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COMMON FIXED POINT THEOREMS USING GENERALIZED CONTRACTIVE CONDITIONS

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Abstract: The aim of this note is to prove some common fixed point theorems in complete metric spaces using generalized contractive conditions.

1. Introduction

In the paper [4], the authors gave some fixed point theorems generalizing and unifying many fixed point theorems obtained by Delbosco in [1], Skof in [8], Rakotch in [5], Reich in [7], and Fisher in [3], (see also the references for other related results). Precisely in [4] the following theorem was established:

1.1 Theorem. *Let T be a self-map of a complete metric space (X, d) and let ϕ be a function verifying*
(i) $\phi : [0, \infty[\to [0, \infty[$ *is continuous and increasing in* $[0, \infty[$, *and*
(ii) $\phi(t) = 0 \Longleftrightarrow t = 0$.

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We suppose that T satisfies the following condition:

$$\begin{aligned}\phi(d(Tx,Ty)) \leq \quad & a(d(x,y))\phi(d(x,y)) + \\ & b(d(x,y))[\phi(d(x,Tx)) + \phi(d(y,Ty))] \\ & +c(d(x,y))\min\{\phi(d(x,Ty)),\phi(d(y,Tx))\} \\ & \forall x,y \in X \quad with \quad x \neq y,\end{aligned} \tag{K}$$

where a, b, c are three decreasing functions from $]0,\infty[$ into $[0,1[$ such that $a(t)+2b(t)+c(t) < 1$, for every $t > 0$. Then T has a unique fixed point.

In the second section of this paper we shall prove some common fixed point theorems for sets of self-mappings verifying contractive conditions close to the relation (K). These results complete and unify the main results obtained in the papers [4] and [6]. In section 3, we establish a common fixed point theorem in compact metric spaces using another type of contractive conditions. We may consider this theorem as a generalization of a theorem established by B. Fisher in [3]. Our generalization is different from that one given in the paper [4] by M.S. Khan, M. Swaleh and S. Sessa.

2. Main theorems

We shall denote by Φ the set of functions ϕ verifying conditions (i) and (ii). Many authors (see the references) were interested by fixed point theorems by altering the distances between the points with the use of functions belonging to the class Φ. The purpose of this section is to contribute in this field of investigations. One of the main results of this paper is the following theorem.

2.1 Theorem. *Let $\phi \in \Phi$ be a convex function and let S, T be two self-maps of a complete metric space (X,d) such that*

$$\begin{aligned}\phi(d(Sx,Ty)) \leq \quad & a(d(x,y))\phi(d(x,y)) + b(d(x,y))\phi(d(x,Sx) \\ & +c(d(x,y))\phi(d(y,Ty)) + e(d(x,y))\phi(\gamma d(x,Ty)) \\ & +f(d(x,y))\phi(\delta d(Sx,y)),\end{aligned} \tag{A}$$

for all distinct x, y in X, where γ, δ are two fixed numbers such that $0 \leq \gamma, \delta \leq \frac{1}{2}$, and a, b, c, e, f are five decreasing functions from $]0, \infty[$ into $[0, 1[$ verifying $a(t)+b(t)+c(t)+e(t)+f(t) < 1$, for every $t > 0$. We suppose also that $\max\{B, C\} < 1$, where, $B := \sup\{b(t) + \delta f(t) : t > 0\}$ and $C := \sup\{c(t)+\gamma e(t) : t > 0\}$. Then S and T have a unique common fixed point $z \in X$. Moreover $Fix(S) = Fix(T) = \{z\}$.

Proof. (I) We shall prove that the pair $\{S, T\}$ has a common fixed point. Let x_0 be some point in X, and define the sequence $\{x_n\}$ by

$$\begin{aligned} x_{2n} &= Sx_{2n-1}, \quad n = 1, 2, ... \\ x_{2n+1} &= Tx_{2n}, \quad n = 0, 1, 2, ... \end{aligned}$$

We put $t_n := d(x_n, x_{n+1})$ for all integer n. (I) is proved if $t_{n_0} = 0$ for some integer n_0. Therefore, we may assume that $t_n > 0$ for all integer n. We see that for an even integer n, we have

$$\phi(t_n) = \phi(d(Sx_{n-1}, Tx_n)) \leq \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5,$$

where

$$\begin{aligned} \Psi_1 &= a(d(x_{n-1}, x_n))\phi(d(x_{n-1}, x_n)), \\ \Psi_2 &= b(d(x_{n-1}, x_n))\phi(d(x_{n-1}, Sx_{n-1})), \\ \Psi_3 &= c(d(x_{n-1}, x_n))\phi(d(x_n, Tx_n)), \\ \Psi_4 &= e(d(x_{n-1}, x_n))\phi(\gamma d(x_{n-1}, Tx_n)), \\ \Psi_5 &= f(d(x_{n-1}, x_n))(\delta d(x_n, Sx_{n-1})) \end{aligned}$$

So

$$\phi(t_n) \leq a(t_{n-1})\phi(t_{n-1})+b(t_{n-1})\phi(t_{n-1})+c(t_{n-1})\phi(t_n)+e(t_{n-1})\phi(\gamma[t_{n-1}+t_n]).$$

Hence by using the convexity of ϕ, we get

$$\phi(t_n) \leq \frac{a(t_{n-1}) + b(t_{n-1}) + \gamma e(t_{n-1})}{1 - c(t_{n-1}) - \gamma e(t_{n-1})}\phi(t_{n-1}) < \phi(t_{n-1}). \tag{1}$$

In a similar manner, one can prove (for the same even integer) that

$$\phi(t_{n-1}) \leq \frac{a(t_{n-2}) + c(t_{n-2}) + \delta f(t_{n-2})}{1 - b(t_{n-2}) - \delta f(t_{n-1})}\phi(t_{n-2}) < \phi(t_{n-2}). \tag{1'}$$

Since ϕ is increasing, (1) and (1′) show that sequence t_n is decreasing. Let t be the limit of t_n. We shall prove that $t = 0$. Indeed, suppose that $t > 0$. Then $t \leq t_{2n}$ and by (1), we have

$$\phi(t_{2n}) \leq \frac{a(t) + b(t) + \gamma e(t)}{1 - c(t) - \gamma e(t)} \phi(t_{2n-1}).$$

Now we let $n \to \infty$ and use the continuity of ϕ to obtain

$$\phi(t) \leq \frac{a(t) + b(t) + \gamma e(t)}{1 - c(t) - \gamma e(t)} \phi(t) < \phi(t),$$

which is a contradiction, hence $t = 0$.

(II) Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. Since $t = 0$ one needs only to see that $\{x_{2n}\}$ is a Cauchy sequence. To get a contradiction, let us suppose that there is a number $\epsilon > 0$ and two sequences $\{2n(k)\}$, $\{2m(k)\}$ with $2k \leq 2m(k) < 2n(k)$, $(k \in \mathbb{N})$ verifying

$$d(x_{2n(k)}, x_{2m(k)}) > \epsilon. \tag{2}$$

For each integer k, we shall denote $2n(k)$ the least even integer exceeding $2m(k)$ for which (2) holds. Then

$$d(x_{2m(k)}, x_{2n(k)-2}) \leq \epsilon \quad \textit{and} \quad d(x_{2m(k)}, x_{2n(k)}) > \epsilon.$$

For each integer k, we shall put $p_k := d(x_{2m(k)}, x_{2n(k)})$, $q_k := d(x_{2m(k)+1}, x_{2n(k)+1})$, and $r_k := d(x_{2m(k)+1}, x_{2n(k)+2})$, then we have

$$\begin{aligned} \epsilon &< p_k = d(x_{2m(k)}, x_{2n(k)}) \\ &\leq d(x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) \\ &\leq \epsilon + t_{2n(k)-2} + t_{2n(k)-1}. \end{aligned} \tag{3}$$

Since the sequence $\{t_n\}$ converges to 0, we deduce from (3) that $\{p_k\}$ converges to ϵ. Furthermore, the sequence $\{q_k\}$ has also ϵ as limit. Indeed, this fact results from the following estimates obtained by triangular inequality

$$\begin{aligned} -t_{2m(k)} - t_{2n(k)} + p_k &< d(x_{2m(k)+1}, x_{2n(k)+1}) \\ &\leq t_{2m(k)} + t_{2n(k)} + p_k \end{aligned} \tag{4}$$

We claim that the sequence $\{r_k\}$ converges to ϵ. Indeed, we get by using triangular inequality the following

$$r_k \leq t_{2m(k)} + p_k + t_{2n(k)} + t_{2n(k)+1}. \tag{5}$$

On the other hand, by definition of the integer $2n(k)$ and by using triangular inequality we have the following

$$\begin{aligned} \epsilon &\leq d(x_{2m(k)}, x_{2n(k)-2}) \\ &\leq t_{2m(k)} + r_k + t_{2n(k)+1} + t_{2n(k)} + t_{2n(k)-1} + t_{2n(k)-2} \end{aligned} \tag{6}$$

(5) and (6) imply our claim. One can deduce that there exists an integer k_0 such that $d(x_{2n(k)+1}, x_{2m(k)}) > 0$, and $t_{2n(k)} < \frac{\epsilon}{2}$ for each integer $k \geq k_0$. Thus, by using (2) and the relation $p_k - t_{2k} \leq d(x_{2n(k)+1}, x_{2m(k)})$, we deduce (for all $k \geq k_0$) that

$$\begin{aligned} \phi(r_k) = \phi(d(x_{2n(k)+2}, x_{2m(k)+1})) &= \phi(d(Sx_{2n(k)+1}, Tx_{2m(k)})) \\ &\leq \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5 \\ &\leq G_1 + G_2 + G_3 + G_4 + G_5, \end{aligned}$$

where

$$\begin{aligned} \Gamma_1 &= a(d(x_{2n(k)+1}, x_{2m(k)}))\phi(d(x_{2n(k)+1}, x_{2m(k)})), \\ \Gamma_2 &= b(d(x_{2n(k)+1}, x_{2m(k)}))\phi(d(x_{2n(k)+1}, x_{2n(k)+2})), \\ \Gamma_3 &= c(d(x_{2n(k)+1}, x_{2m(k)}))\phi(d(x_{2m(k)}, x_{2m(k)+1})), \\ \Gamma_4 &= e(d(x_{2n(k)+1}, x_{2m(k)}))\phi(\gamma d(x_{2n(k)+1}, x_{2m(k)+1})), \\ \Gamma_5 &= e(d(x_{2n(k)+1}, x_{2m(k)}))\phi(\delta d(x_{2m(k)}, x_{2n(k)+2}))\}, \\ G_1 &= a(p_k - t_{2n(k)})\phi(t_{2n(k)} + p_k), \\ G_2 &= \phi(t_{2n(k)+1}), \\ G_3 &= \phi(t_{2m(k)}), \\ G_4 &= e(p_k - t_{2n(k)})\phi(\gamma q_k), \ \ and \\ G_5 &= f(p_k - t_{2n(k)})\phi(\delta[t_{2m(k)} + r_k]). \end{aligned}$$

Let $k \to \infty$. Then by using the continuity of ϕ and the fact that a, b, c, e, f are decreasing on $]0, +\infty[$, we obtain

$$\phi(\epsilon) \leq a(\frac{\epsilon}{2})\phi(\epsilon) + e(\frac{\epsilon}{2})\phi(\gamma\epsilon) + f(\frac{\epsilon}{2})\phi(\delta\epsilon) < [a(\frac{\epsilon}{2}) + e(\frac{\epsilon}{2}) + f(\frac{\epsilon}{2})]\phi(\epsilon) < \phi(\epsilon).$$

This gives a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in a complete metric space (X,d), then one may find a point $z = z(S,T) \in X$ such that $x_n \longrightarrow z$ as $n \longrightarrow \infty$. Next, we shall prove that z is a common fixed point for S and T.

(III) Since $t_n > 0$ for all integer n, we see that both subsequences $(x_{2n})_n$ and $(x_{2n+1})_n$ are not stationary. Therefore, we may find a subsequence $(x_{n(k)})_k$ such that $x_{2n(k)+1} \neq z$ for every integer k. Let us suppose that $Tz \neq z$. In this case we are allowed to apply the inequality (A) and obtain for all $k \in \mathbb{N}$,

$$\begin{aligned}
&\phi(d(x_{2n(k)+2}, Tz)) = \phi(d(Sx_{2n(k)+1}, Tz)) \\
&\leq a(d(x_{2n(k)+1}, z))\phi(d(x_{2n(k)+1}, z)) + b(d(x_{2n(k)+1}, z))\phi(d(x_{2n(k)+1}, x_{2n(k)+2})) \\
&+c(d(x_{2n(k)+1}, z))\phi(d(z, Tz)) + e(d(x_{2n(k)+1}, z))\phi(\gamma d(x_{2n(k)+1}, Tz) \\
&+f(d(x_{2n(k)+1}, z))\phi(\delta d(x_{2n(k)+2}, z)).
\end{aligned} \tag{7}$$

By using the convexity of ϕ, we deduce from (7) the following inequality:

$$\begin{aligned}
\phi(d(x_{2n(k)+2}, Tz)) \leq \quad & \phi(d(x_{2n(k)+1}, z)) + \phi(t_{2n(k)+1}) + \phi(\delta d(x_{2n(k)+2}, z)) \\
& + \Big[c(d(x_{2n(k)+1}, z)) + \gamma e(d(x_{2n(k)+1}, z))\Big]\phi(d(z, Tz)) \\
& + \phi(d(x_{2n(k)+1}, z),
\end{aligned} \tag{8}$$

which gives, after letting $k \longrightarrow \infty$:

$$\begin{aligned}
\phi(d(z, Tz)) \leq \quad & \max\{c(t) + \gamma e(t) : t > 0\}\phi(d(z, Tz)) \\
& = C\phi(d(z, Tz)) < \phi(d(z, Tz)),
\end{aligned} \tag{9}$$

which is a contradiction. Hence $z = Tz$, and in a similar way, it can be shown that $z = Sz$.

(IV) Suppose that there exists another point $\xi \neq z$ fixed, for instance, by S. Then, by inequality (A), we have

$$\begin{aligned}
&\phi(d(\xi, z)) = \phi(d(S\xi, Tz)) \\
&\leq [a(d(\xi, z)) + e(d(\xi, z)) + f(d(\xi, z))]\,\phi(d(\xi, z)) \\
&< \phi(d(\xi, z))
\end{aligned}$$

a contradiction. Therefore, we deduce that there exists a unique point $z \in X$ such that $Fix(S) = \{z\} = Fix(T) = Fix(\{S,T\})$. This completes the proof of our theorem.□

Using the basic ideas in the proof of the previous theorem, one can establish the following theorem which generalizes the main result of the paper [6].

2.2 Theorem. *Let (X, d) be a complete metric space, $\mathcal{A}$ a (finite or infinite) set of self-maps of X and ϕ an element of Φ. We suppose that for all $S, T \in \mathcal{A}$ the following generalized contractive condition holds true:*

$$\begin{aligned}\phi(d(Sx, Ty)) \leq\ & \alpha(d(x, y))\phi(d(x, y)) + \beta(d(x, y))\phi(d(x, Sx)) \\ & + \gamma(d(x, y))\phi(d(y, Ty)) + \theta(d(x, y)) min\{\phi(d(x, Ty)), \\ & \phi(d(y, Sx))\} \quad \forall x, y \in X \quad with \quad x \neq y,\end{aligned}$$

where $\alpha, \beta, \gamma, \theta$ are four decreasing functions from $]0, +\infty[$ into $[0, 1[$ such that $\alpha(t)+\beta(t)+\gamma(t)+\theta(t) < 1$, for every $t > 0$, and $\sup\{\max(\beta(t), \gamma(t)) : t > 0\} < 1$. Then there exists a unique point $z \in X$ such that $Fix(S) = \{z\}$ for all $S \in \mathcal{A}$.

2.3 Remarks.

(a) If we take $\mathcal{A} = \{S, T\}$ and $\beta = \gamma$, then we obtain the result by R. A. Rashwan and A. M. Sadeek in [6].

(b) If we take $\beta = \gamma$ and $\mathcal{A} = \{T\}$, then we obtain one of the main results established by M. S. Khan et al. in the paper [4].

(c) If $\theta = 0$, $\mathcal{A} = \{T\}$ and the functions α, β and γ are constants, then we get the results obtained by D. Delbosco in [1] and by F. Skof in the paper [8].

(d) Example: We give here an example where we discuss the validity of the assumptions of Theorem 2.2. We take $X = \{1, 2, 3, 4\}$ and define a metric d on X by setting $d(1, 2) = 1$, and $d(1, 3) = d(1, 4) = d(2, 3) = d(2, 4) = d(3, 4) = 2$. We put $\mathcal{A} = \{S, T, V\}$, where $S1 = S2 = S3 = S4 = 1$; $T1 = T2 = T3 = 1$, $T4 = 2$; and $V1 = V2 = V4 = 1$, $V3 = 2$. For all $t \geq 0$, we put $\alpha(t) = 2/5$, $\beta(t) = 1/20$, $\gamma(t) = 7/20$, $\theta(t) = 1/6$, and $\phi(t) = t^2$. Then all the conditions of Theorem 2.2 are satisfied for the set $\mathcal{A} = \{S, T, V\}$, which has 1 as unique common fixed point.

The following result is an easy consequence of our Theorem 2.2.

2.4 Corollary. *Let (X, d) a complete metric space, $\mathcal{A}$ a (finite or infinite) set of self-maps of X and ϕ an element of Φ. We suppose that for all $S, T \in \mathcal{A}$ the following generalized contractive condition holds true*

$$\begin{aligned} \phi(d(Sx, Ty)) \leq \; & \alpha(d(x, y))\phi(d(x, y)) + \beta(d(x, y))min\{\phi(d(x, Sx)), \\ & \phi(d(y, Ty))\} + \gamma(d(x, y))min\{\phi(d(x, Ty)), \phi(d(y, Sx))\} \\ & \forall x, y \in X \text{ with } x \neq y, \qquad (C) \end{aligned}$$

where α, β, γ are three decreasing functions from $]0, \infty[$ into $[0, 1[$ such that $\alpha(t)+\mu\beta(t)+\gamma(t) < 1$, for every $t > 0$, where μ is a fixed constant in $]1, +\infty[$. Then there exists a unique point $z \in X$ such that $Fix(S) = \{z\}$ for all $S \in \mathcal{A}$.

3. A fixed point theorem in compact metric spaces

In a paper of Fisher [3], the following theorem has been established:

3.1 Theorem. *Let T be a continuous self-map of a compact metric space (X, d) such that*

$$d(Tx, Ty) < \frac{d(x, Tx) + d(y, Ty)}{2}, \qquad (F)$$

for all distinct x, y in X. Then T has a unique fixed point .

Following the essential idea of our result presented in section 2 we shall generalize Theorem 3.1 as follows:

3.2 Theorem. *Let S, T be two self-maps of a compact metric space (X, d) and let $\phi \in \Phi$ be a convex function. We suppose that $T, S \circ T$ are continuous and that S, T verify for all distinct x, y in X the inequality*

$$\phi(d(Sx, Ty)) < \max\left\{\phi(d(x, y)), \frac{\phi(d(x, Sx) + \phi(d(y, Ty))}{c}, \phi\left(\frac{d(x, Ty) + d(Sx, y)}{c}\right)\right\} \qquad (G)$$

where $c \geq 2$ is a fixed constant. Then S and T have a unique common fixed point $z \in X$. Moreover $Fix(S) = Fix(T) = \{z\}$.

Proof. Let x_0 be an element in X, an associate to it the sequence $(x_n)_n$ given by

$$\begin{aligned} x_{2n} &= Sx_{2n-1}, \quad n = 1, 2, ... \\ x_{2n+1} &= Tx_{2n}, \quad n = 0, 1, 2, ... \end{aligned}$$

Without loss of generality, we may assume that $t_n \neq 0$ for every integer n. In this case, it is easy to see that the sequence $(\phi(t_n))$ is decreasing and therefore it converges. Since X is compact, we may find a subsequence $(x_{2n(k)})_k$ converging to some element $z \in X$. Then by using the continuity of the maps T and ϕ, we get

$$\begin{aligned} \phi(d(z, Tz)) &= \lim_{k\to+\infty} \phi(t_{2n(k)}) = \lim_{k\to+\infty} \phi(t_{2n(k)+1}) \\ &= \lim_{k\to+\infty} \phi(d(x_{2n(k)+1}, x_{2n(k)+2})) \\ &= \lim_{k\to+\infty} \phi(d(Tx_{2n(k)}, (S\circ T)x_{2n(k)})) \\ &= \phi(d(Tz, (S\circ T)z)). \end{aligned} \tag{10}$$

Suppose that $z \neq Tz$, then we can apply the inequality (G) to $x = Tz$ and $y = z$. By using (10) and the fact that ϕ is convex and increasing, we obtain

$$\begin{aligned} \phi(d(z, Tz)) &= \phi(d(S(Tz), Tz)) \\ &< \max\left\{\phi(d(Tz, z)), \tfrac{\phi(d(Tz, STz)) + \phi(d(z, Tz))}{c}, \phi\left(\tfrac{d(STz, Tz) + d(Tz, z)}{c}\right)\right\} \\ &\leq \max\left\{\phi(d(Tz, z)), \tfrac{2\phi(d(z, Tz))}{c}\right\} \\ &\leq \max\left\{1, \tfrac{2}{c}\right\} \phi(d(Tz, z)) \leq \phi(d(Tz, z)). \end{aligned}$$

This is a contradiction. Therefore we must have $Tz = z$. The relation (10) will imply that $Sz = z$. To end the proof, let us suppose that there exists a point $\xi \neq z$ fixed, for instance, by S. Then by applying the inequality (G), we get

$$\phi(d(\xi, z)) = \phi(d(S\xi, Tz)) < \max\left\{\phi(d(\xi, z)), \frac{2}{c}\phi(d(\xi, z))\right\} \leq \phi(d(\xi, z)),$$

which is a contradiction.□

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