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THE KTH VISIT IN SEMI-MARKOV PROCESSES

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Abstract: Let $(X,T) = \{X_n, T_n;\ n = 0,1,2,\cdots\}$ be a Semi-Markov process. Suppose $N_j(k)$ is the step number at which the Markov chain X visits the state j for kth time, and $T_j(k) = T_{N_j(k)}$ is the time of the kth visit to the state j. We study the $U_j(k,t) = N_j(k)1_{[T_j(k)\leq t]}$ which is the step number for the kth visit prior to time t. We derive recursive formula for $q_{ij}^n(k,t) = P_i\{U_j(k,t) = n\} = P[U_j(k,t) = n|X_0 = i]$, and $E[U_j(k,t)|X_0 = i]$. A relation between the generating functions of the sequences $\{q_{ij}^n(k,t)\}_{n=1,2,\ldots;\ k=1,2,\ldots,n}$, $\{q_{ij}^n(1,t)\}_{n=1,2,\ldots}$ and $\{q_{jj}^n(1,t)\}_{n=1,2,\ldots}$ is also presented.

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1 Introduction

The time of the first visit of a given state plays fundamental role in the general theory of Markov and Semi-Markov processes and their applications, Karlin and Taylor(1975), Cinlar (1975). It seems that the time of the kth visit has not yet been treated adequately. The work of the authors on the subject for Markov chains provides some highlights and applications. There is also a recent paper in Markov chains, W. Stadje (1999), concerning the joint distributions of the number of visits. In this work we concentrate on the step number and the time of the kth visit within the transient time t, when the underlying model is a semi-Markov process. Let

$$K_j(n) = \sum_{m=1}^{n} 1_{\{j\}}(X_m)$$

is the number of visits to the state j by Markov chain X during the first n transitions. Then $N_j(k) = min\{n : K_j(n) = k\}$ is the step number at which X visits the state j for k times. Also assume $T_j(k) = T_{N_j(k)}$ is the time of kth visit to state j by the Semi-Markov process. Then $U_j(k,t) = N_j(k)1_{[T_j(k)\leq t]}$ records the step number for the kth visit prior to time t. In this article we present a recursive formula in k and n for

$$q_{ij}^n(k,t) = P(N_j(k) = n, T_j(k) \leq t|X_0 = i) = P[N_j(k) = n, T_n \leq t|X_0 = i],$$

and

$$a_{ij}(k,t) = E[U_j(k,t)|X_0 = i] = \sum_{n=k}^{\infty} n q_{ij}^n(k,t),$$

the mean of the step number for the kth visit of state j in $[0,t]$. We apply our recursive formulas for $q_{ij}^n(k,t)$ to specify the generating functions of the sequence $\{q_{ij}^n(k,t)\}_{n=1,2,\cdots,\ k=1,2,\ldots,n}$ in terms of the generating functions of the sequences $\{q_{jj}^n(1,t)\}_{n=1,2,\ldots}$ and $\{q_{ij}^n(1,t)\}_{n=1,2,\ldots}$.

2 Recursive Formulas

Let $(X,T) = \{X_n, T_n; \quad n = 0,1,2,\cdots\}$ be a Markov renewal process with state space $\mathcal{S} = \{0,1,2,...\}$. Also let $\{Y_t, \quad t \geq 0\}$ denote the associated semi-Markov process. The semi-Markov matrix is denoted by $\mathcal{P}(t) = \| p_{ij}(t) \|, \; i,j \in \mathcal{S}$, where $p_{ij}(t) = P(X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i)$. Also let $p_{ij} = \lim_{t\to\infty} p_{ij}(t)$. It is assumed that Cinlar (1975), that $p_{ij} = \lim_{t\to\infty} p_{ij}(t)$. It is assumed that

$$P\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_0, \cdots, X_n, T_0, \cdots, T_n\}$$

$$= P\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_n\}, \tag{2.1}$$

for all $n \in \{0,1,2,\cdots\}, j \in \mathcal{S}, t \in \mathcal{R}_+ = [0,\infty)$, and

$$p_{ij}^{n+1}(t) = \sum_{k\in\mathcal{S}} \int_0^t p_{ik}(ds) p_{kj}^n(t-s), \tag{2.2}$$

where $p_{ij}^n(t) = P(X_n = j, T_n \leq t | X_0 = i), \quad i,j \in \mathcal{S}, \quad t \geq 0$. Note that $p_{ij}^0(t) = 0, \; i \neq j, \; p_{jj}^0(t) = 1, \; t \geq 0$. We define $N_j = N_j(1)$ to be the step number of the first passage to the state j, hence T_{N_j} is the first passage time to the state j. The joint probability $q_{ij}^n(1,t) = P_i(N_j = n, \; T_n \leq t)$ is the probability of the first visit of state j at the nth transition within the time interval $[0,t]$ given $X_0 = i$, precisely $q_{ij}^n(1,t) = P(X_n = j, X_v \neq j, v = 1,2,\cdots,n-1, \; T_n \leq t | X_0 = i)$. Note that $q_{ij}^n = \lim_{t\to\infty} q_{ij}^n(1,t)$ is the probability that at nth step number the state j will be first visited from the initial state i in the Markov chain $\{X_n\}$. Also for every $i,j \in \mathcal{S}$, we note that $q_{ij}^0(1,t) = 0, \; q_{ij}(1,t) = p_{ij}(t)$. Let us present the following informative formulas.

Proposition 2.1. *The following relations are satisfied.*

$$p_{ij}^n(t) = \sum_{m=0}^{n} \int_0^t q_{ij}^m(1,ds) p_{jj}^{n-m}(t-s), \quad n \geq 0, \tag{2.3}$$

$$p_{ii}^n(t) = \sum_{m=0}^{n} \int_0^t q_{ii}^m(1, ds) p_{ii}^{n-m}(t-s), \quad n \geq 1, \tag{2.4}$$

$$q_{ij}^n(1,t) = \sum_{k \neq j} \int_0^t p_{ik}(ds) q_{kj}^{n-1}(1, t-s), \;\; n > 1, \; t > 0. \tag{2.5}$$

Proof. For (2.3) apply (2.2) to note that

$$\begin{aligned}
p_{ij}^n(t) &= P(X_n = j, T_n \leq t | X_0 = i) \\
&= \sum_{m=1}^{n} P(X_n = j, X_m = j, X_v \neq j, v = 1, \cdots, m-1, \\
&\quad T_n - T_m + T_m \leq t | X_0 = i) \\
&= \sum_{m=1}^{n} \int_0^t P(X_m = j, X_v \neq j, v = 1, \cdots, m-1, T_m = ds | X_0 = i) \\
&\quad P(X_n = j, T_n - T_m \leq t - s | X_m = j, X_v \neq j, v = 1, \cdots, m-1, \\
&\quad X_0 = i, T_n = s) \\
&= \sum_{m=1}^{n} \int_0^t P(X_m = j, X_v \neq j, v = 1, \cdots, m-1, T_m = ds | X_0 = i) \\
&\quad P(X_n = j, T_n - T_m \leq t - s | X_m = j) \\
&= \sum_{m=1}^{n} \int_0^t q_{ij}^m(1, ds) p_{jj}^{n-m}(t-s).
\end{aligned}$$

If $i = j$, then $p_{ii}^0(t) = 1, q_{ii}^0(1,t) = 0$, giving (2.4). for (2.5) we observe that

$$\begin{aligned}
q_{ij}^n(1,t) &= P_i(N_j = n, T_n \leq t) \\
&= \sum_{k \in \mathcal{S}} \int_0^t P_i(N_j = n, T_n \leq t, X_1 = k, T_1 = ds) \\
&= \sum_{k \in \mathcal{S}} \int_0^t P_i(N_j = n, T_n \leq t | X_1 = k, T_1 = ds) \\
&\quad P(X_1 = k, T_1 = ds | X_0 = i) \\
&= \sum_{k \neq j} \int_0^t P_k(N_j = n-1, T_{n-1} \leq t - s) p_{ik}(ds) \\
&= \sum_{k \neq j} \int_0^t p_{ik}(ds) q_{kj}^{n-1}(1, t-s).
\end{aligned}$$

The proof is complete. ■

Similar to (2.3),

$$p_{ij}^n(t) = \sum_{m=k}^{n} \int_0^t q_{ij}^m(k, ds) p_{jj}^{n-m}(t-s). \tag{2.7}$$

Let us present our first recursive formula.

Proposition 2.2. *Suppose $q_{ij}^n(k,t)$ is given by (2.6), then for every $t > 0$ and $n \geq 1$,*

$$q_{ij}^{n+1}(k+1,t) = \sum_{m=k}^{n} \int_0^t q_{ij}^m(k, ds) q_{jj}^{n+1-m}(1, t-s), \;\; 0 < k \leq n, \tag{2.8}$$

and also

$$q_{ij}^{n+1}(k+1,t) = \sum_{m=k}^{n+1-k} \int_0^t q_{ij}^m(1, ds) q_{jj}^{n+1-m}(k, t-s), \;\; 0 < k \leq n. \tag{2.9}$$

Proof. We note that

$$\begin{aligned}
q_{ij}^{n+1}(k+1,t) &= P_i(N_j(k+1) = n+1, T_{n+1} \leq t) \\
&= \sum_{m=k}^{n} \int_0^t P_i(N_j(k+1) = n+1, N_j(k) = m, T_m = ds, \\
&\quad T_{n+1} - T_m \leq t-s) \\
&= \sum_{m=k}^{n} \int_0^t P_i(N_j(k) = m, T_m = ds) P(N_{ij}(k+1) = n+1, \\
&\quad T_{n+1} - T_m \leq t - s | N_j(k) = m, T_m = ds) \\
&= \sum_{m=k}^{n} \int_0^t P_i(N_j(k) = m, T_m = ds) P_j(N_j = n+m-1, \\
&\quad T_{n+m-1} \leq t-s) \\
&= \sum_{m=k}^{n} \int_0^t q_{ij}^m(k, ds) q_{jj}^{n+m-1}(1, t-s),
\end{aligned}$$

arriving at (2.8). ■

Suppose $V_j^n(t)$ is the number of visits of state j from state i by nth transition in the time interval $[0,t]$, thus $V_j^n(t)=\sum_{k=1}^n I_{[X_k=j,\ T_k\le t]}$, where

$$I_{[X_k=j,\ T_k\le t]}=\begin{cases}1 & X_k=j, T_k\le t\\ 0 & \text{otherwise.}\end{cases}$$

Note that $E_i[V_j^n(t)]=\sum_{k=1}^n p_{ij}^k(t)$. Therefore $p_{ij}^n(t)=E_i[V_j^n(t)]-E_i[V_j^{n-1}(t)]$,and $\lim_{n\to\infty}E_i[V_j^n(t)]=\sum_{k=0}^\infty p_{ij}^k(t)=R(i,j,t)$, where $R(i,j,t)$ is the Markov renewal function of the semi-Markov process Y_t.

Lemma 2.1. *The following relations are satisfied.*

$$p_{ij}^n(t)=E_i[V_j^n(t)]-E_i[V_j^{n-1}(t)], \tag{2.10}$$

$$E_i[V_j^n(t)]=\sum_{k=1}^n\sum_{m=k}^n q_{ij}^m(k,t), \tag{2.11}$$

$$R(i,j,t)=\sum_{n=1}^\infty\sum_{k=1}^n q_{ij}^n(k,t). \tag{2.12}$$

Proof. The (2.10) is immediate, (2.12) is followed from (2.11). For (2.11),

$$\begin{aligned}E_i[V_j^n(t)] &= \sum_{k=1}^n P(V_j^n(t)\ge k)\\ &= \sum_{k=1}^n P(T_{N_{ij}(k)}\le t, N_{ij}(k)\le n)\\ &= \sum_{k=1}^n\sum_{m=k}^n P(N_{ij}(k)=m, T_m\le t)\\ &= \sum_{k=1}^n\sum_{m=k}^n q_{ij}^m(k,t).\end{aligned}$$

Proof is complete. ■

Let us present the second recursive formula concerning the mean of the step number for the kth visit.

Proposition 2.3. *Let $a_{ij}(k,t) = \sum_{n=k}^{\infty} n q_{ij}^n(k,t)$ be the mean of the step number for the kth visit of state j from the initial state i within $[0,t]$, then for every $t > 0$,*

$$a_{ij}(k+1,t) = q_{ij}(k,.) * a_{jj}(1,.)(t) + q_{jj}(1,.) * a_{ij}(k,.)(t), \; k \geq 1, \quad (2.13)$$

where $q_{ij}(k,t) = \sum_{n=k}^{\infty} q_{ij}^n(k,t)$, $k \geq 1$, is the probability that the kth visit occurs prior to the time t.

Proof. We apply the recursive formula (2.8) to note that

$$\begin{aligned}
a_{ij}(k+1,t) &= \sum_{n=k}^{\infty}(n+1)q^{n+1}(k+1,t) \\
&= \sum_{n=k}^{\infty}(n+1)\sum_{m=k}^{n}\int_0^t q_{ij}^m(k,ds)q_{jj}^{n+1-m}(1,t-s) \\
&= \int_0^t \sum_{n=k}^{\infty}\sum_{m=k}^{n}(n+1-m)q_{ij}^m(k,ds)q_{jj}^{n+1-m}(1,t-s) \\
&+ \int_0^t \sum_{n=k}^{\infty}\sum_{m=k}^{n} m q_{ij}^m(k,ds)q_{jj}^{n+1-m}(1,t-s) \\
&= \int_0^t \sum_{m=k}^{\infty} q_{ij}^m(k,ds)\sum_{n=m}^{\infty}(n+1-m)q_{jj}^{n+1-m}(1,t-s) \\
&+ \int_0^t \sum_{m=k}^{\infty} m q_{ij}^m(k,ds)\sum_{n=m}^{\infty} q_{jj}^{n+1-m}(1,t-s) \\
&= \int_0^t [\sum_{m=k}^{\infty} q_{ij}^m(k,ds)][\sum_{l=1}^{\infty} l q_{jj}^l(1,t-s)] \\
&+ \int_0^t [\sum_{m=k}^{\infty} m q_{ij}^m(k,ds)][\sum_{l=1}^{\infty} q_{jj}^l(1,t-s)] \\
&= \int_0^t q_{ij}(k,ds)a_{jj}(1,t-s) + \int_0^t a_{ij}(k,ds)q_{jj}(1,t-s) \\
&= q_{ij}(k,.) * a_{jj}(1,.)(t) + q_{jj}(1,.) * a_{ij}(k,.)(t),
\end{aligned}$$

giving the result. ■

Note that by conditioning on (X_1, T_1),

$$\begin{aligned} a_{ij}(k,t) &= E_i[U_j(k,t)] \\ &= \sum_{l\neq j}\int_0^t [1+E_l[U_j(k,t-s)]p_{il}(ds) \\ &+ \int_0^t [1+E_j[U_j(k-1,t-s)]]p_{ij}(ds) \\ &= p_i(t)+\sum_{l\neq j}\int_0^t a_{lj}(k,t-s)p_{il}(ds) \\ &+ \int_0^t a_{jj}(k-1,t-s)p_{ij}(ds), \qquad (2.14) \end{aligned}$$

where $p_i(t)=\sum_j p_{ij}(t)$. The (2.14) does not provide a renewal equation to speecify $a_{ij}(k,t)$. In order to apply the renewal theory, we compromise to let $a_{ij}(k+1,t)\cong a_{ij}(k,t)$, for large k. In this case (2.14) provides that

$$f = g + F * f, \qquad (2.15)$$

where $F(t)=q_{jj}(1,t), f(t)=a_{jj}(k,t)$, and $g(t)=a_{jj}(1,.)*q_{jj}(k,.)(t)$. Therefore $a_{jj}(k,t)\cong g(k,t)+g(k,.)*R_{jj}(t)$, where $R_{jj}(t)=\sum_n q_{jj}^n(1,t)$. As $g(t)$ is not integrable, the Key Renewal Theorem does not apply. The limiting value of $a_{ij}(k,t), t\to\infty$, can be evaluated directly, namely,

$$\begin{aligned} E_i[N_j(k)] &= a_{ij}(k) \\ &= lim_{t\to\infty} a_{ij}(k,t) \\ &= lim_{t\to\infty} E_j[N_j(k)1_{[T_j(k)\le t]}] \\ &= a_{ij}(1)+\frac{k-1}{\pi_j}, \end{aligned}$$

where $\{p_{ij}\}$, and $\{\pi_j\}$, are the transition probabilities and stationary distribution of the embedded Markov chain $\{X_n\}$, see [3] for the derivation given above.

In the following we provide two examples where the distribution of $U_j(k,t)$ can be evaluated directly or by the derived formulas.

Example 2.1. Suppose (X,T) has the transition-sojourn time probability matrix of the form

$$\mathcal{P}(t) = \begin{bmatrix} 0 & 1-e^{-\lambda t} \\ 1-e^{-\lambda t} & 0 \end{bmatrix}$$

then

$$q_{00}^{n}(k,t) = q_{11}^{n}(k,t) = \begin{cases} (1-e^{-\lambda t})^{2k} & n = 2k \\ 0 & n \neq 2k \end{cases}$$

where

$$q_{00}(1,t) = \sum_{n=1}^{\infty} q_{00}^{n}(1,t) = (1-e^{-\lambda t})^2,$$

and, $m_0(t) = \sum_{n=1}^{\infty} n q_{00}^{n}(1,t) = 2(1-e^{-\lambda t})^2$, such that $q_{00}(1,\infty) = 1$, $m_0(\infty) = 2$, also $a_{00}(k,t) = \sum_{n=k}^{\infty} n q_{00}^{n}(k,t) = 2k q_{00}^{2k}(k,t) = 2k(1-e^{-\lambda t})^{2k}$,
$a_{00}(k,\infty) = 2k$, and $q_{00}(k,t) = \sum_{n=k}^{\infty} q_{00}^{n}(k,t) = (1-e^{-\lambda t})^{2k}$, $q_{00}(k,\infty) = 1$,
therefore

$$\begin{aligned} q_{00}^{n+1}(k+1,t) &= \sum_{m=k}^{n} \int_0^t q_{00}^{m}(k,ds) q_{00}^{n+1-m}(1,t-s) \\ &= \int_0^t \sum_{m=k}^{n} q_{00}^{m}(k,ds) q_{00}^{n+1-m}(1,t-s) \\ &= \int_0^t q_{00}^{2k}(k,ds) q_{00}^{2}(1,t-s) \\ &= q_{00}^{2k+2}(k+1,t) = \begin{cases} (1-e^{-\lambda t})^{2k+2} & n = 2k+1, \\ 0 & n \neq 2k+1. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} a_{00}(k+1,t) &= \int_0^t [q_{00}(k,ds)m_0(t-s) + a_{00}(k,ds)q_{00}(1,t-s)] \\ &= 2(k+1)(1-e^{-\lambda t})^{2(k+1)}. \end{aligned}$$

Also

$$a_{00}(k+1,\infty) = a_{00}(1,\infty) + km_0(\infty) = 2 + 2k = 2(k+1),$$

and $E_0N_0(k) = a_{00}(k,\infty) = 2k, a_{11}(k,t) = 2k(1-e^{-\lambda t})^{2k}, E_1N_1(k) = a_{11}(k,\infty) = 2k$. Finally

$$q_{01}^n(k,t) = \begin{cases} (1-e^{-\lambda t})^{2k-1} & n = 2k-1, \\ 0 & n \neq 2k-1, \end{cases}$$

where $a_{01}(k,t) = (2k-1)(1-e^{-\lambda t})^{2k-1}, k \geq 1$.

Example 2.2. Suppose

$$p_{ij}(t) = \frac{e^{-\lambda(t)t}(\lambda(t)t)^{j-i}}{(j-i)!}, j \geq i = 0,1,2,..., \text{where } \lambda(t)t \to \lambda_0, \text{ as } t \to \infty.$$

Then for $k \geq 1$, we have

$$q_{ii}^n(k,t) = \begin{cases} e^{-k\lambda(t)t} & n = k \\ 0 & n \neq k \end{cases}$$

and

$$\begin{aligned} q_{ii}^{n+1}(k+1,t) &= \sum_{m=k}^{n} \int_0^t q_{ii}^m(k,ds)q_{ii}^{n+1-m}(1,t-s) \\ &= \int_0^t \sum_{m=k}^{n} q_{ii}^m(k,ds)q_{ii}^{n+1-m}(1,t-s) \\ &= \int_0^t e^{-k\lambda(t-s)\times(t-s)}d(e^{-\lambda(s)s}) \\ &= e^{-(k+1)\lambda(t)t}, \end{aligned}$$

hence

$$m_i(t) = a_{ii}(1,t) = \sum_{n=1}^{\infty} nq_{ii}^n(1,t) = e^{-\lambda(t)t},\ q_{ii}(k,t) = \sum_{n=k}^{\infty} q_{ii}^n(k,t) = e^{-k\lambda(t)t},$$

and $a_{ii}(k,t) = \sum_{n=k}^{\infty} nq_{ii}^n(k,t) = ke^{-k\lambda(t)t}$, therefore

$$\begin{aligned} a_{ii}(k+1,t) &= \int_0^t [q_{ii}(k,t)m_i(t-s) + a_{ii}(k,ds)q_{ii}(1,t-s)] \\ &= (k+1)\int_0^t e^{-k\lambda(t-s)\times(t-s)} d(e^{-\lambda s}) \\ &= (k+1)e^{-(k+1)\lambda(t)t}. \end{aligned}$$

Also for $j = i+1$ implies that

$$a_{ij}(k,t) = \frac{\lambda(t)te^{-k\lambda(t)t}[k + (1-k)e^{-\lambda(t)t}]}{(1-e^{-\lambda(t)t})^2}.$$

3 The Generating Functions

Suppose $P_{ij}(s,\alpha)$, $Q_{ij}(s,\alpha)$, and $G_{ij}(r,s,\alpha)$, are the generating functions of the sequences $\{p_{ij}^n(t)\}, \{q_{ij}^n(1,t)\}_{n=0,1,\dots,\, t\geq 0}$, and $\{q_{ij}^n(k,t)\}$, respectively define as follows:

$$P_{ij}(s,\alpha) = \sum_{n=0}^{\infty} \int_0^{\infty} p_{ij}^n(t) exp\{-\alpha t\} s^n dt,\ \ |s|<1,\ \alpha>0, \tag{3.1}$$

$$Q_{ij}(s,\alpha) = \sum_{n=0}^{\infty} \int_0^{\infty} exp\{-\alpha t\} s^n q_{ij}^n(dt),\ \ |s|<1,\ \alpha>0, \tag{3.2}$$

and

$$G_{ij}(r,s,\alpha) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \int_0^{\infty} r^k s^k exp\{-\alpha t\} q_{ij}^n(k,dt),\ \ |s|<1,\ |r|<1,\ \alpha>0. \tag{3.3}$$

Theorem 3.1. *The genarating functions satisfy the following relation.*

$$P_{ij}(s,\alpha) = \begin{cases} Q_{ij}(s,\alpha)P_{jj}(s,\alpha) & \text{if } i\neq j, \\ \frac{1}{\alpha[1-Q_{ii}(s,\alpha)]} & \text{if } i=j. \end{cases} \tag{3.4}$$

Proof. For $i \neq j$,

$$\begin{aligned} Q_{ij}(s,\alpha)P_{jj}(s,\alpha) &= (\sum_{n=0}^{\infty}\int_0^{\infty} exp\{-\alpha t\}s^n q_{ij}^n(dt))(\sum_{m=0}^{\infty}\int_0^{\infty} p_{jj}^n(u)exp\{-\alpha u\}s^m du) \\ &= \sum_{n=0}^{\infty}(\sum_{m=0}^{n}\int_0^{\infty}\int_0^{\infty} exp\{-\alpha(t+u)\}q_{ij}^m(dt)p_{jj}^{n-m}(u)du)s^n \\ &= \sum_{n=0}^{\infty}(\sum_{m=0}^{n}\int_0^{\infty}\int_0^{y} exp\{-\alpha y\}q_{ij}^m(dt)p_{jj}^{n-m}(y-t)dy)s^n \\ &= \sum_{n=0}^{\infty}\int_0^{\infty} exp\{-\alpha y\}(\sum_{m=0}^{n}\int_0^{y} q_{ij}^m(dt)p_{jj}^{n-m}(y-t))s^n dy \\ &= \sum_{n=0}^{\infty}\int_0^{\infty} exp\{-\alpha y\}p_{ij}^n(y)s^n dy \\ &= P_{ij}(s,\alpha). \end{aligned}$$

For $i = j$ since $p_{ii}^0(t) = 1, q_{ii}^0(t) = 0$ we obtain that

$$\begin{aligned} Q_{ii}(s,\alpha)P_{ii}(s,\alpha) &= \sum_{n=1}^{\infty}\int_0^{\infty} exp\{-\alpha y\}p_{ij}^n(y)s^n dy \\ &= \sum_{n=0}^{\infty}\int_0^{\infty} exp\{-\alpha y\}p_{ij}^n(y)s^n dy - \int_0^{\infty} exp\{-\alpha y\}dy \\ &= P_{jj}(s,\alpha) - 1/\alpha. \end{aligned}$$

The proof is complete. ■

Theorem 3.2. *Let the generating function of sequence* $\{q_{ij}^n(k,t)\}_{n=1,2,\ldots;k=1,2,\ldots,n}$ *be*

$$G_{ij}(r,s,\alpha) = \sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_0^{\infty} r^k s^k exp\{-\alpha t\}q_{ij}^n(k,dt), \ \ |s|<1, \ |r|<1, \ \alpha>0. \tag{3.5}$$

Then

$$G_{ij}(r,s,\alpha) = \frac{rQ_{ij}(s,\alpha)}{1-rQ_{jj}(s,\alpha)}. \tag{3.6}$$

Proof. Take $a_n = \sum_{k=1}^{n} \int_0^\infty exp\{-\alpha t\} q_{ij}^n(k,t) r^k dt, b_n = \int_0^\infty exp\{-\alpha t\} q_{jj}^n(dt)$, hence

$$\begin{aligned}
c_n &= \sum_{m=1}^{n-1} a_m b_{n-m} \\
&= \sum_{m=1}^{n-1} \sum_{k=1}^{m} \int_0^\infty exp\{-\alpha t\} q_{ij}^m(k,t) r^k dt (\int_0^\infty q_{jj}^{n-m}(du)) \\
&= \sum_{m=1}^{n-1} \sum_{k=1}^{n} \int_0^\infty \int_0^\infty exp\{-\alpha(t+u)\} q_{ij}^m(k,dt) r^k q_{jj}^{n-m}(du) \\
&= \sum_{k=1}^{n-1} \sum_{m=k}^{n-1} \int_0^\infty \int_0^y exp\{-\alpha y\} r^k q_{ij}^m(k,dy) q_{jj}^{n-m}(y-dt) \\
&= \sum_{k=1}^{n-1} \int_0^\infty r^k (\sum_{m=k}^{n-1} \int_0^y q_{ij}^m(k,dy) q_{jj}^{n-m}(y-dt) exp\{-\alpha y\} \\
&= \sum_{k=1}^{n-1} \int_0^\infty r^k exp\{-\alpha y\} q_{ij}^n(k+1,dy) \\
&= \frac{1}{r} \sum_{k=1}^{n-1} \int_0^\infty r^{k+1} exp\{-\alpha y\} q_{ij}^n(k+1,dy).
\end{aligned}$$

Therefore

$$\begin{aligned}
G_{ij}(r,s,\alpha) Q_{jj}(s,\alpha) &= \sum_{n=1}^{\infty} c_n s^n \\
&= \frac{1}{r} \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \int_0^\infty r^{k+1} exp\{-\alpha y\} q_{ij}^n(k+1,dy) \\
&= \frac{1}{r} \sum_{n=1}^{\infty} \sum_{m=2}^{n} \int_0^\infty r^m exp\{-\alpha y\} q_{ij}^n(m,dy) \\
&= \frac{1}{r} \sum_{n=1}^{\infty} [\sum_{m=1}^{n} \int_0^\infty r^m exp\{-\alpha y\} q_{ij}^n(m,dy) \\
&- r \int_0^\infty exp\{-\alpha y\} q_{ij}^n(dy)]
\end{aligned}$$

$$= \frac{1}{r}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\int_0^{\infty} r^m exp\{-\alpha y\} q_{ij}^n(m, dy)$$
$$- \sum_{n=1}^{\infty}\int_0^{\infty} exp\{-\alpha y\} q_{ij}^n(dy)]$$
$$= \frac{1}{r} G_{ij}(r, s, \alpha) - Q_{ij}(s, \alpha),$$

giving the result. ■

Let $\mathcal{P}(\alpha) = \|P_{ij}(\alpha)\|$, where $P_{ij}(\alpha) = \int_0^{\infty} exp(-\alpha t) p_{ij}(dt)$. It is observed that $\mathcal{P}^n(\alpha) = (\mathcal{P}(\alpha))^n$, Cinlar(1975). Now since $P_{ij}(s, \alpha) = \sum_{n=0}^{\infty}\int_0^{\infty} exp(-\alpha t) p_{ij}^n(dt) s^n$, $|s| < 1$, $\alpha > 0$, we observe that in matrix notation we have $\mathcal{P}(s, \alpha) = \|P_{ij}(s, \alpha)\| = \sum_{n=0}^{\infty}(s\mathcal{P}(\alpha))^n$.
Note that $\mathcal{P}(1, \alpha) = \sum_{n=0}^{\infty}(\mathcal{P}(\alpha))^n = \mathcal{R}_\alpha$, as is given in Cinlar 1975.

Proposition 3.1. *If $\mathcal{S}$ is finite, then $\mathcal{P}(s, \alpha) = (I - s\mathcal{P}(\alpha))^{-1}$, and if $\mathcal{S}$ is not finite, then $\mathcal{P}(s, \alpha)$ is minimal solution of equation*

$$(I - s\mathcal{P}(\alpha))M = I, \ M \geq 0.$$

Proof. Since $\mathcal{P}(s, \alpha) = \sum_{n=0}^{\infty}(s\mathcal{P}(\alpha))^n$, hence we have $\mathcal{P}(s, \alpha) = I + s\mathcal{P}(\alpha)\mathcal{P}(s, \alpha)$, or $\mathcal{P}(s, \alpha)[I - s\mathcal{P}(\alpha)] = I$, so that $\mathcal{P}(s, \alpha) = (I - s\mathcal{P}(\alpha))^{-1}$.■

Example 3.1. To apply Proposition 3.1, let

$$\mathcal{P}(t) = \begin{bmatrix} .6(1 - e^{-5t}) & .4(1 - e^{-5t}) \\ .5(1 - e^{-4t}) & .5(1 - e^{-4t}) \end{bmatrix}.$$

Then

$$\mathcal{P}(\alpha) = \begin{bmatrix} \frac{3}{\alpha+5} & \frac{2}{\alpha+5} \\ \frac{2}{\alpha+4} & \frac{2}{\alpha+4} \end{bmatrix},$$

and $\mathcal{P}(s, \alpha) = (I - s\mathcal{P}(\alpha))^{-1}$, provides that

$$\mathcal{P}(s, \alpha) = \begin{bmatrix} \frac{(\alpha+5)(\alpha+4_2 s)}{(\alpha+5-3s)(\alpha+4-2s)-4s^2} & \frac{2s(\alpha+5)}{(\alpha+5-3s)(\alpha+4-2s)-4s^2} \\ \frac{2s(\alpha+4)}{(\alpha+5-3s)(\alpha+4-2s)-4s^2} & \frac{(\alpha+4)(\alpha+5-3s)}{(\alpha+5-3s)(\alpha+4-2s)-4s^2} \end{bmatrix}.$$

Hence it follows from (3.4) that

$$Q(s, \alpha) = \begin{bmatrix} \frac{(\alpha-1)(\alpha+4)-2s\alpha}{\alpha(\alpha+4-2s)} & \frac{2s(\alpha+5)}{(\alpha+4)(\alpha+5-3s)} \\ \frac{2s(\alpha+4)}{(\alpha+5)(\alpha+4-2s)} & \frac{(\alpha-1)(\alpha+5)-3s\alpha}{\alpha(\alpha+5-3s)} \end{bmatrix},$$

and from (3.6) that

$$G(r, s, \alpha) = \begin{bmatrix} \frac{r(\alpha-1)(\alpha+4)-2rs\alpha}{\alpha(\alpha+4-2s)-r(\alpha-1)(\alpha+4)+2rs\alpha} & \frac{2rs\alpha(\alpha+5)}{(\alpha+4)[\alpha(\alpha+5-3s)-r(\alpha-1)(\alpha+5)+3\alpha rs]} \\ \frac{2\alpha rs(\alpha+4)}{(\alpha+5)[\alpha(\alpha+4-2s)-r(\alpha-1)(\alpha+4)+2\alpha rs]} & \frac{r(\alpha-1)(\alpha+5)-3rs\alpha}{\alpha(\alpha+5-3s)-r(\alpha-1)(\alpha+5)+3rs\alpha} \end{bmatrix}.$$

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