$, defined by $A \otimes B = \bigcup_{a \in A, b \in B} a * b$. Conversely an operation on $P^*(H)$ yields a hyperoperation on H, defined by $x * y = \{x\} \otimes \{y\}$.

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Definition 1.10 A hyperstructure (H, *) is called a polygroup if it satisfies the following conditions

(a) x * (y * z) = (x * y) * z, for all x, y and z in H (associative law)

(b) there exists $e \in H$, such that $e * x = x * e = \{x\}$, for all $x \in H$ (identity element)

(c) for all $x \in H$, there exists a unique element x' of H such that $e \in x * x' \cap x' * x$ (inverse element)

(d) for all x, y and z in H we have

 $z \in x * y \Rightarrow x \in z * y' \Rightarrow y \in x' * z$ (reversibility property)

A morphism from (H, *) into (H', *) is defined by a map $f : H \to H'$ such that f(x * y) = f(x) * f(y).

Remark 1.11 In Definition 1.10 (c) the uniqueness of x' is not necessary, in fact we can obtain this property from the other conditions, provided that we replace condition (d) by $z \in x * y \Rightarrow \forall y', x \in z * y' \Rightarrow \forall x', y \in x' * z$.

Theorem 1.12 The collection of polygroups together with morphisms forms a category, which is denoted by PG.

Proof: Straightforward. ■

Definition 1.13 (see [4], Page 16) Let \mathcal{E} be a category with finite products, and $r : A \to B^2$ be a monomorphism in \mathcal{E} (that is r is a relation on B). Let $\alpha, \beta : X \to B$ be morphisms in \mathcal{E} . We say that $\alpha \leq_r \beta$ if there exists a morphism $h : X \to A$ in \mathcal{E} such that $rh = \langle \alpha, \beta \rangle$.

Definition 1.14 (see [4], Page 16) Let $r : A \to B^2$ be a monomorphism in \mathcal{E} . Then we say that r is

(a) reflexive if for every morphism $\alpha : X \to B$ in \mathcal{E} we have $\alpha \leq_r \alpha$.

(b) transitive if for every morphisms α, β and $\gamma : X \to B$ in $\mathcal{E}, \alpha \leq_r \beta$ and $\beta \leq_r \gamma$, implies that $\alpha \leq_r \gamma$.

(c) a preorder if it is reflexive and transitive.

The background category in the definition of a polygroup is the category *Set*. In order to generalize the notion of a polygroup we need to replace the category *Set* by an arbitrary category \mathcal{E} . To achieve this we need to free the definition of a polygroup from element taking, so we make the following observations.

Given a polygroup (H, *), by Remark 1.9, we have an operation on $P^*(H)$. It can be easily verified that this operation is associative if and only if the given hyperoperation on H is associative.

The element $e \in H$ yields the function $E : P^*(H) \to P^*(H)$ taking each set A to the singleton $\{e\}$. Observe that this function factors through the terminal object. Also we have a function $e : H \to H$ taking each element x to e. If $s : H \to P^*(H)$ is the singleton function, that is the function that takes x to $\{x\}$, then it can be easily seen that Es = se. Let us also notice here that the singleton function satisfies the condition: $sx \subseteq sy \Rightarrow x = y$.

The existence of a unique inverse yields a function $i: H \to H$. To interpret $e \in x * x'$ in an arbitrary category, we replace it by $\{e\} \subseteq x * x'$, and since $\{e\}$ and x * x' are elements of $P^*(H)$, we observe that we have a relation " \subseteq " on $P^*(H)$ and that $\{e\}$ is related to x * x'. In other words we have the relation $R = \{(A, B) : A \subseteq B\}$, the inclusion $r : R \to P^{*^2}(H)$, and that $(\{e\}, x * x') \in R$.

So we have a multiple $(H, P^*(H), R, s, r, *, E, i)$ satisfying conditions (a)-(d) of Definition 1.10, rewritten appropriately.

Using these observations we arrive at our definition of a polygroup object in an arbitrary category that is given in the next section.

2. Category of polygroup objects

Definition 2.1 Let \mathcal{E} be a category with finite products. A polygroup object in \mathcal{E} is a multiple (H, P, R, s, r, *, E, i) where H, P and R are objects in \mathcal{E} and s, r, *, E and i are morphisms in \mathcal{E} such that, $s : H \to P$ is a monomorphism and $r : R \to P^2$ is a preorder on P. Moreover for

all morphisms $\alpha, \beta : B \to H$ in \mathcal{E} if $s\alpha \leq_r s\beta$, then $\alpha = \beta$ and: (a) $*: P^2 \to P$ makes the following diagram commutative.

$$\begin{array}{cccc} P^3 & \xrightarrow{1 \times *} & P^2 \\ * \times 1 & \downarrow & \downarrow & * \\ P^2 & \xrightarrow{*} & P \end{array}$$

and hence we say that * is associative.

(b) $E: P \to P$ makes the following diagram commutative.

$$\begin{array}{cccc} P & \stackrel{(1,E)}{\longrightarrow} & P^2 \\ \langle E,1 \rangle \downarrow & \stackrel{1}{\searrow} & \downarrow & * \\ P^2 & \stackrel{-}{\longrightarrow} & P \end{array}$$

That is, 1 * E = E * 1 = 1, and also Es factors through s, i.e., there exists a morphism $e : H \to H$ in \mathcal{E} such that Es = se. Moreover there exist morphisms $\overline{E} : 1 \to P$ and $\overline{e} : 1 \to H$ in \mathcal{E} such that $E = \overline{E}!_P$ and $e = \overline{e}!_H$. We call E the identity.

(c) $i: H \to H$ satisfies $Es \leq_r 1 * i$ and $Es \leq_r i * 1$ where $1 * i = *s^2 \langle 1, i \rangle$ and $i * 1 = *s^2 \langle i, 1 \rangle$. We call i an inverse.

(d) For all morphisms α, β and $\gamma: B \to H$ in \mathcal{E} we have the following implications:

$$s\alpha \leq_r s\beta * s\gamma \Rightarrow s\beta \leq_r s\alpha * si\gamma \Rightarrow s\gamma \leq_r si\beta * s\alpha$$

In this definition we denote the multiple (H, P, R, s, r, *, E, i) by \hat{H} .

Theorem 2.2 Let $\hat{H} = (H, P, R, s, r, *, E, i)$ be a polygroup object in \mathcal{E} and $\star : H^2 \to H$ be a morphism in \mathcal{E} such that $s\star = *s^2$. Then (H, \star, e_H, i) is a group object in \mathcal{E} .

Proof: It is enough to show that $1 \star i = e$. From Definition 2.1(c) it follows that $Es \leq_r s(1 \star i)$ and so $se \leq_r s(1 \star i)$. Thus $1 \star i = e$. Similarly $i \star 1 = e$.

Theorem 2.3 Let (H, *, e, i) be a group object in \mathcal{E} . Set H = P = R and let $r = \Delta = \langle 1, 1 \rangle$ be the diagonal morphism from H

into H^2 , and $s = id_H$ be the identity morphism on H in \mathcal{E} . Then $(H, H, H, id_H, \Delta, *, e, i)$ is a polygroup object in \mathcal{E} .

Proof: Straightforward.

Proposition 2.4 Let \hat{H} be a polygroup object in \mathcal{E} , then the following statements hold:

(i) $Es = se = \overline{E}!_H$ (ii) $s\overline{e} = \overline{E}$, (iii) $1 * \overline{E}!_P = 1$, (iv) $i^2 = 1$, (v) $i\overline{e} = \overline{e}$, (vi) ie = ei = e, (vii) $E^2 = E$, and (viii) E * E = E.

Proof: Straightforward.

Notation Let \mathcal{E} be a category and A be an object in \mathcal{E} . We denote the functor $Hom(-, A) : \mathcal{E}^{op} \longrightarrow Set$ by \overline{A} and if $f : A \to B$ is a morphism in \mathcal{E} we denote the natural transformation $Hom(-, f) : \overline{A} \longrightarrow \overline{B}$, by \overline{f} .

Lemma 2.5 Let $r : R \to P^2$ be a monomorphism in \mathcal{E} . For all objects F and morphisms $f, g : F \to \overline{P}$ in $Set^{\mathcal{E}^{op}}$ we have

$$f \leq_{\overline{r}} g \iff \forall A \in \mathcal{E}, \forall x \in F(A), f_A(x) \leq_r g_A(x).$$

Proof: Let $f \leq_{\overline{r}} g$, so there exists a morphism $h: F \longrightarrow \overline{R}$ in $Set^{\mathcal{E}^{\circ p}}$ such that $\overline{r}h = \langle f, g \rangle$. Thus we get $h_A(x) : A \longrightarrow R$, so $f_A(x) \leq_r g_A(x)$, for all $A \in \mathcal{E}$ and $x \in F(A)$. Thus $f_A(x) \leq_r g_A(x)$.

Now suppose for all $A \in \mathcal{E}$ and $x \in F(A)$ we have $f_A(x) \leq_r g_A(x)$. So there exists a morphism $h_{A,x} : A \to R$ such that $rh_{A,x} = \langle f_A(x), g_A(x) \rangle \, \forall A \in \mathcal{E}$ and $x \in F(A)$. Now define $h_A : F(A) \longrightarrow Hom(A, R)$ by $h_A(x) = h_{A,x}$. It easily follows that $h : F \to \overline{R}$ is a natural transformation and $(\overline{r}h)_A(x) = \langle f, g \rangle_A(x)$, so $f \leq_{\overline{r}} g$. **Lemma 2.6** Let $r : R \to P^2$ be a monomorphism in \mathcal{E} . Then for all morphisms $\alpha, \beta : B \to P$ in \mathcal{E} we have: $\alpha \leq_r \beta \iff \overline{\alpha} \leq_{\overline{r}} \overline{\beta}$.

Proof: First suppose that $\alpha \leq_r \beta$. So there exists a morphism $h : B \to R$ such that $rh = \langle \alpha, \beta \rangle$. If $A \in \mathcal{E}$ and $x : A \to B$ is a morphism in \mathcal{E} , so we get $(\overline{rh})_A(x) = \overline{r}_A(\overline{h}_A(x))$. Thus $\overline{\alpha} \leq_{\overline{r}} \overline{\beta}$.

Now suppose $\overline{\alpha} \leq_{\overline{r}} \overline{\beta}$. Then there exists a morphism $\overline{h} : \overline{B} \longrightarrow \overline{R}$ in $Set^{\mathcal{E}^{op}}$ such that $\overline{rh} = \langle \overline{\alpha}, \overline{\beta} \rangle$. By Yoneda lemma we have $\overline{h} = Hom(-, h)$, where $h = \overline{h}_B(1)$. So we have $(\overline{rh})_B(1) = \langle \alpha, \beta \rangle$, thus $rh = \langle \alpha, \beta \rangle$. Therefore $\alpha \leq_r \beta$.

Remark 2.7 Let $f': \overline{A} \to \overline{B}$ be a morphism in $Set^{\mathcal{E}^{op}}$. By Yoneda Lemma we get $f' = \overline{f}$, where $f = f'_A(1_A)$.

Theorem 2.8 Let H, P, R, s, r, *, E, i be as in the statement of Definition if $\hat{H} = (\overline{H}, \overline{P}, \overline{R}, \overline{s}, \overline{r}, \overline{*}, \overline{e}, \overline{i})$ is a polygroup 2.1. Then $\hat{H} = (H, P, R, s, r, *, E, i)$ is a polygroup object in \mathcal{E} if and only object in Set^{$\mathcal{E}^{\circ p}$}.

Proof: Follows from Lemmas 2.5 and 2.6.

Remark 2.9 Theorem 2.8 is the generalized version of Theorem 1.7.

Definition 2.10 Let \hat{H} and $\hat{H'}$ be polygroup objects in \mathcal{E} . A morphism $\hat{f} : \hat{H} \to \hat{H'}$ is a triple (f_H, f_P, f_R) where $f_H : H \to H'$, $f_P : P \to P'$ and $f_R : R \to R'$ are morphisms in \mathcal{E} such that (a) $f_P s = s' f_H$, (b) $f_P^2 r = r' f_R$, (c) $f_P * = *' f_P^2$, (d) $f_P E = E' f_P$.

Theorem 2.11 If $\hat{f} = (f_H, f_P, f_R) : \hat{H} \to \hat{H}'$ is a morphism, then: (i) $f_H e = e' f_H$, (ii) $f_H i = i' f_H$.

Proof: (i) Straightforward.

(ii) We know that $Es \leq_r 1 * i$, thus:

 $\overline{E}!_H \leq_r 1 * i$; by Proposition 2.4(i)

$$\Rightarrow f_P s \overline{e}!_H \leq_r f_P * s^2 \langle 1, i \rangle ; \text{by Proposition 2.4(ii)}$$

$$\Rightarrow s' f_H \overline{e}!_H \leq_r f_P * s^2 \langle 1, i \rangle$$
; by Definition 2.10(a) and Proposition 2.4(i)

 $\Rightarrow E's'f_H \leq_r f_P * s^2 \langle 1, i \rangle$; by Theorem 2.11(i) and Proposition 2.4(i)

$$\Rightarrow E's'f_H \leq_r *'(s'f_H)^2 \langle 1,i \rangle ; \text{ by Definition 2.10(a)and(c)}$$

 $\Rightarrow \quad s'\overline{e}'!_H \leq_r s'f_H * s'f_H i \text{ ; by Proposition 2.4(i)-(ii)}$

$$\Rightarrow \quad s'f_H i \leq_r s'i'f_H * s'\overline{e}'!_H. ; \text{by Proposition 2.4(ii)-(iii)}$$

Hence $f_H i = i' f_H$.

Lemma 2.12 Let \hat{H} and \hat{G} be two polygroup objects in \mathcal{E} and \hat{f} : $\hat{H} \to \hat{G}$ be a morphism. If $!_H : H \to 1$ is an epimorphism then $f_H \overline{e} = \overline{e'}$.

Proof: Straightforward. ■

Theorem 2.13 The collection of all polygroup objects in \mathcal{E} together with polygroup morphisms forms a category, which is denoted by $PGrp(\mathcal{E})$.

Sketch of proof: For $\hat{f} : \hat{H} \to \hat{H}'$ and $\hat{g} : \hat{H}' \to \hat{H}''$, the composition $\hat{f} \circ \hat{g}$ is defined by $(f_H \circ g_{H'}, f_P \circ g_{P'}, f_R \circ g_{R'})$. The identity morphism $\hat{id} : \hat{H} \to \hat{H}$ is defined to be the triple (id_H, id_P, id_R) , where id_H, id_P , and id_R are the identities on H, P and R respectively in \mathcal{E} .

Notation In the special case, where $\mathcal{E} = Set$, we denote PGrp(Set) by PGrp.

Remark 2.14 By Theorems 2.2 and 2.3, the category $Grp(\mathcal{E})$ can be embedded in the category $PGrp(\mathcal{E})$, i.e. $Grp(\mathcal{E}) \subseteq PGrp(\mathcal{E})$. It then follows that

$$Mon \subseteq Grp \subseteq PG \subseteq PGrp$$

Theorem 2.15 Let $\{\hat{H}_{\alpha}\}_{\alpha \in I}$ be a collection of objects in $PGrp(\mathcal{E})$, where

$$H_{\alpha} = (H_{\alpha}, P_{\alpha}, R_{\alpha}, s_{\alpha}, r_{\alpha}, *_{\alpha}, E_{\alpha}, i_{\alpha})$$

If \mathcal{E} has products then

$$\Pi H_{\alpha} = (\Pi H_{\alpha}, \Pi P_{\alpha}, \Pi R_{\alpha}, \Pi s_{\alpha}, \Pi r_{\alpha}, \Pi *_{\alpha}, \Pi E_{\alpha}, \Pi i_{\alpha})$$

is a product of $\{\hat{H}_{\alpha}\}$ in $PGrp(\mathcal{E})$. In particular $GPGrp(\mathcal{E})$ has finite products.

Proof: Straightforward.

Theorem 2.16 Let \mathcal{E} be a category such that for all objects H in \mathcal{E} , $!_H : H \longrightarrow 1$ is an epimorphism. If \mathcal{E} has equalizers, then so does $PGrp(\mathcal{E})$.

Proof: Let $\hat{H} = (H, P, R, s, r, *, E, i)$ and $\hat{G} = (G, P', R', s', r', *', E', i')$ be two polygroup objects. Given a pair of morphisms $\hat{f}, \hat{g} : \hat{H} \to \hat{G}$ in $PGrp(\mathcal{E})$, let $\epsilon_K : K \to H, \epsilon_{P''} : P'' \to P$, and $\epsilon_{R''} : R'' \to R$ be equalizers of $(f_H, g_H), (f_P, g_P)$, and (f_R, g_R) , respectively. Using the fact that $\epsilon_K, \epsilon_{P''}$, and $\epsilon_{P''}^2$ are equalizers, we obtain the morphisms $s'' : K \to P'', r'' : R'' \to P''^2, *'' : P''^2 \to P'', E'' : P'' \to P'',$ and $i'' : K \to K$. It is straightforward though tedious to show that $\hat{K} = (K, P'', R'', s'', r'', *'', E'', i'')$ is an equalizer of \hat{f}, \hat{g} .

Corollary 2.17 Let \mathcal{E} be a category in which for all objects H in \mathcal{E} , $!_H : H \longrightarrow 1$ is an epimorphism. If \mathcal{E} has limits then so does $PGrp(\mathcal{E})$.

Proof: Since \mathcal{E} has products and equalizers, by Theorems 2.15 and 2.16, so does $PGrp(\mathcal{E})$. Therefore $PGrp(\mathcal{E})$ has limits.

3. Free Polygroup Objects

Proposition 3.1 There exists a faithful functor U_1 from $PGrp(\mathcal{E})$ into $Mon(\mathcal{E})$, where $U_1(\hat{H} \xrightarrow{\hat{f}} \hat{H}') = ((P, *, \overline{E}) \xrightarrow{f_P} (p', *', \overline{E}'))$, that is $PGrp(\mathcal{E})$ is concrete over $Mon(\mathcal{E})$.

Proof: Let $\hat{H} = (H, P, R, s, r, *, E, i)$ and $\hat{H}' = (H', P', R', s', r', *', E', i')$ be two arbitrary objects in $PGrp(\mathcal{E})$ and $\hat{f} = (f_H, f_P, f_R)$ be a morphism from \hat{H} into \hat{H}' in $PGrp(\mathcal{E})$. Then it is easy to see that $U_1 : PGrp(\mathcal{E}) \rightarrow$ $Mon(\mathcal{E})$ is a functor. Now we show that U_1 is faithful. Let $\hat{f}, \hat{g} : \hat{H} \to \hat{H}'$ be morphisms in $PGrp(\mathcal{E})$, such that $U_1(\hat{f}) = U_1(\hat{g})$. Thus $f_P = g_P$. From Definition 2.10(a) we know that $f_P s = s' f_H$ and $g_P s = s' g_H$. So $s' f_H = s' g_H$ and since s' is a monomorphism we get that $f_H = g_H$. Also we have $f_P^2 r = r' f_R$ and $g_P^2 r = r' g_R$, by Definition 2.10 (b), so $r' f_R = r' g_R$. Since r' is a monomorphism, thus $f_R = g_R$. Therefore $\hat{f} = \hat{g}$, that is U_1 is faithful.

Definition 3.2 Let $(P, *, \overline{E})$ be an object in $Mon(\mathcal{E})$ and $\Delta : P \to P^2$ be the diagonal morphism. Then $(1, P, P, \overline{E}, \Delta, *, \overline{E}!_P, id_1)$ is an object in $PGrp(\mathcal{E})$. We denote this object by \hat{P}^1 .

Theorem 3.3 The concrete category $(PGrp(\mathcal{E}), U_1)$ has free objects.

Proof: Let $(P, *, \overline{E})$ be an object in $Mon(\mathcal{E})$. We claim that \hat{P}^1 is a free object over $(P, *, \overline{E})$. For this reason we show that the identity morphism

$$id_{(P,*,\overline{E})}:(P,*,\overline{E})\longrightarrow U_1(\hat{P}^1)=(P,*,\overline{E})$$

is a universal arrow over $(P, *, \overline{E})$. Let $\hat{H}' = (H', P', R', s', r', *', E', i')$ be another object in $PGrp(\mathcal{E})$ and $g: (P, *, \overline{E}) \longrightarrow U_1(\hat{H}') = (P', *', \overline{E})$ be a morphism in $Mon(\mathcal{E})$. Thus by Definition 1.1 we have $g* = *'g^2$ and $g\overline{E}!_P = (\overline{E}'!_{P'})g$. Since \hat{H}' is an object in $PGrp(\mathcal{E})$ thus by Definition 2.1(b) we get $\overline{e}'!_{H'} = e'$ and we know that $g \leq_{\Delta} g$, thus there exists a morphism $g': P \to R'$ such that $r'g' = \langle g, g \rangle = g^2 \Delta$. It is easy to check that $\hat{g} := (\overline{e}', g, g')$ is a morphism from \hat{P} into \hat{H}' in $PGrp(\mathcal{E})$, and $U_1(\hat{g}) = g$, which implies that $U_1(\hat{g}) \circ id_{(P,*,\overline{E})} = g$. Now suppose that $\hat{f} = (f_1, f_2, f_3)$ be an arbitrary morphism from \hat{P} into \hat{H} such that $U_1(\hat{f}) \circ id_{(P,*,\overline{E})} = g$. Therefore we get $U_1(\hat{f}) = g = U_1(\hat{g})$. Since U_1 is faithful, thus $\hat{f} = \hat{g}$. So \hat{g} is a unique morphism such that $U_1(\hat{g}) \circ id_{(P,*,\overline{E})} = g$.

Proposition 3.4 The mapping F_1 defined by

$$F_1((P, *, \overline{E}) \xrightarrow{f} (P', *', \overline{E}')) = (\hat{P}^1 \xrightarrow{(id_1, f, f)} \hat{P'}^1)$$

is a functor from $Mon(\mathcal{E})$ into $PGrp(\mathcal{E})$.

Proof: Straightforward.

Lemma 3.5

(i) $\eta_{1_{(P,*,\overline{E})}} = id_{(P,*,\overline{E})} : (P,*,\overline{E}) \longrightarrow U_1F_1(P,*,\overline{E}) = (P,*,\overline{E})$ defines $\eta_1 : Id_{Mon(\mathcal{E})} \longrightarrow U_1F_1$ as a natural transformation.

(ii) For every object $\hat{H} = (H, P, R, s, r, *, E, i)$ in $PGrp(\mathcal{E})$, the mapping $\epsilon_{1_{\hat{H}}} = (e', id_P, h) : F_1U_1(\hat{H}) = \hat{P}^1 \to \hat{H}$ defines $\epsilon_1 : F_1U_1 \to Id_{PGrp(\mathcal{E})}$ as a natural transformation, where $h : P \to R$ is a morphism in \mathcal{E} such that $rh = \langle id_P, id_P \rangle$.

Proof: Straightforward.

Theorem 3.6 Suppose that U_1 , F_1 , η_1 and ϵ_1 as in Propositions 3.1, 3.4 and Lemma 3.5 respectively. Then we have $(\eta_1, \epsilon_1) : F_1 \vdash U_1 :$ $(PGrp(\mathcal{E}), Mon(\mathcal{E}))$ is an adjoint situation.

Proof: By Lemma 3.5 it is enough to show that $U_1\epsilon_1 \circ \eta_1 U_1 = id_{U_1}$ and $\epsilon_1 F_1 \circ F_1 \eta_1 = id_{F_1}$. We only show the former, the latter is similar. Let

 $\hat{H} = (H, P, R, s, r, *, E, i)$ be an object in $PGrp(\mathcal{E})$. So we have

$$\begin{aligned} (U_1\epsilon_1 \circ \eta_1 U_1)(\hat{H}) &= U_1\epsilon_1(\hat{H}) \circ (\eta_1 U_1)(\hat{H}) \\ &= U_1\epsilon_{1_{\hat{H}}} \circ \eta_{1_{U_1(\hat{H})}} \\ &= id_{(P,*,\overline{E})} \circ id_{(P,*,\overline{E})} \text{ ; by Proposition 3.1} \\ &= U_1(\overline{\epsilon}, id_P, h) \circ id_{(P,*,\overline{E})} \text{ ; by Lemma 3.5 (i)} \\ &= id_{U_1(\hat{H})}. \end{aligned}$$

So $U_1 \epsilon_1 \circ \eta_1 U_1 = i d_{U_1}$.

Theorem 3.7 Let $T_1 = (T_1, \eta_1, \mu_1)$ be the monad associated with the adjoint situation given in Theorem 3.6, where $T_1 = U_1F_1$. Then T_1 is the trivial monad.

Proof: Let x be an object in the Eilenberg-Moore category $(Mon(\mathcal{E})^{T_1}, U_1^{T_1})$ [see 1]. So we have $x: T_1(P, *, \overline{E}) \to (P, *, \overline{E})$, for some object $(P, *, \overline{E})$ in $Mon(\mathcal{E})$, that satisfies

- (a) $x \circ \eta_{1_{(P,*,\overline{E})}} = id_{(P,*,\overline{E})}$, and
- (b) $x \circ T_1 x = x \circ \mu_{1_{(P, *, \overline{E})}}$.

By (a) and Lemma 3.5 (i) we have $x = id_{(P,*,\overline{E})}$. Since $T_1 = U_1F_1$, so we get $T_1 = Id_{Mon(\mathcal{E})}$, and by condition (b) we conclude that $\mu_1 = id_{T_1}$. Thus we have $\mu_1 = \eta_1 = id_{T_1}$. Then the monad T_1 is trivial monad $(Id_{Mon(\mathcal{E})}, id_{T_1}, id_{T_1})$.

Corollary 3.8 The Eilenberg-Moore category $(Mon(\mathcal{E})^{T_1}, U_1^{T_1})$ is concretely isomorphic to $(Mon(\mathcal{E}), Id_{Mon(\mathcal{E})})$.

Proof: Straightforward.

Theorem 3.9 Let $(\eta_2, \epsilon_2) : F_2 \to U_2 : Mon(\mathcal{E}) \to \mathcal{E}$ be an adjoint situation and $T_2 = (T_2, \eta_2, \mu_2)$ be its associated monad. Then the Eilenberg-Moore category $(\mathcal{E}^{T_2}, U_2^{T_2})$ is concretely isomorphic to the category $Mon(\mathcal{E})$.

Sketch of Proof: If $x : T_2P \to P$ is a T_2 -algebra, multiplication in P is defined by $* = x \circ U_2 *_2 \circ l_P \times l_P$ where $l_P : P \to U_2F_2(P)$ is an universal arrow and $F_2(P) = (F_2P, *_2, \overline{E}_2)$ is a free object over P in $Mon(\mathcal{E})$. And \overline{E} defined by $\overline{E} = x \circ U_2\overline{E}_2$. Then $(P, *, \overline{E})$ is an object in $Mon(\mathcal{E})$.

Let $(\eta_1, \epsilon_1) : F_1 \vdash U_1 : (PGrp(\mathcal{E}), Mon(\mathcal{E}))$ be adjoint situation as in Theorem 3.6. Let $U = U_2U_1$, $F = F_1F_2$, $\eta = U_2\eta_1F_2 \circ \eta_2$, $\epsilon = \epsilon_1 \circ F_1\epsilon_2U_1$ and $\mu = U\epsilon F$. Since composition of adjoint situations is an adjoint situations (see Proposition 19.13[1]), so we have $(\eta, \epsilon) : F \vdash U$: $(PGrp(\mathcal{E}), \mathcal{E})$ is an adjoint situation. Suppose that $T = (T, \eta, \mu)$ be its associated monad, then we have the following theorem:

Theorem 3.10 The Eilenberg-Moore category (\mathcal{E}^T, U^T) is concretely isomorphic to the $Mon(\mathcal{E})$.

Proof: The proof is obvious by Theorems 3.8 and 3.9. ■

Proposition 3.11 If the category \mathcal{E} has finite products, then the functor

 $F_1: Mon(\mathcal{E}) \to PGrp(\mathcal{E})$ given in Proposition 3.4, preserves finite products.

Proof: Let $\{(P_{\alpha}, *_{\alpha}, \overline{E}_{\alpha})\}_{\alpha \in I}$ be a finite family of objects in $Mon(\mathcal{E})$. Then by Theorem 1.4 the object $(\Pi P_{\alpha}, \Pi *_{\alpha}, \Pi \overline{E}_{\alpha})$ is a product of the above family in $Mon(\mathcal{E})$. Let $\Delta : \Pi P_{\alpha} \to (\Pi P_{\alpha})^2$ be the diagonal morphism, then $Pr_{\alpha}^2 \Delta = \Delta_{\alpha} Pr_{\alpha}$, for all $\alpha \in I$, where $Pr_{\alpha} : \Pi P_{\alpha} \to P_{\alpha}$ is the canonical projection morphism. By Theorem 2.15 $F(\Pi P_{\alpha}, \Pi *_{\alpha}, \Pi \overline{E}_{\alpha}) = \Pi \widehat{P_{\alpha}}^1 = (1, \Pi P_{\alpha}, \Pi P_{\alpha}, \Pi \overline{E}_{\alpha}, \Delta, \Pi *_{\alpha}, \Pi \overline{E}_{\alpha}!_{\Pi P_{\alpha}}, id_1)$ together with the canonical projection $\widehat{P_{r_{\alpha}}} = (id_1, Pr_{\alpha}, Pr_{\alpha})$ is a product of the family $\{F(P_{\alpha}, *_{\alpha}, \overline{E}_{\alpha}) = \widehat{P_{\alpha}}^1 = (1, P_{\alpha}, P_{\alpha}, \overline{E}_{\alpha}, \Delta_{\alpha}, *_{\alpha}, \overline{E}_{\alpha}!_{P_{\alpha}}, id_1)\}$ in $PGrp(\mathcal{E})$.

Proposition 3.12 Let \mathcal{E} have equalizers. If the following diagram

$$(P, *, \overline{E}) \xrightarrow{f} (P', *', \overline{E}') \xrightarrow{g}_{\overrightarrow{h}} (P'', *'', \overline{E}'') (1)$$

is an equalizer in $Mon(\mathcal{E})$, then

$$F_1(P, *, \overline{E}) = \hat{P}^1 \xrightarrow{\hat{f} = (id_1, f, f)} F_1(P', *', \overline{E}') = \hat{P'} \xrightarrow{\hat{I}^{g = (id_1, g, g)}}_{\hat{h} = (id_1, h, h)} F_1(P'', *'', \overline{E}'') = \hat{P''}^1$$

is an equalizer in $PGrp(\mathcal{E})$.

Proof: Since gf = hf, so $\hat{g}\hat{f} = \hat{h}\hat{f}$. Let $\hat{H}_1 = (H_1, P_1, R_1, s_1, r_1, *_1, E_1, i_1)$ be an object in $PGrp(\mathcal{E})$ and $\hat{k}_1 = (k_{H_1}, k_{P_1}, k_{R_1}) : \hat{H}_1 \to \hat{P'}^1$ be a morphism in $PGrp(\mathcal{E})$, such that $\hat{g}\hat{k}_1 = \hat{h}\hat{k}_1$. By Theorem 2.13 we have $gk_{P_1} = hk_{P_1}$ and $gk_{R_1} = hk_{R_1}$.

Since $(P_1, *_1, \overline{E}_1)$ is an object in $Mon(\mathcal{E})$ and the diagram (1) is an equalizer in $Mon(\mathcal{E})$, so we get the unique morphism $t_{P_1} : (P_1, *_1, \overline{E}_1) \rightarrow (P, *, \overline{E})$ in $Mon(\mathcal{E})$, such that $ft_{P_1} = k_{P_1}$. Let $(F_2(R_1), *_2, \overline{E}_2)$ be the free monoid over R_1 in $Mon(\mathcal{E})$, so we have the unique morphism $h' : (F_2(R_1), *_2, \overline{E}_2) \rightarrow (P', *', \overline{E'})$ in $Mon(\mathcal{E})$ such that $U_2(h') \circ l = k_{R_1}$, where $l : R_1 \rightarrow U_2F_2(R_1)$ is a U_2 -universal morphism in \mathcal{E} . Thus we

have $h' \circ l = k_{R_1}$. Since $g : (P', *', \overline{E}') \to (P'', *'', \overline{E}'')$ is a morphism in $Mon(\mathcal{E})$, then we get gh' = hh'. So by diagram (I) we get a unique morphism $t' : (F_2(R_1), *_2, \overline{E}_2) \to (P, *, \overline{E})$ in $Mon(\mathcal{E})$ such that ft' = h'. Now, define $t_{R_1} := U_2(t') \circ l$, that is, $t_{R_1} = t' \circ l$. Since the concrete category $(Mon(\mathcal{E}), U_2)$ over \mathcal{E} , has free objects, and f is a monomorphism in $Mon(\mathcal{E})$, then by Theorem 8.38 [1], we have $U_2(f) = f$ is a monomorphism in \mathcal{E} . Now it easily follows that $\hat{t}_1 = (!_{H_1}, t_{P_1}, t_{R_1}) : \hat{H}_1 \to \hat{P}$ is a morphism in $PGrp(\mathcal{E})$. Also $\hat{f}\hat{t}_1 = \hat{k}_1$, and \hat{t}_1 is a unique morphism such that $\hat{f}\hat{t}_1 = \hat{k}_1$.

Theorem 3.13 If \mathcal{E} has finite limits, then the pair (F_1, U_1) where $F_1 \vdash U_1$ is a geometric morphism.

Proof: The proof is obvious by Propositions 3.11 and 3.12, see also page 26 of [4]. \blacksquare

Remark 3.14 By Theorem 3.3, the concrete category $(PGrp(\mathcal{E}), U_1)$ over

 $Mon(\mathcal{E})$ has free objects. So by Theorem 8.38 [1], we get that U_1 preserves and reflects monomorphisms, and by Proposition 7.44 [1], U_1 reflects epimorphisms.

Example 3.15 We know that $(Z, 0, \overline{E})$ and $(Q, 0, \overline{E}')$ are objects in *Mon*. Let f be the inclusion homomorphism from $(Z, 0, \overline{E})$ into $(Q, 0, \overline{E}')$, i.e., f(n) = n for all $n \in Z$. We have that f is an epimorphism, by some manipulation (see Example 7.40 (5) of [1]). But $f: Z \hookrightarrow Q$ as a function in *Set* is not an epimorphism. We know that \hat{Z}^1 and \hat{Q}^1 are objects in *PGrp*, and $\hat{f} = (id_1, f, f) : \hat{Z}^1 \to \hat{Q}^1$ is a morphism in *PGrp*. It is easy to see that \hat{f} is an epimorphism in *PGrp*. But $U(\hat{f}) = f: Z \hookrightarrow Q$, as a morphism in *Set*, is not an epimorphism.

Notation The full subcategory of PGrp whose objects are $(1, P, R, s, r, *, E, id_1)$ is denoted by P_1Grp .

Theorem 3.16 If $\hat{f} = (f_1, f_2, f_3) : \hat{P} \to \hat{P}'$ is an epimorphism in P_1Grp , then $U_1(\hat{f}) = f_2$ is an epimorphism in Mon.

Proof: Let $(P, *, \overline{E}) \xrightarrow{f_2} (P', *', \overline{E}') \xrightarrow{g_2}{h_2} (P'', *'', \overline{E}'')$ be a diagram in Mon, such that $g_2f_2 = h_2f_2$. We know that $r' : R' \to {P'}^2$ and r'(R') is the set defined by $\{(p'_1, p'_2) | \exists x \in R', \text{ such that } r'(x) = (p'_1, p'_2)\}$. Now let R'' be the smallest preorder relation on the set

$$\{(g_2(p_1'),g_2(p_2'))|(p_1',p_2')\in r'(R')\}\cup\{(h_2(p_1'),h_2(p_2'))|(p_1',p_2')\in r'(R')\}$$

and $r'': R'' \hookrightarrow {P''}^2$ be the inclusion map. Define $s'': 1 \to P''$ by $s'' = g_2 s'$. Thus s'' is a monomorphism in Set. Since for any morphisms $\alpha, \beta: B \to 1$ in Set, we have $\alpha = \beta$, therefore if $s'' \alpha \leq_{r''} s'' \beta$, then $\alpha = \beta$. Now define $\hat{P}'' = (1, P'', R'', s'', r'', *'', \overline{E}''_{P''}, id_1)$. It is easy to check that \hat{P}'' is an object in $P_1 Grp(Set)$. It can easily be checked that $\hat{g} = (id_1, g_2, g_2^2 r')$ and $\hat{h} = (id_1, h_2, h_2^2 r')$ are morphisms in $P_1 Grp$ from \hat{P}' to \hat{P}'' . Therefore $\hat{h}\hat{f} = \hat{g}\hat{f}$, and since \hat{f} is an epimorphism in $P_1 Grp$ is an epimorphism in Mon.

Proposition 3.17 Let $\hat{f} = (f_H, f_P, f_R) : \hat{H} \to \hat{H}'$ be a morphism in $PGrp(\mathcal{E})$. \hat{f} is an isomorphism in $PGrp(\mathcal{E})$ if and only if f_H , f_P and f_R are isomorphisms in \mathcal{E} .

Proof: Straightforward. ■

Remark 3.18 In Example 3.15, we showed that $\hat{f} = (id_1, f, f)$: $\hat{Z}^1 \rightarrow \hat{Q}^1$ is an epimorphism in *PGrp*. Since the concrete category (*PGrp*, *U*) over *Set* has free objects, so by Theorem 8.38 [2] we get \hat{f} is also a monomorphism in *PGrp*. But \hat{f} is not an isomorphism in *PGrp*, because if \hat{f} is an isomorphism in *PGrp*, then $U(\hat{f}) = f : Z \hookrightarrow Q$ is an isomorphism in *Set*, which is a contradiction. So the category *PGrp* is not balanced, and so it is not a topos.

4. Essentially Algebraic Property

We start this section by assuming that the forgetful functor $U_2 : Mon(\mathcal{E}) \to \mathcal{E}$ is (generating, Mono-source)-factorizable. For example, we know that

the forgetful functor $U_2 : Mon \to Set$ is an adjoint, and so by Proposition 18.3 [1], it is (generating,-)-factorizable, hence it is (generating, Mono-source)-factorizable.

Throughout this Section we use $U = U_2U_1 : PGrp(\mathcal{E}) \to \mathcal{E}$, where U_1 and U_2 are the functors introduced in section 3.

Theorem 4.1: The functor $U : PGrp(\mathcal{E}) \to \mathcal{E}$ is (generating, Monosource)-factorizable.

Proof: Let $\{\hat{H}_j = (H_j, P_j, R_j, s_j, r_j, *_j, E_j, i_j)\}_{j \in I}$ be a collection of objects in $PGrp(\mathcal{E})$, and $(X \xrightarrow{f_j} U(\hat{H}_j) = P_j)_{j \in I}$ be a U-structure source. Since the functor U_2 : $Mon(\mathcal{E}) \rightarrow \mathcal{E}$ is (generating, Monosource)-factorizable, thus there exists a U_2 -generating morphism $e: X \to$ $U_2(G, *, \overline{E})$ and a mono-source $((G, *, \overline{E}) \xrightarrow{m_j} (P_j, *_j, \overline{E}_j))_{j \in I}$ in $Mon(\mathcal{E})$, such that $f_j = U_2(m_j) \circ e$; for all $j \in I$. On the other hand, $U_2(G, *, \overline{E}) =$ $U_2U_1F_1(G, *, \overline{E}) = U(\hat{G}^1)$. Therefore we have the morphism $e: X \to U_2U_1F_1(G, *, \overline{E}) = U(\hat{G}^1)$. $U(\hat{G}^1)$. We claim that e is U-generating. To show this, let $\hat{G}^1 \stackrel{i}{\longrightarrow} \hat{K}$ be morphisms in $PGrp(\mathcal{E})$, such that $U(\hat{g}) \circ e = U(\hat{h}) \circ e$. So $U_2(U_1(\hat{g})) \circ e =$ $U_2(U_1(\hat{h})) \circ e$, since e is U_2 -generating thus $U_1(\hat{g}) = U_1(\hat{h})$. Therefore $\hat{g} = \hat{h}$, because U_1 is faithful. Now we want to get a mono-source $(\hat{G}^1 \xrightarrow{\hat{m}_j} \hat{H}_j)_{j \in I}$ in $PGrp(\mathcal{E})$. Since $r_j : R_j \to P_j^2$ is reflexive, for all $j \in I$, thus $m_j \leq_{r_j} m_j$. Hence for all $j \in I$, there exists a morphism $h_j : G \to R_j$ in \mathcal{E} , such that $r_j h_j = m_j^2 \Delta$. Now define $\hat{m}_j : \hat{G}^1 \to \hat{G}^1$ \hat{H}_j by $\hat{m}_j = (\hat{e}_j, m_j, h_j)$. By Proposition 2.4, it is easy to check that \hat{m}_j is a morphism in $PGrp(\mathcal{E})$. Now we show that $(\hat{G}^1 \xrightarrow{\hat{m}_j} \hat{H}_j)_{j \in I}$ is a mono-source in $PGrp(\mathcal{E})$. Let $\hat{k} = (k_1, k_2, k_3)$ and $\hat{g} = (g_1, g_2, g_3)$ be morphisms in $PGrp(\mathcal{E})$ from \hat{k} to \hat{G}^1 such that $\hat{m}_i \circ \hat{k} = \hat{m}_i \circ \hat{g}$, for all $j \in I$. So we have $U(\hat{m}_j \circ \hat{k}) = U(\hat{m}_j \circ \hat{g})$, for all $j \in I$. Therefore $m_j \circ k_2 = m_j \circ g_2$, for all $j \in I$, and since $(m_j)_{j \in I}$ is a mono source, we have $k_2 = g_2$. By Proposition 3.1, we get $\hat{k} = \hat{g}$.

Thus $(\hat{G}^1 \xrightarrow{\hat{m}_j} \hat{H}_j)_{j \in I}$ is a mono-source in $PGrp(\mathcal{E})$, and we have:

 $U(\hat{m}_{j}) \circ e = U_{2}(U_{1}(\hat{m}_{j})) \circ e = U_{2}(m_{j}) \circ e = f_{j}$

We conclude that every U-structure has a (generating, Mono-source)-factorization. \blacksquare

Theorem 4.2 The forgetful functor $U : PGrp(\mathcal{E}) \longrightarrow \mathcal{E}$, creates isomorphisms.

Proof: Let $\hat{H} = (H, P, R, s, r, *, E, i)$ be an object in $PGrp(\mathcal{E})$, and $f: X \to U(\hat{H}) = P$ be an \mathcal{E} -isomorphism. Define $\hat{X}_H = (H, X, R, s', r', *', E', i)$, where $s' = f^{-1}s$, $r' = f^{-2}r$, $*' = f^{-1} * f^2$, $\overline{E}' = f^{-1}\overline{E}$ and $E' = f^{-1}Ef$. It is easy to check that \hat{X}_H is an object in $PGrp(\mathcal{E})$ and $\hat{f} = (id_H, f, id_R) : \hat{X}_H \to \hat{H}$ is an isomorphism in $PGrp(\mathcal{E})$. Also we have $U(\hat{f}) = f$, and since U is faithful, thus \hat{f} is unique morphism in $PGrp(\mathcal{E})$ such that $U(\hat{f}) = f$.

Corollary 4.3 The concrete category $(PGrp(\mathcal{E}), U)$ over \mathcal{E} is essentially algebraic.

Proof: By Theorem 4.1 and 4.2, we have U is essentially algebraic, so the concrete category $(PGrp(\mathcal{E}), U)$ over \mathcal{E} , is essentially algebraic.

Corollary 4.4

- (i) The concrete category $(PGrp(\mathcal{E}), U)$ has equalizers.
- (ii) The functor U detects colimits.
- (iii) The functor U preserves and creates limits.
- (iv) If \mathcal{E} is complete, then $PGrp(\mathcal{E})$ is complete.
- (v) If \mathcal{E} has coproducts, then $PGrp(\mathcal{E})$ is cocomplete.
- (vi) If \mathcal{E} is wellpowered, then $PGrp(\mathcal{E})$ is wellpowered.

Proof: The proof is concluded by Corollary 23.10, Theorem 23.11, and Proposition 23.12 of [1].

Remark 4.5 By Corollaries 4.3 and 4.4 we get that PGrp is complete, cocomplete and wellpowered.

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