SHAPE-MEASURE METHOD FOR SOLVING ELLIPTIC OPTIMAL SHAPE PROBLEMS (FIXED CONTROL CASE)

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> Abstract: The aim of this paper is to introduce a new method for solving optimal shape problems which are defined with respect to a pair of geometrical elements. The problem is to find the optimal domain for a given functional that is involved with the solution of a linear or nonlinear elliptic equation with a boundary condition over a domain. By transferring the problem into a measure-theoretical form the shape-measure method, in Cartesian coordinates, will be used to find the optimal solution in two steps. First we will find the solution of the elliptic problem for a given domain by using the embedding method. Then the Shape-Measure method will be applied to find the optimal solution. Two examples are given for the linear and nonlinear cases of the elliptic problem.

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1. Introduction

Consider the optimal shape (OS) or optimal shape design (OSD) problems in which they are defined with respect to a pair of geometrical elements; this pair conists in a measurable set (in \mathbb{R}^2), which can be regarded as a domain, and a simple closed curve containing a given point, which is the boundary of the set. Based on the simple property of curves, the related OSD problem depends on the geometry which is used. We solved the appropriate OS in [2] by introducing shape-measure method in Polar coordinates. But in Cartesian coordinates, it is difficult to introduce a linear condition which determines the property of a closed curve being simple; thus in this paper we consider those measurable sets D which its boundary consists in a variable part Γ and a fixed part between two given points, to be sure it is simple.

This paper deals with solving an OS or OSD problem with a fixed control, which is to find the optimal domain like D for a given function, I, that is involved with the solution of a linear or nonlinear elliptic partial differential equation with a boundary condition over D. The process of solution is achieved in two stages. First for a fixed domain, by using the idea of approximating a curve by broken lines, Γ can be determined with fixed number of M points. Then D, any integral on D and the variational form of elliptic equations can be considered as a function of M variables. By means of a well-known process of embedding, we transfer the problem into a measure-theoretical one. The history of this idea can be found, for instance, in [2] and [10]. Then we enlarge the underlying space to reach an infinite linear system of equations that the unknown is a measure. By the use of total sets and putting an appropriate discretization, one can approximate the solution of the problem with the solution of a finite linear system of equations. Hence the value of I is calculated as a function of M variables for any given domain D. In the second stage, considering the previous one, a vector function $J: D \in \mathcal{D}_M \to I(D)$ is set up. Using a standard minimization algorithm on J, gives the minimizer domain; then Theorem 1, proves that this minimizer, is the optimal solution for the problem. Finally, two

examples for the linear and nonlinear cases of elliptic problem are given.

2. Problem

Let $D \subset \mathbb{R}^2$ be a bounded domain with a piecewise-smooth, closed and simple boundary ∂D . We assume that some part of ∂D is fixed and the rest, Γ , with the given initial and final points A and B respectively, is not fixed. Here we suppose that the fixed part of ∂D is made by three segments, parts of lines y = 0, x = 0 and y = 1 between points A(1,0), (0,0), (0,1), B(1,1) (see Figure 1). For more general case, the reader is advised to see [1]. Thus we choose an appropriate (variable) curve Γ joining A and B, so that D is well-defined. Let $X \in D \longrightarrow$ $u(X) \in \mathbb{R}$, that $X = (x, y) \in \mathbb{R}^2$, is a bounded solution of the following elliptic partial differential equation with the boundary condition on the domain D:

$$\Delta u(X) + f(X, u) = v(X), \ u_{|_{\partial D}} = 0, \tag{1}$$

where $X \in D \longrightarrow v(X) \in \mathbb{R}$ is a bounded fixed control function; the function f is assumed to be a bounded and continuous real-valued function in $L_2(D \times \mathbb{R})$. A domain D as above, is called an *admissible domain* if the elliptic equation (1) has a bounded solution on D; we denote by \mathcal{D} as the set of all such admissible domains. We are going to solve the problem of minimizing the functional $I(D) = \int_D f_0(X, u) dX$, on the set \mathcal{D} where f_0 is a given continuous, nonnegative, real-valued function on $D \times \mathbb{R}$. To calculate the value of I(D) for a given domain D, it is necessary, first, to identify the solution of the partial differential equations (1).

3. Weak Solution and Metamorphosis

In general, it is difficult and sometimes impossible to identify a classical solution for the problem like (1); thus usually one tries to find a generalized or *weak* solution of them which is more applicable than the classical one in some branches. In our method, especially whenever one wants to change the problem into a measure-theoretical form,

this kind of solution is more appropriate. Hence the variational form of the problem (1) is introduced in the following proposition. We remind the reader that here $H_0^1(D) = \{\psi \in H^1(D) : \psi_{|_{\partial D}} = 0\}$, where $H^1(D) = \{h \in L_2(D) : \frac{\partial h}{\partial x} \in L_2(D), \frac{\partial h}{\partial y} \in L_2(D)\}$ is the Sobolev space of order 1.

Proposition 1: Let u be the classical solution of (1), then we have the following equality:

$$\int_{D} (u\Delta\psi + \psi f) \, dX = \int_{D} \psi v \, dX \; ; \; \forall \psi \in H^1_0(D).$$
⁽²⁾

Proof: Multiplying (1) by the function $\psi \in H_0^1(D)$ and then integrating over D, with use of the Green's formula (see for instance [4]) gives:

$$\int_{D} \left(u \Delta \psi + \psi f - \psi v \right) dX = \int_{\partial D} \left(\psi \frac{\partial u}{\partial n} - u \frac{\partial \psi}{\partial n} \right) dS$$

where *n* is the unit vector normal to the boundary ∂D and directed outward with respect to *D*. Because $\psi_{|_{\partial D}} = 0$ and $u_{|_{\partial D}} = 0$, then (2) is satisfied. \Box

Definition: A function $u \in H^1(D)$ is called a bounded weak solution of the problem (1) when it bounded and satisfies in the equality (2) for all functions $\psi \in H^1_0(D)$.

We remind the reader that conditions for the existence of the classical and of the weak solution of the problem (1), and also other properties of them such as boundedness and uniqueness, have been considered in many references, like [4] and [3].

Now we can apply our *Shape-Measure* method for solving the problem. The bounded weak solution can be represented by a positive Radon measure. Hence instead of looking for the weak solution on the given domain D, one prefers to seek for its related measure, defined on the appropriate space. For the rest of the paper, we suppose $\Omega \equiv U \times \overline{D}$, where $U \subset \mathbb{R}$ is the smallest bounded set in which the bounded weak solution $u(\cdot)$ takes values. By applying the Riesz Representation Theorem [11], similar to the Proposition 3.1 in [2], one can prove the following proposition.

Shape-measure method for solving....

Proposition 2: Let u(X) be a bounded generalized solution of (1); there exists a unique positive Radon measure, say μ_u , in $\mathcal{M}^+(\Omega)$ so that:

$$\mu_u(F) \equiv \int_{\Omega} F \, d\mu_u = \int_D F(X, u) \, dX \; ; \; \forall F \in C(\Omega). \tag{3}$$

Thus the equality (2) changes into the following:

$$\mu_u(F_{\psi}) = \gamma_{\psi} \qquad ; \qquad \forall \psi \in H_0^1(D) \tag{4}$$

where $F_{\psi} = u\Delta\psi + f\psi$ and $\gamma_{\psi} = \int_{D} \psi v \, dX$. Also, $I(D) = \mu_u(f_{\circ})$. Because the measure μ_u projects on the (x, y)-space as the respective Lebesgue measure, we should have $\mu_u(\xi) = a_{\xi}$, where $\xi : \Omega \to \mathbb{R}$ depends only on variable X (i.e. $\xi \in C_1(\Omega)$), and a_{ξ} is the Lebesgue integral of ξ over D. Therefore the problem can be described as follows:

Find a measure $\mu_u \in \mathcal{M}^+(\Omega)$ so that it satisfies the following constraints:

$$\mu_u(F_{\psi}) = \gamma_{\psi}, \qquad \forall \psi \in H_0^1(D);$$

$$\mu_u(\xi) = a_{\xi}, \qquad \forall \xi \in C_1(\Omega).$$
(5)

As Rubio did in [9], to be sure that we do not miss any solution, consider a more general version of the problem by extending the underlying space; instead of finding a $\mu_u \in \mathcal{M}^+(\Omega)$, defined by Proposition 2, satisfying equalities (5), we seek a measure $\mu \in \mathcal{M}^+(\Omega)$ which satisfies just the conditions

$$\mu(F_{\psi}) = \gamma_{\psi}, \qquad \forall \psi \in H_0^1(D);$$

$$\mu(\xi) = a_{\xi}, \qquad \forall \xi \in C_1(\Omega).$$
(6)

Hence we have $I(D) = \mu(f_{\circ})$. The system (6) is linear because all the functions in the right-hand-side of equations are linear functions in their argument μ . But the number of equations and the underlying space are not finite.

4. Approximation

We shall develop the system (6) by requiring that only a finite number of the constraints are satisfied. This will be achieved by choosing countable sets of functions whose linear combinations are dense in the appropriate spaces. First we try to approximate the unknown part of the boundary, Γ , just by the finite number of points.

Approximating ∂D with broken lines: The idea of selecting a finite set of points instead of the curve Γ , comes from the approximation of a curve by broken lines. In general the curve ∂D , and hence Γ , can be regarded as an infinite set of points. More specifically, by applying the density property, one can regard Γ as a countable set. For the given D and hence for the given Γ , let $A_m = (x_m, y_m), m = 0, 1, 2, \ldots, M$, be a finite number of these points (we suppose $A_0 = A$). We link together each pair of consecutive points A_m and A_{m+1} for $m = 0, 1, \ldots, M - 1$ and close this curve by joining the points A_M and B together. Now the resulted shape, which is denoted by ∂D_M , is an approximation for ∂D ; we also call D_M to the domain which introduced by its boundary ∂D_M . The domain D_M is called a *M*-approximated domain of D (domains D, D_M and their boundaries are shown in Figure 1).

It is possible that by increasing the number of points, M, the curve ∂D_M will become closer and closer (in the Euclidean metric) to the curve ∂D , and hence one may conclude that the minimizer of I over \mathcal{D}_M , if one exists, tends to the minimizer of I over \mathcal{D} , if one exists. In the Appendix, we have explained some of the difficulties that arise. Thus, we will fix the number of points (M) and look for the minimizer of I(D) amongst all admissible D_M 's.

Here we have actually 2M unknowns to determine, x_1, x_2, \ldots, x_M ,

 y_1, y_2, \ldots, y_M . It would be more convenient if one, somehow, could reduce the number of unknowns, without losing the generality. For a given positive integer M, let the value of the components y_1, y_2, \ldots, y_M , be fixed. Because x_m is a free term, the point A_m could be anywhere on the line $y = Y_m, x \ge 0$ for every m (see Figure 1). Therefore points A_m

and A_{m+1} can be chosen so that they belong to Γ and hence the part of Γ between the lines $y = Y_m$ and $y = Y_{m+1}$ can be approximated by the segment $A_m A_{m+1}$ (especially whenever the number M is large). It means, we do not lose generality. Thus, from now on, we fix the components y_1, y_2, \ldots, y_M with the values Y_1, Y_2, \ldots, Y_M , respectively. Indeed the set $\{A_M = (x_m, Y_m), m = 1, 2, \ldots, M\}$, which is called *M*-representation of D, determines the M-approximation domain D_M .

First set of functions: We are going to introduce the set

 $\{\psi_i \in H_0^1(D) : i = 1, 2, ...\}$ so that the linear combinations of the functions ψ_i 's are uniformly dense - that is, dense in the topology of the uniform convergence - in the space $H_0^1(D)$. We know that the vector space of polynomials with the variable x and y, P(x, y), is dense in $C^{\infty}(\overline{D})$; therefore the set $P_0(x, y) = \{p(x, y) \in P(x, y) \mid p(x, y) = 0, \forall (x, y) \in \partial D\}$, is dense (uniformly of course) in the following space:

$$\left\{h \in C^{\infty}(\overline{D}) : h_{|_{\partial D}} = 0\right\} \equiv C_0^{\infty}(\overline{D}).$$

So the set $Q(x,y) = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \ldots\}$ is a countable base for the vector space P(x, y) and hence each elements of P(x, y)and also $P_0(x, y)$, is a linear combination of the elements in Q(x, y). By theorem 3 of Mikhailov [4] page 131, the space $C^{\infty}(\overline{D})$ is dense in $H^1(D)$; thus the space $C_0^{\infty}(\overline{D})$ will be dense in $H_0^1(D)$. Consequently, the space $P_0(x, y)$ is uniformly dense in $H_0^1(D)$. We define the function ψ_i for each *i* as:

$$\psi_i(x,y) = xy(y-1) \prod_{l=1}^M (x-x_l+y-y_l)q_i(x,y),$$
(7)

where q_i is an element of the countable set Q(x, y). Therefor $\psi_{|_{\Gamma}} = 0$ and the set $\{\psi_i(x, y) : i = 1, 2, ...\}$, is total in $H_0^1(D)$.

<u>Second set of functions</u>: Let L be a given positive integer number and divide D into L (not necessary equal) parts D_1, D_2, \ldots, D_L , so that by increasing L the area of each $D_s, s = 1, 2, \ldots, L$, will be decreased. Then, for each $s = 1, 2, \ldots, L$, we define:

$$\xi_s(x, y, u) = \begin{cases} 1 & \text{if } (x, y) \in D_s \\ 0 & \text{otherwise.} \end{cases}$$

These functions are not continuous, but each of them is the limit of an increasing sequence of positive continuous functions, $\{\xi_{s_k}\}$; then if μ is any positive Radon measure on Ω , $\mu(\xi_s) = \lim_{k\to\infty} \mu(\xi_{s_k})$. Now consider the set $\{\xi_j : j = 1, 2, ..., l\}$ of all such functions, for all positive integer L. The linear combination of these functions can approximate a function in $C_1(\Omega)$ arbitrary well (see [9] chapter 5).

As a result, the problem (6) can be replaced by another one in which we are looking for the measure $\mu \in \mathcal{M}^+(\Omega)$, so that it satisfies the following constraints:

$$\mu(F_i) = \gamma_i, \qquad i = 1, 2, \dots; \mu(\xi_j) = a_j, \qquad j = 1, 2, \dots$$
(8)

where $F_i \equiv F_{\psi_i}$, $\gamma_i \equiv \gamma_{\psi_i}$, $a_j \equiv a_{\xi_j}$. To approximate the system of equations in (8) with a finite system of equations, first we choose a finite number of equations as follows:

$$\mu_{M_1,M_2}(F_i) = \gamma_i, \qquad i = 1, 2, \dots, M_1; \mu_{M_1,M_2}(\xi_j) = a_j, \qquad j = 1, 2, \dots, M_2,$$
(9)

where M_1 and M_2 are two positive integers. If we denote by $Q(M_1, M_2)$ the set of positive Radon measures in $\mathcal{M}^+(\Omega)$ which satisfy equalities (6), and also denote by Q the set of positive Radon measures in $\mathcal{M}^+(\Omega)$ which satisfy equalities (6), by regarding the property of the total sets one can easily prove the following Proposition by considering the proof of Proposition III.1 in [9].

Proposition 3: If $M_1, M_2 \longrightarrow \infty$; then $Q(M_1, M_2) \longrightarrow Q$, hence for the large enough numbers M_1 and M_2 the set Q can be identified by $Q(M_1, M_2)$. But even if the number of equations in (6) is finite, the underlying space $Q(M_1, M_2)$ is still infinite-dimensional. It is possible to define a finite linear system whose solutions can be used to approximate that for (6).

Discritization: By a result of Rosenbloom [8], which was proved in Theorem A.5 Appendix in [9], that μ_{M_1,M_2} in (6) can be characterized as $\mu_{M_1,M_2} = \sum_{n=1}^{M_1+M_2} \alpha_n \delta(Z_n)$, with triples $Z_n \in \Omega$ and the coefficients $\alpha_n \geq 0$ for $n = 1, 2, \ldots, M_1 + M_2$, where $\delta(z) \in \mathcal{M}^+(\Omega)$ is supposed to be a unitary atomic measure with support the singleton set $\{z\}$. This structural result points the way toward a further approximation scheme; the measure problem is equivalent to a nonlinear one in which the unknowns are the coefficients α_n and supports $\{Z_n\}$. It would be more convenient if one could find the solution only with respect to the coefficients α_n ; this would be a finite linear system of equations (a type of linear programming problem). The answer lies in approximating this support, by introducing a set dense in Ω . Proposition *III.3* of [9] Chapter 3, states that the measure μ_{M_1,M_2} has the following form

$$\mu_{M_1,M_2} = \sum_{n=1}^{N} \alpha_n \delta(Z_n), \tag{10}$$

where $Z_n, n = 1, 2, ..., N$, belongs to a dense subset of Ω .

Now let put a discretization on Ω , with the nodes $Z_n = (x_n, y_n, u_n)$, in a dense subset of Ω ; then we can set up the following linear system in which the unknowns are the coefficients α_n :

$$\alpha_{n} \geq 0, \qquad n = 1, 2, \dots, N;$$

$$\sum_{n=1}^{N} \alpha_{n} F_{i}(Z_{n}) = \gamma_{i}, \qquad i = 1, 2, \dots, M_{1};$$

$$\sum_{n=1}^{N} \alpha_{n} \xi_{j}(Z_{n}) = a_{j}, \qquad j = 1, 2, \dots, M_{2}.$$
(11)

We remind the reader that the solution of (11) is not necessary unique, (even if the problem (1) satisfies the necessary conditions for having a unique bounded weak solution, because of the approximation schame. Each solution introduces a measure μ_{M_1,M_2} via the equality (10) which has the same properties (approximately) as the measure μ_u , the representative measure for the weak solution u(X). Indeed we achieve an approximate solution for the elliptic problem in the given domain D. Therefore we are able to calculate the value of I(D) for each given domain D. In the next, we shall explain how one can find the optimal domain for the mentioned OS problem in \mathcal{D}_M by applying the above results.

5. The optimal solution

The main aim of the present section is to find an optimal domain $D^* \in \mathcal{D}_M$ so that the value of $I(D^*)$ will be the minimum on the set \mathcal{D}_M . By applying the result of the previous section, a solution of (1) can be found. This solution is approximated by a solution of the linear system (11) according to the variables, $x_m, m = 1, 2, \ldots, M$. As mentioned, this solution is not necessary unique. Let us to specify one of them for each D; there are some possibilities, for example, by solving the following linear programming problem, one may chose that one in which the value of $\int_D f_\circ(X, u) dX$ (for a given D) is minimum according to the variables $\alpha_n, n = 1, 2, \ldots, N$:

$$\begin{aligned} Minimize: & \sum_{n=1}^{N} \alpha_n f_0(Z_n) \\ Subject to: & \alpha_n \ge 0, & n = 1, 2, \dots, N; \\ & \sum_{n=1}^{N} \alpha_n F_i(Z_n) = \gamma_i, & i = 1, 2, \dots, M_1; \\ & \sum_{n=1}^{N} \alpha_n \xi_j(Z_n) = a_j, & j = 1, 2, \dots, M_2. \end{aligned}$$

As a result, for each D, the value $I(D) = \int_D f_o(X, u) dX \equiv \mu(f_o) \simeq \mu_{M_1, M_2}(f_o)$, is defined uniquely in terms of the variables $x_m, m = 1, 2, \dots, M$.

So, we set up a function, J, on \mathcal{D}_M defined by

$$J : D \in \mathcal{D}_M \longrightarrow I(D) \cong \mu_{M_1, M_2}(f_\circ) \in \mathbb{R};$$
(13)

where $\mu_{M_1,M_2}(f_\circ) = \sum_{n=1}^N \alpha_n f_\circ(Z_n)$. By regarding the definition of M-representation of D, clearly J is a function of the variables x_1, x_2, \ldots, x_M , and hence can be regarded as a vector function:

$$J : (x_1, x_2, \dots, x_M) \in \mathbb{R}^M \longrightarrow \mu_{M_1, M_2}(f_\circ) \in \mathbb{R}.$$
 (14)

It is not possible in general to ascertain continuity properties of this function (see for instance [6]); we can say, however, that, since this is a real-valued function which is bounded below, and is defined on a compact set (since constraints are to be put in the variables), it is possible to find a sequence of points so that the value of the function along the sequence tends to the (finite) infimum of the function. The coordinate values corresponding to the points in the sequence are of course finite.

Now, suppose that $(x_1^*, x_2^*, \ldots, x_M^*)$ is the minimizer of the vector function J; it can be identified by using one of the related minimization methods (for instance the method introduced by Nelder and Mead, see [12]and [5]). For this, one can apply standard Algorithms and Routines (like AMOEBA [7] or EO4JAF-NAG Library Routine). The introduced domain by the minimizer $(x_1^*, x_2^*, \ldots, x_M^*)$ is denoted by D^* . We assume in the following theoretical result that the minimization algorithm which is used, (such as AMOEBA) is perfect; that is, it comes out with the global minimum of J in its (compact) domain.

Theorem 1: Let M, M_1 and M_2 be the given positive integer numbers which were defined in section 4, and D^* be the minimizer of (14) as mentioned above. Then D^* is the minimizer domain of the functional I over \mathcal{D}_M and the value of $I(D^*)$ can be approximated by $J(D^*)$; moreover $J(D^*) \longrightarrow I(D^*)$ as M_1 and M_2 tend to infinity.

Proof: Suppose D^* is not the minimizer of I; hence there exists a domain, call D', in \mathcal{D}_M so that $I(D') < I(D^*)$. Proposition 2 shows that there is a unique measure, call μ' , in $\mathcal{M}^+(\Omega)$ so that $I(D') = \mu'(f_\circ)$, and also Proposition 2 states that for sufficiently large numbers M_1 and M_2 , $\mu'(f_\circ)$ can be approximated by $\mu'_{M_1,M_2}(f_\circ)$ in $Q(M_1,M_2)$. Thus, $I(D') \cong \mu'_{M_1,M_2}(f_\circ) = J(D')$. In the same way, one can show that $J(D^*)$ approximates $I(D^*)$; so $I(D^*) \cong \mu^*_{M_1,M_2}(f_\circ) = J(D^*)$. Hence $J(D') < J(D^*)$. which is contrary with the fact that D^* is the minimizer of J. Moreover, from Proposition 2 it follows that $J(D^*)$ tends to $I(D^*)$ as $M_1, M_2 \longrightarrow \infty$. \Box

6. Numerical Examples

For the next two examples, we consider the elliptic equations (1) for which the function v(x, y) (the fixed control function) is defined as:

$$v(x,y) = \begin{cases} 1 & \text{if } (x,y) \in D \cap C \\ 0 & \text{otherwise,} \end{cases}$$

where C is the square $[\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}]$ (see Figure 2). We also take M = 8 and suppose Y_1, Y_2, \ldots, Y_8 are 0.15, 0.25, ..., 0.85, respectively. By extra constraints on $x_2, x_3, \ldots, x_7, x_m \geq \frac{3}{4}$, $m = 2, 3, \ldots, 7$, the value of γ_i for any $D \in \mathcal{D}_M$ is defined as

$$\gamma_i = \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \psi_i(x, y) \, dx \, dy \; ; \; i = 1, 2, \dots, M_1.$$

We also assume that the function $u(\cdot)$ takes value in the bounded set U = [-1, 1] (one may obtain the set U by trial and error so as to be sure that the appropriate finite linear system in (11) has a solution.

Our way to find an optimal domain is an iterative method. For a given set of variables $x_1 = X_1, x_2 = X_2, \ldots, x_8 = X_8$, we will set up the linear system (11) and calculate the value of I(D) according to the X_m 's. Then the standard minimization algorithm changes these X_1, X_2, \ldots, X_8 , to new ones for which the value of I(D) is supposed to be less than previous; these new values introduce a new domain. Again, in the next iteration, an appropriate linear system for the new domain will be solved to calculate the value of I(D) and see whether I(D) is smaller than the previous on in the former iteration or not. If the value is not smaller, the Algorithm changes the domain with the suitable one; if it has been smaller, the Algorithm seeks again for the other domain like $D' \in \mathcal{D}_M$ with the smaller value of I(D') than I(D). The iteration will be stopped whenever the optimal domain is obtained; note that we assume in this discussion that the standard minimization Algorithm (AMOEBA) is qualified to obtain the global minimizer without any restriction (see Appendix C of [1]).

6.1. Nods and Equations

To establish the linear system (11) it is necessary to put a discretization on Ω ; because our method is iterative, the discretizations depends on the values X_1, X_2, \ldots, X_8 at each iteration. Thus, we select N = 740 nodes $Z_n = (x_n, y_n, u_n)$ in Ω , so that each component is a rational number; hence these nodes belong to a dense subset of Ω . Since $u_{|_{\partial D}} = 0$, for each $(x_n, y_n) \in \partial D$, we should have $Z_n = (x_n, y_n, 0)$. This fact has been taken into account in the discretization by choosing 36 related nodes. The rest of the nodes are related to the interior points of D. We consider $Z_n = (x_n, y_n, u_n) \in D$ for n = 36 + 88(i - 1) + 11(j - 1) + k as

$$x_n = \frac{(i+0.5)X_j}{10}, y_n = Y_j, u_n = \frac{2(k-1)}{10} - 1$$

that $1\leq i\leq 8$, $1\leq j\leq 8$, $1\leq k\leq 11.$

To set up the mentioned linear system in (11) we select $M_1 = 10$ and $M_2 = 8$, and consider the polynomial $q_i(x, y)$ form the set $\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$. Also the domain D is divided into 8 parts, say D_1, D_2, \ldots, D_8 , as follows: D_1 is the region of D between the lines y = 0 and y = 0.2 (OAe_1o_2 in Figure 2), D_2 is the region of D between the lines y = 0.2 and y = 0.3 ($o_1e_1e_2o_2$ in Figure 2), and similarly D_3, D_4, \ldots, D_7 ; we define D_8 as the region of D between the lines y = 0.8 and y = 1 (o_7e_7BE in Figure 2), where $x_{e_1} = \frac{1}{2}(X_{l+1} - X_l) + X_l$; $l = 1, 2, \ldots, 7$. Therefore $a_j = \int_D \xi_j(x, y) dX = area of Dj, \forall j = 1, 2, \ldots, 8$.

Hence in our case, the linear system (11) is

$$\alpha_n \ge 0, \qquad n = 1, 2, \dots, 740;
\sum_{n=1}^{740} \alpha_n F_i(Z_n) = \gamma_i, \qquad i = 1, 2, \dots, 10;
\sum_{n=1}^{740} \alpha_n \xi_j(Z_n) = a_j, \qquad j = 1, 2, \dots, 8.$$
(15)

To find the nonnegative unknowns α_n 's we apply the E04MBF - NAG Library Routine Document. The result shows a nonnegative value for each α_n , $n = 1, 2, \ldots, 740$, that satisfy the linear system. By applying these values in (10), one can calculate the value of I(D) for a given function f_o , which is a function of the variables X_1, X_2, \ldots, X_8 ; thus we have set up the function J in (14). By applying a standard minimization algorithm on J, the optimal domain in \mathcal{D}_M is obtained. We remind the reader that the functions F_i and the values of γ_i , $i = 1, 2, \ldots, 10$, have been calculated by the package "Maple V.3".

6.2. Minimization

In minimization, we apply the Downhill Simplex Method in Multidimension by using the Subroutine AMOEBA (see [7]) with the conditions $X_1 \ge 0, X_8 \ge 0$ and $X_m \ge 0.75, m = 2, 3, ..., 7$; besides, we also consider an upper bound for variables (suppose they are not higher than 2). These conditions are applied by means of a penalty method to change the constraint minimization problem into an unconstrainted one (for instance see [12]).

To start, AMOEBA needs an initial value for variables X_m , when m = 1, 2, ..., 8, (a given domain). At any iteration the new domain is illustrated and the new value for J is calculated; comparing this value with the previous one leads the algorithm to find a domain with a smaller value. This procedure is repeating till the optimal domain is characterized.

In the next, two examples are given; one for the linear case and the other for the nonlinear case of the elliptic equation. We chose the function f_{\circ} as $f_{\circ} = (u - 0.1)^2$, this function, indeed, can be considered as a distribution of heat in the surface for the system governed by an elliptic equations.

6.3. Example 1

In the linear case defined by the partial differential equations (1) and f(x, y, u) = 0, the function F_i in (15) is $F_i = u\Delta\psi_i$; i = 1, 2, ..., 10. We used the initial values $X_m = 1.0, m = 1, 2, ..., 8$, and the stopping tolerance for the program (variable *ftol* in the Subroutine *AMOEBA*) has been chosen as 10^{-7} . Here are the results:

- The optimal value of I = 0.70469099432415;
- The number of iterations = 827;
- The value of the variables in the final step: $X_1 = 1.033028, X_2 = 1.390598, X_3 = 1.422364, X_4 = 0.97706,$ $X_5 = 1.017410, X_6 = 0.958974, X_7 = 1.018387, X_8 = 0.951333.$

These values represent the optimal domain. The initial and the final domain has been shown in the Figure 3, and also the alteration of the objective function, according to the number of iterations, has been plotted in the Figure 4.

6.4. Example 2

For the nonlinear case of the partial differential equations (1), we have taken $f(x, y, u) = 0.25u^2$, and used the same initial values and stopping tolerance as *Example 1*. The obtained results are:

- The optimal value of I = 0.45467920356379;
- The number of iterations = 502;
- The value of the variables in the final step:
 - $X_1 = 1.050197, X_2 = 1.085212, X_3 = 0.750001, X_4 = 0.768701,$
 - $X_5 = 1.129861, X_6 = 1.137751, X_7 = 0.977838, X_8 = 1.615668,$

which represent the optimal domain, shown in the Figure 5. Also the change of the objective function, according to the number of iterations, has been plotted in the Figure 6.

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7. Appendix

Why \mathcal{D}_M instead of \mathcal{D} ?

Based on the approximation of a closed and simple curve in \mathbb{R}^2 by a set of broken lines, we decided to consider \mathcal{D}_M as the underlying space in which the minimization takes place. Indeed we approximated the variable part of any domain $D \in \mathcal{D}_M$, Γ , by M number of segments (in other words by M + 1 corners). As $M \longrightarrow \infty$, if an appropriate optimal shape design problem in \mathcal{D}_M has a minimizer, then this may tend in some topology to the minimizer over \mathcal{D} if such exists. However things can go wrong; for instance: There may be no minimizer over \mathcal{D}_M , there may be no minimizer over \mathcal{D} (or both \mathcal{D} and \mathcal{D}_M), the sequence of minimizer over \mathcal{D}_M may not be convergent or may tend in some sense towards a curve that does not define a shape.

On the other hand, let $D_M^* \in \mathcal{D}_M$ be the optimal solution of the appropriate problem over \mathcal{D}_M , and $\eta_M^* \in \mathcal{M}^+(\omega)$ be the optimal measure which represents the boundary of $D_M^*(\partial D_M^*)$; then because $\mathcal{M}^+(\omega)$ is compact, the sequence $\{\eta_M^*\}_{M=1}^{\infty}$ and hence $\{\partial D_M^*\}_{M=1}^{\infty}$, have a convergent subsequence even they are not convergent. Young in [13] has shown that their related subsequences of broken lines, tends to an infinitesimal zigzag (generalized curve). This is not (necessarily) an admissible curve (see [13] Chapter VI). So the solution over \mathcal{D}_M does not tend to the solution over \mathcal{D} , even in the weakly*-sense. Also, there is the important point that too oscillatory boundaries (like the infinitesimal zigzag) sometimes cause problem; Pironneau in [6] shows some of these problems.

So, we prefer to fix the number of M in this paper, and search for the optimal solution of the appropriate optimal shape design problems over \mathcal{D}_M . 8. Figures

Figure 1: D and ∂D in the defined assumption.

Figure 2: An admissible domain D under the assumptions of the numerical work

Figure 3: The initial and the optimal domain for the linear case of elliptic equation.

Shape-measure method for solving....

Figure 4: Changes of the objective function according to iterations in the linear case.

Figure 5: The initial and the optimal domain for nonlinear case of elliptic equations.

 $Shape-measure\ method\ for\ solving....$

Figure 6: Changes of the objective function according to iterations in the nonlinear case.