Bulletin of the Iranian Mathematical Society Vol. 34 *No.* 2 (*2008*), *pp 121-139.*

ON SOME PROPERTIES OF ANALYTIC SPACES CONNECTED WITH BERGMAN METRIC BALL

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Communicated by Fereidoun Ghahramani

Abstract. We obtain some new sharp results for some new analytic functional spaces defined with the help of Bergman metric ball.

1. Introduction and notations

Let B denote the unit ball of $\mathbb{C}^n$. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in $\mathbb{C}^n$. We write,

$$\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n,\ |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

Thus, $B = \{z \in \mathbb{C}^n : |z| < 1\}$. Let S be the unit sphere of $\mathbb{C}^n$. Let dv be the normalized Lebesgue measure on B and $d\sigma$ be the normalized rotation invariant Lebesgue measure on S. We denote by $H(B)$ the class of all holomorphic functions on B. Let $r > 0$ and $z \in B$. The Bergman metric ball at z is defined as:

$$D(z, r) = \Big\{w \in B : \beta(z, w) = \frac{1}{2}\log\frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|} < r\Big\}.$$

Here, the involutions φ_z has the form,

$$\varphi_z(w) = \frac{z - P_z w - s_z Q_z w}{1 - \langle w, z\rangle},$$

MSC(2000): Primary 32A10; Secondary 32A37
Keywords: Carleson measure, weighted Bergman space, Bergman metric
Received: 10 September 2007, Accepted: 5 April 2008
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where $s_z = (1-|z|^2)^{1/2}$, $P_z w = \frac{\langle w,z\rangle z}{|z|^2}$, $P_0 w = 0$ and $Q_z = I - P_z$ (see, e.g., [15]).

Let $0 < p < \infty$ and $\alpha > -1$. Recall that the weighted Bergman space A^p_α consists of those functions $f \in H(B)$ for which

$$\|f\|^p_{A^p_\alpha} = \int_B |f(z)|^p dv_\alpha(z) = C_\alpha \int_B |f(z)|^p (1-|z|^2)^\alpha dv(z) < \infty,$$

where $C_\alpha = \Gamma(n+\alpha+1)/(n!\Gamma(\alpha+1))$.

During the past decade, the theory of Bergman spaces has developed in a variety of directions. About the Bergman spaces theory in the unit disk and the unit ball, we refer the reader to [10, 15].

One of the goals of this paper is to extend some results of weighted Bergman spaces in the unit ball (see [15]) to the case of more general $A(p,q,\alpha)$ class.

Famous Muckenhoupt weights are well known in the study of problems connected with the boundedness of various operators (for example, Maximal operator, Hilbert operator, etc.), acting in or from one weighted space to another one (see, e.g., [8]). Nevertheless such type of weights in higher dimensions (unit ball and polydisk) are less known. With the help of Bergman metric ball, we introduce new analogues of Muckenhoupt weights in the ball and prove two estimates for them generalizing previously known inequalities. Some results of the main section of this note can be transferred without big difficulties to the so called mixed norm spaces and holomorphic Triebel-Lizorkin spaces in the unit ball; see [13, 14]. For the simplicity of exposition, we present complete proofs only in the linear case of weighted Bergman spaces; sketches of more general "mixed norm" case will be presented.

Throughout the paper, constants are denoted by C and $C_i, i = 1, 2, \cdots$, are positive and may not be the same at each occurrence.

2. **Preliminaries**

Here, we collect some known estimates and results connected with the Bergman metric ball. We will use them in the final section for the proof of the main theorems of this note. We also provide several new inequalities for spaces defined with the help of Bergman metric ball. The following three results can be found in [15].

Lemma 2.1. *There exists a positive integer N such that for any $0 < r \le 1$ we can find a sequence $\{a_k\}$ in B with the following properties:*
(1) $B = \cup_k D(a_k, r)$.
(2) The sets $D(a_k, r/4)$ are mutually disjoint.
(3) Each point $z \in B$ belongs to at most N of the sets $D(a_k, 2r)$.

Such a sequence will be called a sampling sequence.

Lemma 2.2. *For each $r > 0$, there exists a positive constant C_r such that*

$$C_r^{-1} \le \frac{1-|a|^2}{1-|z|^2} \le C_r, \; C_r^{-1} \le \frac{1-|a|^2}{|1-\langle z, a\rangle|} \le C_r,$$

for all a and z such that $\beta(a,z) < r$. Moreover, if r is bounded above, then we may choose C_r independent of r.

Lemma 2.3. *Suppose $r > 0$, $p > 0$ and $\alpha > -1$. Then, there exists a constant $C > 0$ such that*

$$|f(z)|^p \le \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)|^p dv_\alpha(w)$$

for all $f \in H(B)$ and $z \in B$.

Using properties of Bergman metric balls contained in Lemmas 2.1 and 2.3, we get

$$\begin{aligned} \|f\|^p_{A^p_\alpha} &= \int_B |f(w)|^p dv_\alpha(w) \asymp \sum_{k=1}^{\infty} \max_{z \in D(a_k,r)} |f(z)|^p v_\alpha(D(a_k,r)) \\ &\asymp \sum_{k=1}^{\infty} \int_{D(a_k,2r)} |f(z)|^p (1-|z|)^\alpha dv(z), \; 0 < p < \infty, \; \alpha > -1. \end{aligned} \tag{2.1}$$

Motivated by (2.1), we introduce a new space as follows.

Definition 2.4. Let μ be a positive Borel measure in B, $0 < p, q < \infty$ and $s > -1$. Fix an $r \in (0, \infty)$ and a sampling sequence $\{a_k\}_{k \in \mathbb{N}}$. The space $A(p,q,d\mu)$ is the space of all holomorphic functions f such that

$$\|f\|^q_{A(p,q,d\mu)} = \sum_{k=1}^{\infty} \Big(\int_{D(a_k,r)} |f(z)|^p d\mu(z) \Big)^{q/p} < \infty. \tag{2.2}$$

If $d\mu = (1-|z|^2)^s dv(z)$, then we will denote by $A(p,q,s)$ the space $A(p,q,d\mu)$. It is clear that $A(p,p,s) = A^p_s$.

Remark 2.5. From (2.2), we see that the definition of $A(p,p,s)$ space is independent from $\{a_k\}$ and r. But, in the general case of $A(p,q,s)$, the answer is unknown. Therefore, the quazinorm $\|f\|_{A(p,q,s)}$, in general, should be written as $\|f\|_{A(p,q,s,a_k,r)}$. For simplicity, we denote $\|f\|_{A(p,q,s,a_k,r)}$ by $\|f\|_{A(p,q,s)}$.

It is known (see, e.g., [7]) that for every δ, there exists a sampling sequence $\{a_j\}$ such that $d(a_j,a_k) > \delta/5$ if $j \neq k$ and

$$\sum_{k=1}^{\infty} \chi_{D(a_k,5\delta)}(z) \leq C. \tag{2.3}$$

Using (2.3) and the inequality,

$$\Big(\sum_{k=1}^{\infty} x_k\Big)^{p/q} \leq \sum_{k=1}^{\infty} x_k^{p/q}, \ q \geq p, \tag{2.4}$$

we have,

$$\begin{aligned} \|f\|^q_{A(p,q,s)} &\leq \sum_{k=1}^{\infty}\bigg(\int_B \chi_{D(a_k,5\delta)}(z)|f(z)|^p(1-|z|)^s dv(z)\bigg)^{q/p} \\ &\leq C\Big(\int_B |f(z)|^p(1-|z|^2)^s dv(z)\Big)^{q/p} \\ &= \|f\|^q_{A^p_s}, \ q \geq p, s > -1, \end{aligned} \tag{2.5}$$

that is,

$$\|f\|_{A(p,q,s)} \leq C\|f\|_{A^p_s}, \ q \geq p, s > -1. \tag{2.6}$$

Motivated by (2.6), we pose the following very natural and more general problem.

Problem I. Let μ be a positive Borel measure and $\{a_k\}_{k\in\mathbb{N}}$ be a sampling sequence. Let X be a quazinormed subspace of $H(B)$ and $0 < r,p,q < \infty$. Describe all positive Borel measures such that

$$\|f\|_{A(p,q,d\mu)} \leq C\|f\|_{\mathbb{X}}. \tag{2.7}$$

Proposition 2.6. *Let* $\{a_k\}_{k\in\mathbb{N}}$ *be a sampling sequence and* μ *be a positive Borel measure on* B, $0 < q < p < \infty$. *Then, for any* f *belonging to Lorentz space* $L^{p,\infty}(B, d\mu)$, *we have,*

$$\Big(\int_{D(a_k,r)} |f(z)|^q d\mu\Big)^{1/q} \le C\Big(\mu(D(a_k,r))\Big)^{(1/q-1/p)} \|f\|_{L^{p,\infty}}. \quad (2.8)$$

Proof. If $f \in L^{p,\infty}(B, d\mu)$, $0 < q < p < \infty$, then from [9] we see that

$$\int_{D(a_k,r)} |f|^q d\mu \;\; \le \;\; C\Big(\frac{p}{p-q}\Big)\mu(D(a_k,r))^{1-q/p}\|f\|^q_{L^{p,\infty}}.$$

Thus, the result follows immediately.

Remark 2.7. Let $f \in H(B)$, $0 < q < p < \infty$, and

$$c_k = \Big(\mu(D(a_k,r))\Big)^{(1/q-1/p)\tilde{p}}.$$

As a consequence of Proposition 2.6, we have,

$$\sum_{k=1}^{\infty}\Big(\int_{D(a_k,r)} |f(z)|^q d\mu\Big)^{\tilde{p}/q} \le C\sum_{k=1}^{\infty} c_k\Big(\int_{D(a_k,r)} |f(z)|^p d\mu(z)\Big)^{\tilde{p}/p}. \quad (2.9)$$

Problem II. Describe all $\{c_k\}_{k\in\mathbb{N}}$ sequences such that (2.9) holds.

Definition 2.8. A positive locally integrable function $V(z)$ on B is said to belong to the $MH(p)$ class if

$$\sup_{D(z,r)}\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} V(w)dv(w)\Big)\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} V^{\frac{-q}{p}}(w)dv(w)\Big)^{\frac{p}{q}} < \infty \quad (2.10)$$

for any Bergman metric ball $D(z,r)$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

The $MH(p)$ class that we defined can be considered as natural analogues of the so called Muckenhoupt class A_p, $p > 1$, that was introduced in [11] (see also [2, 3, 8]). A_p is defined as the class of locally integral nonnegative function w satisfying

$$\sup_I \Big(\frac{1}{|I|}\int_I w dx\Big)\Big(\frac{1}{|I|}\int_I (\frac{1}{w})^{\frac{1}{p-1}} dx\Big)^{p-1} < \infty.$$

Fix two real parameters $a, r > 0$ and $b > -1$. Let f be a locally integral function and $\{a_k\}_{k\in\mathbb{N}}$ be a sampling sequence in B. Consider the integral operator defined by

$$S^{a,b}_{a_k,r} f(z) = (1-|z|^2)^a \int_{D(a_k,r)} \frac{(1-|w|^2)^b f(w) dv(w)}{|1-\langle z,w\rangle|^{n+1+a+b}}, \ z \in B. \quad (2.11)$$

The following assertion, a consequence of Schur's test, shows that the $S^{a,b}_{a_k,r}$ operators are invariant in $A(p,s,t)$ spaces. Note that for $p = s$, the result easily follows from Theorem 2.10 of [15].

Proposition 2.9. *Let $\{a_k\}_{k\in\mathbb{N}}$ be a sampling sequence in B, $0 < s < \infty$, $r > 0$, $1 \le p < \infty$, $t \in (-1,\infty)$ and $-pa < t+1 < p(b+1)$. Then,*

$$\sum_{k=1}^{\infty}\Big(\int_{D(a_k,r)} |S^{a,b}_{a_k,r} f(z)|^p dv_t(z)\Big)^{s/p} \le C \sum_{k=1}^{\infty}\Big(\int_{D(a_k,r)} |f(z)|^p dv_t(z)\Big)^{s/p}.$$

Proof. If $p = 1$, then the result follows from Fubini's theorem. Now, we consider the case of $1 < p < \infty$. Let $1/p + 1/q = 1$. Fix

$$\tilde{s} \in (-\frac{b+1}{p}, \frac{a}{q}) \bigcap (-\frac{a+1+t}{p}, \frac{b-t}{p}).$$

Let $h(z) = (1-|z|^2)^{\tilde{s}}$, $z \in B$. Then, by Lemma 2.2 we have,

$$\int_{D(a_k,r)} \frac{(1-|w|^2)^{\tilde{s}q}(1-|z|^2)^a(1-|w|^2)^b dv(w)}{|1-\langle z,w\rangle|^{n+1+a+b}} \le C(1-|z|^2)^{\tilde{s}q},$$

and

$$\int_{D(a_k,r)} \frac{(1-|z|^2)^{\tilde{s}p}(1-|z|^2)^a(1-|w|^2)^b dv(z)}{|1-\langle z,w\rangle|^{n+1+a+b}} \le C(1-|w|^2)^{\tilde{s}p}.$$

Using Theorem 2.9 of [15], we obtain:

$$\int_{D(a_k,r)} |S^{a,b}_{a_k,r} f(z)|^p dv_t(z) \le C \int_{D(a_k,r)} |f(z)|^p dv_t(z).$$

Then, the result follows.

3. Main results and proofs

Here, we give partial solutions of problems I and II. Various sharp embedding theorems for different holomorphic spaces on the unit ball are well known (see, e.g., [4, 13, 15]). The following sharp result was proved in [6].

Theorem A. *Let μ be a positive Borel measure on B and $\{a_k\}_{k\in\mathbb{N}}$ be a sampling sequence. If $q \geq p$ and $\alpha > -1$. We have,*

$$\left(\int_B |f|^q d\mu\right)^{1/q} \leq C\|f\|_{A^p_\alpha}$$

if and only if

$$\mu(D(a_k, r)) \leq C(1 - |a_k|)^{(\frac{n+1+\alpha}{p})q}.$$

Here, we also provide a new sharp embedding with the help of the Bergman metric ball. The following assertion concerns Problem I.

Theorem 3.1. *Let $0 < q, p < \infty$, $0 < s \leq p < \infty$ and $\beta > -1$. Let $\{a_k\}_{k\in\mathbb{N}}$ be a sampling sequence in B and μ be a positive Borel measure on B. We have,*

$$\|f\|_{A(q,p,d\mu)} \leq C\|f\|_{A^s_\beta} \tag{3.1}$$

if and only if

$$\mu(D(a_k, r)) \leq C(1 - |a_k|^2)^{\frac{q(n+1+\beta)}{s}}. \tag{3.2}$$

Proof. Suppose that (3.2) holds. Using Lemmas 2.1 and 2.3, we have,

$$\begin{aligned}
&\Big(\sum_{k=1}^{\infty}\Big(\int_{D(a_k,r)}|f(z)|^q d\mu(z)\Big)^{p/q}\Big)^{s/p}\\
\leq\ & C\Big(\sum_{k=1}^{\infty}\max_{z\in D(a_k,r)}|f(z)|^p(1-|a_k|)^{\frac{p(n+1+\beta)}{s}}\Big)^{s/p}\\
\leq\ & C\sum_{k=1}^{\infty}\max_{z\in D(a_k,r)}|f(z)|^s(1-|a_k|)^{n+1+\beta}\\
\leq\ & C\int_B|f(z)|^s(1-|z|)^{\beta}dv(z)\leq C\|f\|^s_{A^s_\beta},\ \beta>-1,\ 0<s<\infty.
\end{aligned}$$

The result follows immediately from the above inequality.

Conversely, suppose that (3.1) holds. Set the family of test functions,

$$f_k(z)=\left(\frac{(1-|a_k|^2)^{n+\beta+1}}{(1-\langle z,a_k\rangle)^{2(n+\beta+1)}}\right)^{1/s},\ z\in B,\ k=1,2,\cdots. \tag{3.3}$$

It is easy to check, using Theorem 1.12 of [15], that $\sup_k\|f_k\|_{A^s_\beta}\leq C$.

On the other hand, the following estimates are obvious:

$$\begin{aligned}
\Big(\int_{D(a_k,r)}|f(z)|^q d\mu(z)\Big)^{1/q} &\leq C\Big(\sum_{k=1}^{\infty}\Big(\int_{D(a_k,r)}|f(z)|^q d\mu(z)\Big)^{p/q}\Big)^{1/p}\\
&\leq C\|f\|_{A^s_\beta}.
\end{aligned}\tag{3.4}$$

Substituting the test functions f_k into (3.4) we get the desire result. Thus, the proof of the theorem is complete.

Remark 3.2. It is obvious that Theorem 3.1 is a generalization of Theorem A.

Let $0<p,q<\infty$, $\alpha>-1$ and $f\in H(B)$. Recall that $f\in F^{p,q}_\alpha$, called the holomorphic Triebel-Lizorkin spaces, if

$$\|f\|^p_{F^{p,q}_\alpha}=\int_S\left(\int_0^1|f(r\xi)|^q(1-r)^{\alpha}dr\right)^{p/q}d\sigma(\xi)<\infty.$$

Define for the same values of parameters the holomorphic Besov type spaces (see [13]),

$$B^{p,q}_\alpha=\{f\in H(B):\|f\|^q_{B^{p,q}_\alpha}=\int_0^1 M^q_p(f,r)(1-r)^{\alpha}dr<\infty\},$$

where,

$$M_p^p(f,r) = \int_S |f(r\xi)|^p d\sigma(\xi),\ 0 < p < \infty,\ r \in (0,1).$$

The result of Theorem 3.1 can be extended to the case of mixed norm space and the so called holomorphic Triebel-Lizorkin space. Thus, the following result holds. We have,

$$\|f\|_{A(q,p,d\mu)} \leq C\|f\|_{F^{s,q}_{\frac{\beta+1}{s}q-1}}$$

if and only if

$$\|f\|_{A(q,p,d\mu)} \leq C\|f\|_{B^{s,q}_{\frac{\beta+1}{s}q-1}}$$

or equivalently, if and only if

$$\mu(D(a_k,r)) \leq C(1-|a_k|^2)^{\frac{q(n+1+\beta)}{s}}.$$

Here, $0 < \max\{s,q\} \leq p < \infty, \beta > -1$.

The proof for the necessity of the above mentioned statement make use of the same type of standard test function as used in Theorem 3.1. The proof of the sufficiency is based on the following embeddings (see [14]):

$$\int_S \int_0^1 |f(r\xi)|^s (1-r)^\beta dr d\sigma(\xi)$$

$$\leq C \int_S \left(\int_0^1 |f(r\xi)|^q (1-r)^{\frac{\beta+1}{s}q-1} dr \right)^{\frac{s}{q}} d\sigma(\xi),$$

and

$$\int_S \int_0^1 |f(r\xi)|^s (1-r)^\beta dr d\sigma(\xi)$$

$$\leq C \left(\int_0^1 \left(\int_S |f(r\xi)|^s d\sigma(\xi) \right)^{\frac{q}{s}} (1-r)^{\frac{\beta+1}{s}q-1} dr \right)^{s/q},$$

for $0 < q \leq s < \infty$, $\beta > -1$ and $f \in H(B)$.

In the following theorem, we will give a complete characterization of the "weighted" $A(p,q,s)$ spaces, when $q/p > 1$.

Theorem 3.3 *Let $0 < p < q < \infty$ and $\alpha > 0$. Let $\{a_k\}_{k\in\mathbb{N}}$ be a sampling sequence in B and μ be a positive Borel measure on B. Then, the following two statements are equivalent:*

(a)

$$\int_B \Big(\int_B \Big(\frac{(1-|z|)^n}{|1-\langle\lambda, z\rangle|^{2n}} \Big)^{1+\alpha q} d\mu(z) \Big)^{\frac{q}{q-p}} dv_{\alpha q n-1}(\lambda) < \infty. \tag{3.5}$$

(b)

$$\sum_{k=1}^{\infty} (1-|a_k|)^{-(\alpha q n+n)\frac{p}{q-p}} \Big(\mu(D(a_k, r)) \Big)^{\frac{q}{q-p}} < \infty. \tag{3.6}$$

Proof. $(a) \Rightarrow (b)$. Using Lemma 2.2, we have the following chain of estimates,

$$\begin{aligned}
&\int_B \Big(\int_B \Big(\frac{(1-|z|)^n}{|1-\langle\lambda, z\rangle|^{2n}} \Big)^{1+\alpha q} d\mu(z) \Big)^{\frac{q}{q-p}} dv_{\alpha q n-1}(\lambda) \\
&\geq C \sum_{k=1}^{\infty} \int_{D(a_k,r)} \Big(\sum_{k=1}^{\infty} \int_{D(a_k,r)} \Big(\frac{(1-|z|)^n}{|1-\langle\lambda, z\rangle|^{2n}} \Big)^{1+\alpha q} d\mu(z) \Big)^{\frac{q}{q-p}} dv_{\alpha q n-1}(\lambda) \\
&\geq C \sum_{k=1}^{\infty} (1-|a_k|)^{-(\alpha q n+n)\frac{p}{q-p}} \Big(\mu(D(a_k, r)) \Big)^{\frac{q}{q-p}}.
\end{aligned}$$

$(b) \Rightarrow (a)$. Using Lemma 2.3,

$$|f(z)| = \Big(|f(z)|^{q/p} \Big)^{p/q} \leq C \Big(\frac{1}{(1-|a_k|)^{n+1}} \int_{D(a_k,r)} |f|^{q/p} dv \Big)^{p/q}.$$

Using the above inequality, Lemmas 2.1 and 2.2, we have,

$$\begin{aligned}
&\int_B |f| d\mu = \sum_{k=1}^{\infty} \int_{D(a_k,r)} |f| d\mu \leq \sum_{k=1}^{\infty} \max_{z\in D(a_k,r)} |f(z)| \mu(D(a_k, r)) \\
\leq\ & C \sum_{k=1}^{\infty} \Big(\int_{D(a_k,2r)} |f(z)|^{q/p} dv_{\alpha q n-1}(z) \Big)^{p/q} (1-|a_k|)^{\frac{-p(n+q n\alpha)}{q}} \mu(D(a_k, r)).
\end{aligned}$$

Using Hölder's inequality, we get finally for any subharmonic function f,

$$\int_B |f|d\mu \leq C\|f\|_{A^{q/p}_{\alpha qn-1}} \left[\sum_{k=1}^{\infty} \Big((1-|a_k|)^{\frac{-(n+qn\alpha)p}{q}} \mu(D(a_k,r))\Big)^{\frac{q}{q-p}}\right]^{\frac{q-p}{q}}. \tag{3.7}$$

Let us consider the following function,

$$f(z) = (Sg)(z) = \int_B \left(\frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}}\right)^{1+t} g(w)(1-|w|)^{tn-1}dv(w). \tag{3.8}$$

Then, f is subharmonic. From Theorem 2.10 of [15], we see that

$$\|Sg\|_{L^{q/p}(B,dv_{tn-1})} \leq C\|g\|_{L^{q/p}(B,dv_{tn-1})},\ q>p. \tag{3.9}$$

So substituting this fixed function into (3.7), we get

$$\begin{aligned} &\int_B\int_B \left(\frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}}\right)^{1+\alpha q} g(w)(1-|w|)^{\alpha qn-1}dv(w)d\mu(z) \\ \leq\ & C\|g\|_{L^{q/p}_{\alpha qn-1}} \left(\sum_{k=1}^{\infty} \Big((1-|a_k|)^{\frac{-(n+qn\alpha)p}{q}} \mu(D(a_k,r))\Big)^{\frac{q}{q-p}}\right)^{\frac{q-p}{q}}. \end{aligned}$$

Now, using duality argument we get the desired result. The proof of the theorem is complete.

Remark 3.4. The condition (3.5) first appeared in [5]. The approach we used in the proof of Theorem 3.3 is based on theorems on projections in weighted Bergman spaces, which can be used for estimates of more general integrals such as

$$\int_S \left\{\int_0^1 \Big(\int_B \Big(\frac{(1-|z|^2)^n}{|1-\langle \lambda,z\rangle|^{2n}}\Big)^{1+\alpha q} d\mu(z)\Big)^{q/(q-p)} (1-|\lambda|)^{\alpha qn-1} d|\lambda|\right\}^{p_1} d\sigma,$$

and

$$\int_0^1 \left\{\int_S \Big(\int_B \Big(\frac{(1-|z|^2)^n}{|1-\langle \lambda,z\rangle|^{2n}}\Big)^{1+\alpha q} d\mu(z)\Big)^{q/(q-p)} d\sigma\right\}^{p_1} (1-|\lambda|)^{\alpha qn-1} d|\lambda|.$$

We should use projection theorems for mixed norm and holomorphic Triebel-Lizorkin spaces from [1, 13], instead of projection theorems for classical weighted Bergman spaces. Here, $\alpha > 0, q > p$ and $p_1 > 1$.

Remark 3.5. Let us consider the particular case in Theorem 3.2 when $d\mu = |f(z)|^{q-p}dv_\alpha(z)$. It is easy to see that for the limit case

$p = 0$, the assertion of Theorem 3.3 with this particular value of $d\mu$ is a simple fact concerning Bergman $A(q, q, \alpha)$ classes. Hence, our result can be considered as an extension of an already known result on Bergman $A(q, q, \alpha)$ spaces to the case of $A(q - p, q, s)$ classes.

Theorem 3.6. *Let $0 < q, s, r < \infty$, $q \geq s$ and $\alpha > -1$. Let $\{a_k\}_{k\in\mathbb{N}}$ be a sampling sequence in B and μ a positive Borel measure on B. We have,*

$$\int_B |f(z)|^q d\mu(z) \leq C \int_B \Big(\int_{D(w,r)} |f(z)|^s dv_\alpha(z) \Big)^{q/s} dv(w) \qquad (3.10)$$

if and only if

$$\mu(D(a_k, r)) \leq C(1 - |a_k|^2)^{q(\frac{n+1+\alpha}{s} + \frac{n+1}{q})},\ k \in \mathbb{N}, \qquad (3.11)$$

for some constant $C > 0$.

Proof. Suppose that (3.11) holds. By Lemma 2.1, we have,

$$\begin{aligned} \int_B |f(z)|^q d\mu(z) &\leq C \sum_{k=1}^{\infty} \max_{z\in D(a_k,r)} |f(z)|^q \mu(D(a_k, r)) \\ &\leq C \sum_{k=1}^{\infty} \Big(\max_{z\in D(a_k,r)} |f(z)|^s \Big)^{\frac{q}{s}} (1 - |a_k|^2)^{q(\frac{n+1+\alpha}{s} + \frac{n+1}{q})}. \end{aligned}$$

Using Lemma 2.3,

$$\int_{D(a_k,2r)} |f(z)|^s dv(z) \leq C \int_{D(a_k,2r)} \Big(\int_{D(z,r)} |f(\widetilde{w})|^s dv_\alpha(\widetilde{w}) \Big) \frac{dv(z)}{(1-|z|)^{n+1+\alpha}}.$$

Therefore, we have the following chain of estimates:

$$\begin{aligned} &\int_B |f(z)|^q d\mu(z) \\ &\leq C \sum_{k=1}^{\infty} \Big(\int_{D(a_k,2r)} |f(z)|^s dv(z) \frac{1}{(1-|a_k|)^{n+1}} \Big)^{q/s} (1 - |a_k|^2)^{q(\frac{n+1+\alpha}{s} + \frac{n+1}{q})}. \\ &\leq C \sum_{k=1}^{\infty} \Big(\int_{D(a_k,2r)} |f(z)|^s dv(z) \Big)^{q/s} (1 - |a_k|^2)^{n+1+q\alpha/s} \\ &\leq C \sum_{k=1}^{\infty} \Big(\int_{D(a_k,2r)} \int_{D(z,r)} |f(\widetilde{w})|^s dv_\alpha(\widetilde{w}) \frac{dv(z)}{(1-|z|)^{n+1}} \Big)^{q/s} (1 - |a_k|^2)^{n+1}. \end{aligned} \qquad (3.12)$$

Using Hölder inequality,

$$\begin{aligned}&\Big(\int_{D(a_k,2r)}\int_{D(z,r)}|f(\widetilde{w})|^s dv_\alpha(\widetilde{w})\frac{dv(z)}{(1-|z|)^{n+1}}\Big)^{q/s}\\ \leq\ &\int_{D(a_k,2r)}\Big(\int_{D(z,r)}|f(\widetilde{w})|^s dv_\alpha(\widetilde{w})\Big)^{q/s}(1-|a_k|^2)^{-(n+1)}. \quad (3.13)\end{aligned}$$

Combining (3.12) with (3.13), we get the desired result.

Conversely, suppose that (3.10) holds. For any β which is big enough, define,

$$f_k(z)=\frac{(1-|a_k|)^{\beta-\frac{n+1+\alpha}{s}-\frac{n+1}{q}}}{(1-\langle z,a_k\rangle)^{\beta}},\ \ a_k,z\in B,k=1,2,\cdots.$$

Then,

$$\begin{aligned}&\int_B\Big(\int_{D(w,r)}|f_k(z)|^s dv_\alpha(z)\Big)^{q/s}dv(w)\\ \leq\ \ &C(1-|a_k|)^{\beta q-q\frac{n+1+\alpha}{s}-n+1}\times\int_B\frac{dv(w)}{|1-\langle w,a_k\rangle|^{\beta q-q\frac{n+1+\alpha}{s}}}\leq C,\end{aligned}$$

and

$$\int_B|f(z)|^q d\mu(z)\geq\mu(D(a_k,r))(1-|a_k|)^{-q(\frac{n+1+\alpha}{s}+\frac{n+1}{q})}.$$

Combining the last two inequalities we get the desired result. The proof of the theorem is now complete.

Remark 3.7. For $f\in H(B)$ and $z\in B$, let

$$\nabla f(z)=\Big(\frac{\partial f}{\partial z_1}(z),\cdots,\frac{\partial f}{\partial z_n}(z)\Big)$$

denote the complex gradient of f. Let $\widetilde{\nabla}f$ denote the invariant gradient of B; i.e., $(\widetilde{\nabla}f)(z)=\nabla(f\circ\varphi_z)(0)$.

Let $1<p<\infty$. Recall that the Möbius invariant Besov space B_p consists of those holomorphic functions f for which the $\widetilde{\nabla}f$ are p-integrable functions with respect to the invariant measure $d\lambda(z)$, where $d\lambda(z)=(1-|z|^2)^{-n-1}dv(z)$.

Fix any radius $r>0$ and $f\in H(B)$, and define,

$$I_rf(z)=\int_{D(z,r)}|\widetilde{\nabla}f(w)|d\lambda(w),\ \ z\in B.$$

From [12] or [15], $f \in B_p$ if and only if $I_r f \in L^p(B, d\lambda)$. We define $\widetilde{\nabla}^{-1}(\widetilde{\nabla} f) = f$. Then, from the above result we see that $\widetilde{\nabla}^{-1} f \in B_q$ if and only if

$$\int_{D(z,r)} |f(w)| d\lambda(w) \in L^q(B, d\lambda), \ 1 < q < \infty.$$

This means that for $s = 1$ and $\alpha = 0$, estimate (3.10) in Theorem 3.6 is equivalent to:

$$\int_B |f(z)|^q d\mu(z) \leq C \|(\widetilde{\nabla}^{-1} f) \times (1 - |w|)^{(n+1)+\frac{n+1}{q}} \|_{B_q}, \ 1 < q < \infty.$$

Theorem 3.8. *Let $0 < r < \infty$ and $f \in H(B)$. Let $\{a_k\}_{k \in \mathbb{N}}$ be a sampling sequence in B. Then, the following two statements hold.*

(a) If $0 < s < \infty, \alpha > -1$, $V \in MH(p)$, $p > 1$, then

$$\sum_{k=1}^{\infty} \Big(\int_{D(a_k,r)} \Big(S^{0,\alpha}_{a_k,r} f \Big)^p V(z) dv(z) \Big)^{s/p}$$

$$\leq C \sum_{k=1}^{\infty} \Big(\int_{D(a_k,r)} |f(w)|^p V(z) dv(z) \Big)^{s/p}.$$

(b) If $V^p \in MH(p/q)$, $p > q$ and

$$\int_{D(a_k,r)} |f(z)|^p dv(z)$$

$$\times (1 - |a_k|)^{-\frac{(n+1)(p-q)}{q}} \left(\int_{D(a_k,r)} [V^{-p}(z)]^{\frac{q}{p-q}} dv(z) \right)^{\frac{q-p}{q}} \tag{3.14}$$

$$\leq \int_{D(a_k,r)} |f(z)|^p V^{-p}(z) dv(z),$$

then

$$\Big(\int_B |f(z)|^p V^p(z) dv(z) \Big)^{q/p}$$

$$\leq \sum_{k=1}^{\infty} (1 - |a_k|)^{(n+1)(p-q)(\frac{1}{p} - \frac{1}{q})} \Big(\int_{D(a_k,2r)} |f(z)|^p V^{-p}(z) dv(z) \Big)^{q/p}.$$

Proof. *(a).* Note that

$$\begin{aligned} M &= \int_{D(a_k,r)} \Big(\int_{D(a_k,r)} \frac{|f(w)|(1-|w|)^{\alpha} dv(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}} \Big)^p V(z) dv(z) \\ &\le (1-|a_k|)^{-(n+1)p} \int_{D(a_k,r)} V(z) dv(z) \Big(\int_{D(a_k,r)} |f(w)| dv(w) \Big)^p. \end{aligned}$$

Using Hölder inequality, we get,

$$\begin{aligned} &\Big(\int_{D(a_k,r)} |f(w)| dv(w) \Big)^p \\ \le\ & \int_{D(a_k,r)} |f(w)|^p V(w) dv(w) \cdot \Big(\int_{D(a_k,r)} (V(w))^{-\frac{q}{p}} dv(w) \Big)^{p/q}. \end{aligned}$$

Since $V \in MH(p)$, we obtain,

$$M \le \int_{D(a_k,r)} |f(w)|^p V(z) dv(w).$$

Thus, the result follows.

(b). By Lemmas 2.1-2.3, we have,

$$\begin{aligned} K &= \Big(\int_B |f(z)|^p V^p(z) dv(z) \Big)^{q/p} \\ &\le C \sum_{k=1}^{\infty} \Big(\int_{D(a_k,r)} V^p(z) |f(z)|^p dv(z) \Big)^{q/p} \\ &\le C \sum_{k=1}^{\infty} \Big(\int_{D(a_k,r)} V^p(z) dv(z) \sup_{z\in D(a_k,r)} |f(z)|^p \Big)^{q/p} \\ &\le C \sum_{k=1}^{\infty} \frac{1}{(1-|a_k|)^{(n+1)q/p}} \Big(\int_{D(a_k,2r)} |f(z)|^p dv(z) \Big)^{q/p} \\ &\qquad \Big(\int_{D(a_k,r)} V^p(z) dv(z) \Big)^{q/p}. \end{aligned}$$

From the conditions that $V^p \in MH(p/q)$, $p > q$ and (3.14), we have,

$$\begin{aligned} K &\leq C\sum_{k=1}^{\infty}\frac{(1-|a_k|)^{n+1}}{(1-|a_k|)^{(n+1)q/p}}\Big[\frac{1}{(1-|a_k|)^{(n+1)p/q}}\int_{D(a_k,r)}V^p(z)dv(z)\\ &\quad\times\Big(\int_{D(a_k,r)}[V^{-p}(z)]^{\frac{q}{p-q}}dv(z)\Big)^{\frac{p-q}{q}}\Big]^{q/p}\\ &\quad\times\Big(\int_{D(a_k,2r)}|f(z)|^p dv(z)\Big)^{q/p}\Big(\int_{D(a_k,r)}[V^{-p}(z)]^{\frac{q}{p-q}}dv(z)\Big)^{\frac{q-p}{p}}\\ &\leq C\sum_{k=1}^{\infty}(1-|a_k|)^{(n+1)(p-q)(\frac{1}{p}-\frac{1}{q})}\Big(\int_{D(a_k,2r)}|f(z)|^p V^{-p}(z)dv(z)\Big)^{q/p}. \end{aligned}$$

The proof of the theorem is complete.

Remark 3.9. Theorem 3.8 shows also that our introduced $S^{0,\alpha}_{a_k,r}$ operators are invariant in some sense in some Bergman type spaces with $MH(p)$ weights. Such type of a result was previously proved by Bekollé (see [2]). To be more precise, in [2], in particular, the boundedness of Bergman projections in Bergman spaces with Muckehoupt weights was obtained.

Remark 3.10. For $q = p$, $V(z) = C$, condition (3.14) vanishes and the second estimate is well known; see [15]. The first estimate in Theorem 3.8 for $V(z) = C$ is contained in Proposition 2.9.

In the following assertions, a partial solution of Problem II is presented.

Theorem 3.11. *Let $\{a_k\}_{k\in\mathbb{N}}$ be a sampling sequence in B, $t_k > 1$,$f_k \in H(B)$, $k \in \mathbb{N}$ and $\alpha > -1$. If $1 \leq p \leq q < \infty$ and $q_1 \in (0,\infty)$, then*

$$\begin{aligned} &\Big(\sum_{k=1}^{\infty}\Big(\int_{D(a_k,r)}|f_k(z)|^{q_1}dv_\alpha(z)\Big)^{\frac{q}{q_1}}\Big)^{1/q}\\ &\leq C\Big(\sum_{k=1}^{\infty}t_k\Big(\int_{D(a_k,r)}|f_k(z)|^{q_1}dv_\alpha(z)\Big)^{\frac{p}{q_1}}\Big)^{1/p} \qquad (3.15) \end{aligned}$$

if and only if

$$\sup_k t_k^{-1/p} < \infty. \tag{3.16}$$

Proof. Suppose that (3.16) holds. Let

$$y_0 = 0, y_k = \Big(\int_{D(a_k,r)} |f_k(z)|^{q_1} dv_\alpha(z)\Big)^{1/q_1},\ k = 1, 2, \cdots .$$

Since we have,

$$|y_j| \le \sup_{j\in\mathbb{N}}(t_j^{-1/p})\Big(\sum_{k=j-1}^{j+1} t_k|y_k|^p\Big)^{1/p},$$

then it follows:

$$\sum_{j=1}^{\infty}|y_j|^q \le C\Big(\sup_n t_j^{-1/p}\Big)^q \sum_{j=1}^{\infty}\Big(\sum_{k=j-1}^{j+1} t_k|y_k|^p\Big)^{q/p}$$

$$\le C\Big(\sum_{j=1}^{\infty}\Big(\sum_{k=j-1}^{j+1} t_k|y_k|^p\Big)\Big)^{q/p}.$$

Hence,

$$\Big(\sum_{j=1}^{\infty}|y_j|^q\Big)^{1/q} \le C\Big(\sum_{k=1}^{\infty} t_k|y_k|^p\Big)^{1/p}.$$

Conversely, suppose that (3.15) holds. Fix $f_k = 0, k \ne j$ and

$$f_j(z) = \frac{(1-|a_j|)^\beta}{|1-\langle z, a_j\rangle|^\gamma}, k = j, \beta = \gamma - \frac{\alpha+n+1}{q_1} > 0.$$

Substituting these into a fixed vector $f = (f_k)$ and then substituting f into (3.15), using standard properties of $\{a_k\}_{k\in\mathbb{N}}$, we get what we need. Indeed, since $y_k = 0$, for all $k \ne j$, we see that (3.15) is equivalent to:

$$\Big(\int_{D(a_j,r)} |f_j(z)|^{q_1} dv_\alpha(z)\Big)^{1/q_1} \le C t_j^{1/p}\Big(\int_{D(a_j,r)} |f_j(z)|^{q_1} dv_\alpha(z)\Big)^{1/q_1},$$

$$j = 1, 2, \cdots .$$

It remains to note that

$$C_1 \leq \left(\int_{D(a_j,r)} |f_j(z)|^{q_1} dv_\alpha(z) \right)^{1/q_1} \leq C_2,\ j = 1, 2, \cdots .$$

Remark 3.12. Let $t_k > 1$, $\alpha > -1$, $1 \leq p \leq q < \infty$ and $\sup_k t_k^{-1/p} < \infty$. Substitute in estimate (3.15) of Theorem 3.11, $f_k = f$ for all $k = 1, 2, \cdots$, $q = q_1$. Then,

$$\|f\|_{A^q_\alpha} \leq C \left[\sum_{k=1}^{\infty} t_k \Big(\int_{D(a_k,r)} |f(z)|^q dv_\alpha(z) \Big)^{p/q} \right]^{1/p}.$$

Remark 3.13. Using Theorem 3.11, various assertions can be obtained for various concrete $\{t_k\}$ for example,

$$t_k = \sup_{z \in D(a_k,r)} |f_k(z)|^q > 1 \text{ or } t_k = \int_{D(a_k,r)} |f_k(z)|^q dv_\alpha(z) > 1,\ f_k \in H(B).$$

Acknowledgment

The first author is supported partly by the Natural Science Foundation of Guangdong Province of China (grant No.7300614).

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