ON FINITE GROUPS WITH TWO IRREDUCIBLE CHARACTER DEGREES

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Abstract. We characterize non-abelian finite groups with only two irreducible character degree and prime number of non-linear irreducible characters.

1. Introduction

Let \( \text{Irr}(G) \) be the set of irreducible complex characters of a finite group \( G \) and \( \text{cd}(G) = \{ \chi(1) \mid \chi \in \text{Irr}(G) \} \) be the set of irreducible character degrees of \( G \). The question:

Given the set \( \text{cd}(G) \), what can be said about the structure of \( G \)?

have been studied by several people. In case \( \text{cd}(G) = \{ 1, m \} \), for some integer \( m \), the basic tools for studying the question can be found in Chapter 12 of the well known book of Isaacs [8]. Isaacs proved in Theorem 12.5 and Corollary 12.6 the following results.

**Theorem A.** Let \( G \) be a finite group and \( \text{cd}(G) = \{ 1, m \} \). Then, at least one of the followings occurs:
(a) \( G \) has an abelian normal subgroup of index \( m \).
(b) \( m = p^r \) for some prime \( p \) and \( G \) is a direct product of a \( p \)-group and an abelian group.

**Theorem B.** Let \( G \) be a finite group and \( \text{cd}(G) = \{1, m\} \). Then, \( G' \) is an abelian group, where \( G' \) is the derived subgroup of \( G \).

If \( G \) is non-nilpotent and \( \text{cd}(G) = \{1, m\} \), then clearly (a) of Theorem A holds. Recently, Bianchi et al. [4], characterized non-nilpotent groups \( G \) with \( \text{cd}(G) = \{1, m\} \), and proved the following result.

**Theorem C.** Let \( G \) be a finite non-nilpotent group. Then, \( \text{cd}(G) = \{1, m\} \) if and only if \( G' \) is abelian and one of the following holds:

1. \( m \) is a prime, and \( F(G) \), the fitting subgroup of \( G \), is abelian of index \( m \) in \( G \).
2. \( G' \cap Z(G) = 1 \) and \( G/Z(G) \) is a Frobenius group with kernel \( G' \times Z(G) \) and a cyclic complement of order \( |G : G' \times Z(G)| = m \), where \( Z(G) \) denotes the center of \( G \).

Our notations and terminologies are standard and mainly taken from [8]. We also use the following notations:

- \( \text{Lin}(G) = \) the set of linear characters of \( G \).
- \( \text{Irr}_1(G) = \text{Irr}(G) \setminus \text{Lin}(G) \).
- \( ES(c, 2) = \) the extra-special group of the order \( 2^{2c+1} \).
- \( (H, N) = \) the Frobenius group with kernel \( N \) and complement \( H \).
- \( C_n = \) the cyclic group of order \( n \).
- \( E(q^n) = \) the elementary abelian \( q \)-group of rank \( n \).
- \( Z_i = Z_i(G) = \) the \( i \)-th center of \( G \).

Here, we characterize groups \( G \) with two irreducible character degrees, that is \( \text{cd}(G) = \{1, m\} \), such that the number of irreducible non-linear characters is a prime \( p \). In [1], Berkovich characterized solvable groups for which the degrees of the characters are distinct, except for one pair (having the same degrees). Thus, by [1] we may assume that \( p \geq 3 \). Firstly, we consider nilpotent groups and prove the following result.

**Theorem 1.** Let \( G \) be a finite nilpotent group, \( p \geq 3 \), a prime and \( m \), a positive integer. Then \( \text{cd}(G) = \{1, m\} \) and \( G \) has \( p \) non-linear
irreducible characters of degree \( m \) if and only if one of the followings holds:

(a) \( G = C_p \times ES(c, 2) \); that is, \( G \) is the direct product of a cyclic group of order \( p \) and an extra-special group of order \( 2^{2c+1} \).

(b) \( G \) is an special \( 2 \)-group of order \( m^2(p + 1) \), and for each subgroup \( M \) of index \( 2 \) of \( G' \), \( G/M \cong ES(c, 2) \), where \( 2^c = m \).

(c) \( G \) is dihedral, semidihedral or a generalized quaternion \( 2 \)-group and \( |G| = 4(p + 1) \).

If \( cd(G) = \{1, m\} \), then by Theorem B, \( G \) is metabelian. Theorem C characterizes non-nilpotent groups \( G \) with \( cd(G) = \{1, m\} \). We complete the characterization of groups \( G \) with \( cd(G) = \{1, m\} \) having \( p \) non-linear characters of degree \( m \), by considering the non-nilpotent case in the following theorem.

**Theorem 2.** Let \( G \) be a finite non-nilpotent group, \( p \geq 3 \), a prime and \( m \), a positive integer. Then \( cd(G) = \{1, m\} \) and \( G \) has \( p \) non-linear characters of the degree \( m \) if and only if \( G' \) is abelian and one of the followings holds:

(a) \( G = (C_m, G') \), where \( G \) is a Frobenius group with kernel of order \( mp + 1 \) and a cyclic complement of order \( m \).

(b) \( G/Z(G) = (C_m, (G/Z(G))') \), \( Z(G) = C_p \) and \( G' = E(q^n) \), where \( q^n - 1 = m \). Therefore, \( G = C_p \times (C_{q^n-1}, E(q^n)) \) or \( G = C_p(q^n-1) \times E(q^n) \); that is, \( G \) is a semidirect product with kernel \( E(q^n) \).

(c) \( m \) is a prime, \( G' \cap Z_{t-1} = m^{t-2} = p+1 \) and \( G/Z_{t-1} \cong (C_m, (G/Z_{t-1})') \), a Frobenius group with kernel of order \( b \), such that \( |G| = m^tb \) and \( \gcd(m, b) = 1 \).

2. Proofs

To prove Theorem 1, firstly we prove the following Lemma for \( p \)-groups.

**Lemma 1.** Let \( G \) be a finite \( p \)-group, where \( p \) is a prime, \( cd(G) = \{1, m\} \) and \( |\text{Irr}_1(G)| \) is odd. Then, at least one of the followings holds:
(a) $G$ is an special 2-group, the index of $G'$ is $m^2$, and for each subgroup $M$ of index 2 of $G'$, $G/M \simeq ES(c, 2)$ such that $2^c = m$.

(b) $G$ is dihedral, semidihedral or a generalized quaternion 2-group and $|G| = 4(p + 1)$.

**Proof.** Let $p \neq 2$. Then, by [8, Exercise 3.16], the number of non-linear characters of $G$ is even, which is a contradiction. Thus, $|G| = 2^n$ and $m = 2^c$. Let $|\text{Irr}_1(G)| = x$. From

$$|G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2$$

we have

$$2^n = x2^{2c} + |G : G'| = x2^{2c} + 2^k,$$

for some positive integer $k$. Since $x$ is odd, then $2c = k$ and $|G| = (x + 1)2^{2c}$. Hence, $x + 1 = 2^r = |G'|$ and $|G : G'| = 2^{2c}$. Now, we distinguish two cases below.

**Case 1:** $G' \leq Z(G)$. Since $G$ is nilpotent then $2^{2c} = m^2 = \chi(1)^2$ divides $|G : Z(G)|$, for all $\chi \in \text{Irr}_1(G)$ (see [6, Theorem 8.2]). Since $|G : G'| = 2^{2c}$, it follows that $Z(G) = G'$. Now, let $M$ be an arbitrary normal subgroup of index 2 in $G'$. Note that $M$ is normal in $G$, since $G'$ is central. If $|\text{Irr}_1(G/M)| = y$, then

$$|G/M| = 2^{2c+1} = y2^{2c} + 2^{2c}.$$ 

So $y = 1$ and by [2, Main Theorem], $G/M \simeq ES(c, 2)$. Thus, $\Phi(G/M) = (G/M)'$, where $\Phi(G)$ is the Frattini subgroup of $G$, and

$$\frac{G/M}{\Phi(G/M)} \simeq G/G'$$

is elementary. So, $\Phi(G)G'/G' = \Phi(G/G') = 1$, and hence $\Phi(G) \leq G'$. Therefore, $\Phi(G) = G'$. Now, by [5, Lemma III, 3.14], $G^2 = (G')^2 \leq Z(G)$. So for each $g_1, g_2 \in G$, $[g_1, g_2]^2 = [g_1^2, g_2^2] = 1$, and hence $G'$ is an elementary abelian group. Thus, $G$ is an special 2-group. Therefore, (a) holds.

**Case 2:** $G' \nsubseteq Z(G)$. Let $c \neq 1$. By [8, Exercise 5.14], each subgroup of index $2^c$ in $G$ contains $G'$. Then, each $\chi \in \text{Irr}_1(G)$ can be induced from a linear character of some normal subgroup of $G$. Thus, $G$ is an $nM$-group and by [7, Theorem 3], $G'$ is an elementary abelian subgroup.
(Recall that a group is said $nM$-group if all its irreducible characters are induced from a linear character of some normal subgroup of $G$.)

If $N$ is a normal subgroup of $G$ such that $G/N$ is a non-abelian group, since $\text{cd}(G/N) = \{1, 2^c\}$, by [6, Theorem 8.2], $2^{2c}$ divides $|G/N|$. So, from $|G/N| = \sum_{\chi \in \text{Irr}(G/N)} \chi(1)^2$, we have that $2^{2c}$ divides $t = |G/N| : (G/N)'$. But $t = |G : G'| \leq |G : G'| = 2^{2c}$, so that $t = 2^{2c}$ and $N \leq G'$. Therefore, $Z(G) \leq G'$ and for $\chi \in \text{Irr}(G)$ if $K = \ker(\chi)$, then $K \leq G'$.

Now, suppose that $\bar{G} = G/K$. Since $\text{cd}(\bar{G}) = \{1, 2^c\}$, then from $|\bar{G}| = \sum_{\tilde{\chi} \in \text{Irr}(\bar{G})} \tilde{\chi}(1)^2$ we have that $|\text{Irr}(\bar{G})|$ is odd. By a similar argument for the group $\bar{G}$ instead of $G$, we have $Z(\bar{G}) \leq G' / K$. Since $\bar{G}$ is an $nM$-group for each $\tilde{\chi} \in \text{Irr}(\bar{G})$, then there exists a normal subgroup $\bar{H}$ of $\bar{G}$ of index $2^c$ and $\bar{\theta} \in \text{Lin}(\bar{H})$ such that $\tilde{\chi} = \bar{\theta} \bar{G}$. Let $\ker(\bar{\theta}) \neq 1$. From $Z(\bar{G}) \leq G' \leq \bar{H}$, we have $\ker(\bar{\theta}) \cap Z(\bar{G}) \neq 1$, and so $\bigcap_{x \in G}(\ker(\bar{\theta}))^x \neq 1$. Thus, $\ker(\tilde{\chi}) \neq 1$ (see [8, Lemma 5.11]). But, for each $xK \in \bar{G}$ we have $\tilde{\chi}(xK) = \chi(x)$ and thus $\ker(\tilde{\chi}) = 1$, which is a contradiction. Therefore, $\ker(\bar{\theta}) = 1$ and $\bar{H}' = \bigcap\{\ker(\lambda) \mid \lambda \in \text{Lin}(\bar{H})\} = 1$. So, $\bar{H}$ is abelian and by [8, Theorem 2.32], $\bar{H}$ is cyclic. Since $\bar{G}' \leq \bar{H}'$, then $\bar{G}'$ is cyclic. Thus, $\bar{G}'$ is a cyclic elementary abelian group and hence $|\bar{G}'| = 2$. Therefore, $Z(\bar{G}) = \bar{G}'$ and $\bar{G}' = Z(\chi)$, where $Z(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\}$. So,

$$G' \leq \bigcap_{\chi \in \text{Irr}(G)} Z(\chi) = Z(G),$$

which is a contradiction. Therefore, $c = 1$, $|G/G'| = 4$, and by [5, Chapter III, Theorem 11.9(a)], $G$ is dihedral, semidihedral or generalized quaternion $2$-group. So, (b) holds. \hfill \Box

Now, we are ready to prove Theorem 1.

**Proof of Theorem 1:** Since $G$ is a non-abelian nilpotent group, then $G = Q \times A$, for some non-abelian Sylow $q$-subgroup $Q$ and a subgroup $A$ of $G$. Let $A$ be non-abelian. Let $\chi_1 \in \text{Irr}(Q)$, $\chi_2 \in \text{Irr}(A)$ and $\lambda \in \text{Lin}(A)$. If $\chi_1(1) = x_1$ and $\chi_2(1) = x_2$, then $\chi_1 \lambda(1) = x_1$ and $\chi_1 \chi_2(1) = x_1 x_2$. So, $x_1, x_1 x_2 \in \text{cd}(G)$, which is a contradiction. Thus $A$ is abelian.
Now, every irreducible character of $G$ is of the form $\chi\lambda$, where $\chi \in \text{Irr}(Q), \lambda \in \text{Irr}(A)$. Since the number of non-linear characters is equal to $p$, then either $Q$ has only one non-linear character and $A$ has $p$ linear characters or else $A = 1$.

If $Q$ has only one non-linear character then by [2, Main theorem], $Q \simeq ES(c, 2)$, for some positive integer $c$. Thus, $G \simeq ES(c, 2) \times C_p$ and (a) holds.

If $A = 1$ then $G$ is a $q$-group and by by Lemma 1, $G$ satisfies (b) or (c).

Conversely, suppose $G = C_p \times ES(c, 2)$. Since $ES(c, 2)$ has only one non-linear character, then $G$ has $p$ non-linear characters of the same degree. If $G$ satisfies (c), then $G$ has a normal abelian subgroup of index 2, and thus $\text{cd}(G) = \{1, 2\}$ and $|\text{Irr}_1(G)| = p$.

If $G$ is an special 2-group such that $|G : G'| = m^2 = 2^{2c}$, then $G'$ has $2^r - 1$ distinct subgroups of index 2, where $|G'| = 2^r$ for some positive integer $r$. If $M$ is one of such subgroups, then $G/M \simeq ES(c, 2)$ has only one non-linear character. Then, for $\chi \in \text{Irr}_1(G)$ there exists only one maximal subgroup $M_\chi$ of $G'$ such that $M_\chi \subseteq \ker\chi$. Thus, $G$ has at least $2^r - 1$ non-linear characters of degree $2^r$ and $|G : G'| = 2^{2c}$. Then, from $\sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = |G|$, we have $\text{cd}(G) = \{1, 2^r\}$ and $|\text{Irr}_1(G)| = p$. \hfill \Box

**Proof of Theorem 2:** Since $G$ has only $p$ non-linear characters of degree $m$, then

$$|G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = pm^2 + |G : G'|.$$  

Since $G'$ is an abelian normal subgroup $G$, then by Ito’s Theorem, see [8, Theorem 6.15], $\chi(1)$ divides $|G : G'|$. If $|G : G'| = mp^2$, then $|G'| = 2$ and $G$ is nilpotent, which is a contradiction. So, $|G : G'|$ is one of the followings:

(i) $|G : G'| = my$, such that $y \neq p$ and $y$ divides $m$. If $m$ is not a prime, then by [3, Lemma 2] we have $G = (C_m, G')$. If $m$ is not a prime, then by Theorem C, $G' \cap Z(G) = 1$ and $G/Z(G) = (C_m, \frac{G' \times Z(G)}{Z(G)})$. So,
\[ |Z(G)| = y. \text{ If } |\text{Irr}_1(G/Z(G))| = x, \text{ then} \]
\[ |G/Z(G)| = \frac{pm^2 + my}{y} = xm^2 + \left| \frac{G}{Z(G)} : \frac{G'}{Z(G)} \right| = xm^2 + m. \]

So \( y = 1 \) and \( G = (C_m, G') \) and (a) holds.

(ii) \( |G : G'| = pm^2, \) such that \( y \) divides \( m \). We have \( G = pm^2 + pm^2. \) If \( y \neq 1 \) then \( |G'| \leq m \), and by [8, Exercise 5.14], \( G \) is nilpotent, which is a contradiction. So, \( |G'| = pm(m+1). \) Suppose \( m \) is a prime. By Theorem A, \( G \) has a normal abelian subgroup \( M \) of index \( m \). By [8, Theorem 12.12], \( |M \cap Z(G)| = p. \) If \( Z(G) \not\subseteq M \), then \( G = Z(G)M \). So, \( G \) is abelian, which is a contradiction. So, \( Z(G) \leq M \) and \( Z(G) = C_p \). Now, suppose \( m \) is not a prime. Then, by Theorem C, \( G' \cap Z(G) = 1 \) and \( G/Z(G) = \left( C_m, \frac{G'}{Z(G)} \right). \) Since \( |G'| = pm(m+1) \), then \( |Z(G)| = p \) and \( Z(G) = C_p \). If \( Z(G) \not\subseteq G' \), then \( \frac{G}{Z(G)} : \frac{(G/Z(G))'}{Z(G)} = pm \). Hence, by the above argument, \( m(m+1) = xm^2 + pm \) and \( m+1 = xm + p \), which is a contradiction. So, \( Z(G) \not\subseteq G' \) and \( |\frac{G}{Z(G)}| : (\frac{G}{Z(G)})' = m. \) Let \( x \) be the number of non-linear characters of \( G/Z(G) \). Then,
\[ |G/Z(G)| = \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(1)^2 = xm^2 + m, \]
and hence \( x = 1 \). Therefore, by [2, Main Theorem], \( G/Z(G) = (C_{q^n-1}, E(q^n)) \) is a Frobenius group with elementary abelian kernel \( E(q^n) = (G/Z(G))' \approx G' \) and cyclic complement \( C_{q^n-1} \), for some prime \( q \) and integer \( n \) such that \( m = q^n - 1 \).

Now \( G' \) is a minimal normal subgroup of \( G \). Since \( H \) is a proper subgroup of \( G' \) which is normal in \( G \), then \( |\frac{G}{H} : (G/H)'| = |G : G'| = mp. \) Hence, if \( G/H \), has \( r \) non-linear characters of degree \( m \), then \(|G/H| = rm^2 + pm). \) So, \( (rm^2 + pm) \mid mp(m+1) = |G| \) and \( (rm + p) \mid m(p-r) \) so \( (rm + p) \mid (m+1) \), which is a contradiction. Since \( G \) is a non-nilpotent group, then there exists one maximal normal subgroup \( M \) of \( G \). Now, \( G' \) is a minimal normal abelian subgroup of \( G \), and thus \( G' \cap M = 1 \), and \( G = M \times G' \). Now, \( G/Z(G) = (C_{q^n-1}, E(q^n)) \) and \( G' \approx (C_{q^n-1}, E(q^n)). \) Hence, \( G = Z(G) \times (C_{q^n-1}, E(q^n)) \) or \( G = C_{p(q^n-1)} \times E(q^n) \), a semidirect product with kernel \( G' = E(q^n) \). So, (b) holds.
(iii) $|G : G'| = m^2$. We have $|G| = m^2(p + 1)$. Let $m$ be a prime. Now $G$ has a normal abelian group $A$ of index $m$ and by [8, Lemma 12.2], $|Z(G) \cap A| = m$. If $Z(G) \neq A$, then $G$ is abelian, and so $Z(G) \leq A$ and $|Z(G)| = m$. Suppose that $|G| = m^2b$, such that $\gcd(m, b) = 1$. From $\sum_{\chi \in \text{Irr}(G/Z(G))} \chi(1)^2 = |G/Z(G)|$, we have $mb = xm^2 + |\frac{G}{Z(G)} : (\frac{G}{Z(G)})'|$, where $|\text{Irr}_1(G/Z(G))| = x$. Thus, $m^2$ divide $|\frac{G}{Z(G)} : (\frac{G}{Z(G)})'| = |\frac{G}{Z(G)}|$. 

Now, $|G : G'| = m^2$, and so $Z(G) \leq G'$. For $1 \leq i < t$, $G/Z_i$ is a non-nilpotent group and $\text{cd}(G/Z_i) = \{1, m\}$. Thus, by the above argument, $|Z(G/Z_i)| = m$. Hence, $Z_{t-1}(G)$ is a subgroup of order $m^{t-1}$ of $G$. If $G = G'Z_{t-1}$, then $G/Z_{t-1}$ is abelian, which is a contradiction. Therefore $|G' \cap Z_{t-1}| = m^{t-2}$.

Now, from $\sum_{\chi \in \text{Irr}(G/Z_{t-1})} \chi(1)^2 = |G/Z_{t-1}|$, we have $mb = xm^2 + |\frac{G}{Z_{t-1}} : (\frac{G}{Z_{t-1}})'|$, where $|\text{Irr}_1(G/Z_{t-1})| = x$. Hence, $|\frac{G}{Z_{t-1}} : (\frac{G}{Z_{t-1}})'| = m$. Thus, by [3, Lemma 2], $G/Z_{t-1} = (C_m, (G/Z_{t-1}))$.

Now, let $m$ be not a prime. Then, by Theorem C, $G/Z(G) = (C_m, \frac{G' \times Z(G)}{Z(G)})$, and $|\frac{G' \times Z(G)}{Z(G)}| = p + 1$. Then, $m \mid p$ and $m = p$ is prime, which is a contradiction.

Conversely, if $G$ satisfies (a) or (b), then $Z(G) \cap G' = 1$ and $G/Z(G) = (C_m, \frac{G' \times Z(G)}{Z(G)})$. So, by Theorem C, $\text{cd}(G) = \{1, m\}$ and $|\text{Irr}_1(G)| = p$. If $G$ satisfies (c), then $|G| = m^2b$ and $G'$ is a normal abelian subgroup of index $m^2$ of $G$. So, $G'$ contains $N$, a normal abelian subgroup of order $b$ of $G$, and thus $NZ_{t-1}$ is a normal abelian subgroup of index $m$ of $G$. So, by [8, Theorem 6.15], $\text{cd}(G) = \{1, m\}$. Now, since $|G : G'| = m^2$, then $|\text{Irr}_1(G)| = p$. This completes the proof. \qed

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References

[1] Y. Berkovich, Finite solvable groups in which only two non-linear irreducible characters have equal degree, J. Algebra 184 (1996) 584-603.
On finite groups with two irreducible character degrees


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