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CONVERGENCE THEOREMS FOR TWO FINITE FAMILIES OF UNIFORMLY L-LIPSCHITZIAN MAPPINGS

X. QIN, M. SHANG AND S. M. KANG*

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Abstract. Here, we consider a composite iterative process for two finite families of Lipschitzian mappings in a real Banach space. Our results mainly improve and extend the recent results of [S.S. Chang, Some results for asymptotically pseudo-contractive mappings and asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 129 (2001) 845-853], [Y.J. Cho, J.I. Kang, H. Zhou, Approximating common fixed points of asymptotically nonexpansive mappings, Bull. Korean Math. Soc. 42 (2005) 661-670] and some others.

1. Introduction and preliminaries

Throughout the paper, we assume that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2,\ \|f\| = \|x\|\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^*. The single-valued normalized duality mapping is denoted by j.

Let $T : C \to C$ be a mapping. Recall the following definitions.

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∗Corresponding author

(1) T is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that for any $x, y \in C$,

$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad \forall n \geq 1.$$

(2) T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for any given $x, y \in C$,

$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \quad \forall n \geq 1.$$

(3) T is said to be asymptotically pseudo-contractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for any $x, y \in C$, there exists $j(x - y) \in J(x - y)$,

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k_n\|x - y\|^2, \quad \forall n \geq 1.$$

Remark 1.1. It is easy to see that if T is an asymptotically nonexpansive mapping, then T is a uniformly L-Lipschitzian mapping, where $L = \sup_{n\geq 1} k_n$. Every asymptotically nonexpansive mapping is asymptotically pseudo-contractive, but the inverse is not true, in general.

Example 1.2 [13]. Let $E = R$ and $C = [0, 1]$ and let the mapping $T : C \to C$ be defined by

$$Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$$

for all $x \in C$. It can be proved that T is not Lipschitzian, and so it is not asymptotically nonexpansive. Since T is monotonically decreasing and $T \circ T = I$, the identity mapping, then we have

$$(T^n x - T^n y)(x - y) = \begin{cases} |x - y|^2, & \text{if } n \text{ is even,} \\ (Tx - Ty)(x - y) \leq |x - y|^2, & \text{if } n \text{ is odd.} \end{cases}$$

This implies that T is an asymptotically pseudo-contractive mapping with a constant sequence $\{1\}$.

Remark 1.3. The concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [6], while the concept of asymptotically pseudo-contractive mapping was introduced by Schu [15] in 2001.

The iterative approximation problems for asymptotically nonexpansive mappings and asymptotically pseudo-contractive mappings were studied extensively by Chang [2], Cho et al. [3,4], Chidume [5], Goebel and Kirk [6], Khan et al. [7,8], Ofoedu [11], Osilike and Aniagbosor

[12], Rhoades [13], Qin et al. [14], Schu [15] and Xu [17] in the setting of Hilbert spaces or Banach spaces. Recently, Chang [2] proved the following theorem in the framework of uniformly smooth Banach spaces.

Theorem 1.4. *Let E be a uniformly smooth Banach space, D be a nonempty bounded closed convex subset of E, $T : D \to D$ be an asymptotically pseudo-contractive mapping with a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and $F(T) \neq \emptyset$, where $F(T)$ is the set of fixed points of T in D. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be four sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\alpha_n + \gamma_n \leq 1$, $\beta_n + \delta_n \leq 1$,
(ii) $\alpha_n \to 0$, $\beta_n \to 0$, $(n \to \infty)$,
(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \gamma_n = \infty$.

Let $x_0 \in D$ be any given point and let $\{x_n\}$ and $\{y_n\}$ be the modified Ishikawa iterative sequence errors defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \gamma)x_n + \alpha_n T^n y_n + \gamma_n u_n, \\ y_n = (1 - \beta_n - \delta_n)x_n + \beta_n T^n x_n + \delta_n v_n, \quad n \geq 0. \end{cases}$$

If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x_n - x^*, j(x_n - x^*)\rangle \leq k_n \|x_n - x^*\|^2 - \phi(\|x_n - x^*\|), \quad \forall n \geq 0,$$

where $x^* \in F(T)$ is some fixed point of T in D, then $x_n \to x^*$ as $n \to \infty$.

In this paper, motivated by Chang [2], Khan et al. [7,8], Ofoedu [11] and Schu [15], we consider an Ishikawa type iterative process [9] for two families of Lipschitzian mappings instead of the assumptions that the mappings are uniformly Lipschitzian and asymptotically pseudo-contractive in a real Banach space. Our results extend and improve upon the corresponding results announced by several others.

In order to prove our main results, we need the following lemmas.

Lemma 1.5 [1]. *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

Lemma 1.6 [10]. *Let $\{\theta_n\}$ be a sequence of nonnegative real numbers and $\{\lambda_n\}$ be a real sequence satisfying the following conditions:*

$$0 \leq \lambda_n \leq 1, \quad \sum_{n=0}^{\infty} \lambda_n = \infty.$$

If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ such that

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer and $\{\sigma_n\}$ is a sequence of nonnegative numbers such that $\sigma_n = o(\lambda_n)$, then $\theta_n \to \infty$ as $n \to \infty$.

Lemma 1.7 [16]. *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1 + \lambda_n) a_n + b_n, \quad \forall n \geq n_0,$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=0}^{\infty} \lambda_n < \infty$. If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

2. **Main results**

Theorem 2.1. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N : C \to C$ be a finite family of uniformly L_i-Lipschitzian mappings and $\{S_i\}_{i=1}^N : C \to C$ be a finite family of uniformly K_i-Lipschitzian mappings such that $F = \cap_{i=1}^N F(T_i) \bigcap \cap_{i=1}^N F(S_i) \neq \emptyset$, where $F(T_i)$ is the set of fixed points of T_i in C and $F(S_i)$ is the set of fixed points of S_i in C. Let x^* be a point in F. Let $\{k_n\} \subset [1, \infty)$ be a sequence with $k_n \to 1$ as $n \to \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$,
(iii) $\sum_{n=0}^{\infty} \alpha_n \beta_n < \infty$,
(iv) $\sum_{n=1}^{\infty} \alpha_n (k_n - 1) < \infty$.

For any $x_0 \in C$, let $\{x_n\}$ be the iterative sequence defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n) x_n + \alpha_n S_{r_n}^n y_n, \\ y_n = (1 - \beta_n) x_n + \beta_n T_{r_n}^n x_n, \quad n \geq 0, \end{cases} \tag{2.1}$$

where $r_n = n \mod N$. *If there exists a strictly increasing function* $\phi : [0, \infty) \to \infty[0, \infty)$ *with* $\phi(0) = 0$ *such that*

$$\langle T_i^n x - x^*, j(x - x^*)\rangle \le k_n \|x - x^*\|^2 - \phi(\|x - x^*\|)$$

and

$$\langle S_i^n x - x^*, j(x - x^*)\rangle \le k_n \|x - x^*\|^2 - \phi(\|x - x^*\|)$$

for all $j(x - x^*) \in J(x - x^*)$ *and* $x \in C$, $i = 1, 2, \ldots, N$, *then* $\{x_n\}$ *converges strongly to* x^*.

Proof. Put $M = \max\{L_i,\ K_i\}$ for all $1 \le i \le r$. First, we prove that the sequence $\{x_n\}$ defined by (2.1) is bounded. Indeed, it follows from (2.1) and Lemma 1.5 that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(S_{r_n}^n y_n - x^*)\|^2 \\
&\le (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle S_{r_n}^n y_n - x^*, j(x_{n+1} - x^*)\rangle \\
&\le (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle S_n^n x_{n+1} - x^*, j(x_{n+1} - x^*)\rangle \\
&\quad + 2\alpha_n \langle S_{r_n}^n y_n - S_{r_n}^n x_{n+1}, j(x_{n+1} - x^*)\rangle \\
&\le (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \{k_n \|x_{n+1} - x^*\|^2 - \phi(\|x_{n+1} - x^*\|)\} \\
&\quad + 2\alpha_n M \|y_n - x_{n+1}\| \|x_{n+1} - x^*\|.
\end{aligned} \tag{2.2}$$

On the other hand, we have,

$$\begin{aligned}
&\|x_{n+1} - y_n\| \\
&= \|(1 - \alpha_n)(x_n - y_n) + \alpha_n(S_{r_n}^n y_n - y_n)\| \\
&\le (1 - \alpha_n)\|x_n - y_n\| + \alpha_n \|S_{r_n}^n y_n - x^* + x^* - y_n\| \\
&\le (1 - \alpha_n)\|x_n - y_n\| + \alpha_n(1 + M)\|y_n - x^*\| \\
&\le (1 - \alpha_n)\|x_n - y_n\| + \alpha_n(1 + M)(\|y_n - x_n\| + \|x_n - x^*\|) \\
&= (1 + M\alpha_n)\|x_n - y_n\| + \alpha_n(1 + M)\|x_n - x^*\| \\
&= (1 + M\alpha_n)\beta_n \|x_n - T_{r_n}^n x_n\| + \alpha_n(1 + M)\|x_n - x^*\| \\
&\le (1 + M\alpha_n)\beta_n(1 + M)\|x_n - x^*\| + \alpha_n(1 + M)\|x_n - x^*\| \\
&= t_n ||x_n - x^*||,
\end{aligned} \tag{2.3}$$

where,

$$t_n = (1 + M)\{(1 + M\alpha_n)\beta_n + \alpha_n\}.$$

By the conditions (ii) and (iii), we know that

$$\sum_{n=0}^{\infty} \alpha_n t_n < \infty. \tag{2.4}$$

Substituting (2.3) into (2.2), we have,

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\{k_n\|x_{n+1} - x^*\|^2 - \phi(\|\|x_{n+1} - x^*\|)\} \\ &\quad + 2\alpha_n M t_n\|x_n - x^*\|\|x_{n+1} - x^*\| \\ &\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\{k_n\|x_{n+1} - x^*\|^2 - \phi(\|x_{n+1} - x^*\|)\} \\ &\quad + \alpha_n M t_n\{\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\}. \end{aligned} \tag{2.5}$$

It follows that

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\leq \frac{1 - 2\alpha_n + \alpha_n^2 + \alpha_n M t_n}{1 - (2\alpha_n k_n + \alpha_n M t_n)}\|x_n - x^*\|^2 - \frac{2\alpha_n \phi(\|x_{n+1} - x^*\|)}{1 - (2\alpha_n k_n + \alpha_n M t_n)} \\ &= \{1 + \frac{2\alpha_n(k_n - 1) + 2Mt_n\alpha_n + \alpha_n^2}{1 - (2\alpha_n k_n + \alpha_n M t_n)}\}\|x_n - x^*\|^2 \\ &\quad - \frac{2\alpha_n \phi(\|x_{n+1} - x^*\|)}{1 - (2\alpha_n k_n + \alpha_n M t_n)}. \end{aligned} \tag{2.6}$$

Since $\alpha_n \to 0$ as $n \to \infty$, then there exists a positive integer n_0 such that

$$\frac{1}{2} < 1 - (2\alpha_n k_n + \alpha_n M t_n) \leq 1$$

for all $n \geq n_0$. From (2.6), we have,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \leq \{1 + 2[2\alpha_n(k_n - 1) + 2Mt_n\alpha_n + \alpha_n^2]\}\|x_n - x^*\|^2 \\ - 2\alpha_n \phi(\|x_{n+1} - x^*\|) \end{aligned} \tag{2.7}$$

for all $n \geq n_0$. Since $\phi(x) \geq 0$ for all $x \geq 0$, then for all $n \geq n_0$, we obtain:

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \leq \{1 + 2[2\alpha_n(k_n - 1) \\ + 2Mt_n\alpha_n + \alpha_n^2]\}\|x_n - x^*\|^2. \end{aligned} \tag{2.8}$$

By the condition (ii), (iii) and (2.4), we have

$$2\sum_{n=0}^{\infty}[2\alpha_n(k_n - 1) + 2Mt_n\alpha_n + \alpha_n^2] < \infty.$$

It follows from Lemma 1.7 that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Therefore, the sequence $\{\|x_n - x^*\|\}$ is bounded. Without loss of generality, we can assume that $\|x_n - x^*\|^2 \le M'$, where M' is an appropriate positive constant. Considering (2.7) again, we obtain,

$$\|x_{n+1}-x^*\|^2 \le \|x_n-x^*\|^2-2\alpha_n\phi(\|x_{n+1}-x^*\|)+2[2\alpha_n(k_n-1)+2Mt_n\alpha_n+\alpha_n^2]M'$$

for all $n \ge n_0$. By the conditions (i)-(iii), we know that all the conditions in Lemma 1.6 are satisfied. Therefore, we obtain $\lim_{n\to\infty}\|x_n - x^*\| = 0$. This completes the proof. □

Remark 2.2. We note that Theorem 2.1 carries over trivially to the iterative formula (2.1) with errors. One can repeat the argument of this paper under suitable conditions.

Having the proof of Theorem 2.1, we can obtain the following results immediately.

Corollary 2.3. *Let E be a real Banach space and C a nonempty closed convex subset of E. Let $T, S : C \to C$ be two uniformly L_i-Lipschitzian mappings such that $F = F(T)\bigcap F(S) \neq \emptyset$, where $F(T)$ is the set of fixed points of T in C and $F(S)$ is the set of fixed points of S in C, respectively. Let x^* be a point in F. Let $\{k_n\} \subset [1,\infty)$ be a sequence with $k_n \to 1$ as $n \to \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$,
(iii) $\sum_{n=0}^{\infty} \alpha_n\beta_n < \infty$,
(iv) $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.

For any $x_0 \in C$, let $\{x_n\}$ be the iterative sequence defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \qquad n \ge 0. \end{cases}$$

If there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \le k_n\|x - x^*\|^2 - \phi(\|x - x^*\|)$$

and

$$\langle S^n x - x^*, j(x - x^*)\rangle \le k_n\|x - x^*\|^2 - \phi(\|x - x^*\|)$$

for all $j(x - x^*) \in J(x - x^*)$ *and* $x \in C$*, then* $\{x_n\}$ *converges strongly to* x^**.*

Corollary 2.4. *Let* E *be a real Banach space and* C *a nonempty closed convex subset of* E*. Let* $S : C \to C$ *be a uniformly L-Lipschitzian mapping such that* $F(S) \neq \emptyset$*, where* $F(S)$ *is the set of fixed points of* T *in* C*. Let* x^* *be a point in* $F(S)$*. Let* $\{k_n\} \subset [1, \infty)$ *be a sequence with* $k_n \to 1$ *as* $n \to \infty$*. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences in* $[0, 1]$ *satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$,
(iii) $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.

For any $x_0 \in C$*, let* $\{x_n\}$ *be the iterative sequence defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^n x_n, \qquad n \geq 0.$$

If there exists a strictly increasing function $\phi : [0,\ \infty) \to [0,\ \infty)$ *with* $\phi(0) = 0$ *such that*

$$\langle S^n x - x^*, j(x - x^*)\rangle \leq k_n\|x - x^*\|^2 - \phi(\|x - x^*\|)$$

for all $j(x - x^*) \in J(x - x^*)$ *and* $x \in C$*, then* $\{x_n\}$ *converges strongly to* x^**.*

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Xiaolong Qin
Department of Mathematics and the RINS, Gyeongsang National University, Jinju, 660-701, Korea.
Email: `qxlxajh@163.com`
Email: `ljjhqxl@yahoo.com.cn`

Meijuan Shang
Department of Mathematics, Shijiazhuang University, Shijiazhuang, 050035, China.
Email: `meijuanshang@yahoo.com.cn`

Shin Min Kang
Department of Mathematics and the RINS, Gyeongsang National University, Jinju, 660-701, Korea.
Email: `smkang@nongae.gsnu.ac.kr`