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PROPERTIES OF MULTIVALENT FUNCTIONS ASSOCIATED WITH A CERTAIN INTEGRAL OPERATOR

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ABSTRACT. Let $A(p)$ denote the class of functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ $(p \in N = \{1, 2, 3, \cdots\})$, which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$. By making use of a certain integral operator, we obtain some interesting properties of multivalent analytic functions.

1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in N = \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic in the open unit disk $U = \{z\colon z \in C \text{ and } |z| < 1\}$.

Recently, Jung, et al. [2] introduced the following integral operator, $Q^{\alpha}_{\beta,1} : A(1) \to A(1)$:

$$Q^{\alpha}_{\beta,1} f(z) = \begin{pmatrix} \alpha+\beta \\ \beta \end{pmatrix} \frac{\alpha}{z^{\beta}} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)dt \qquad (\alpha > 0, \beta > -1; f \in A(1)). \tag{1.2}$$

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Some interesting subclasses of analytic functions, associated with the operator $Q^{\alpha}_{\beta,1}$ and its many special cases, have been considered by Jung et al. [2], Auof et al. [1], Liu [3, 4, 5], Liu and Owa [6] and Patel and Rout [8].

Motivated by Jung, et al.'s work [2], we now introduce a linear operator, $Q^{\alpha}_{\beta,p} : A(p) \to A(p)$, as follows:

$$\begin{aligned} Q^{\alpha}_{\beta,p} f(z) &= \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^{\beta}} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)dt \\ &\quad (\alpha \geq 0, \beta > -1; f \in A(p)). \end{aligned} \tag{1.3}$$

We note that

$$Q^{\alpha}_{\beta,p} f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+\beta)\Gamma(p+\alpha+\beta)}{\Gamma(k+\alpha+\beta)\Gamma(p+\beta)} a_k z^k. \tag{1.4}$$

It is easily verified from the definition (1.1),(1.2),(1.3) and (1.4) that

$$z\left(Q^{\alpha+1}_{\beta,p} f(z)\right)' = (\alpha+\beta+p) Q^{\alpha}_{\beta,p} f(z) - (\alpha+\beta) Q^{\alpha+1}_{\beta,p} f(z). \tag{1.5}$$

When $p = 1$, the identity (1.5) was given by Jung et al. [2]. Here, we shall derive certain interesting properties of the linear operator $Q^{\alpha}_{\beta,p}$.

2. **Main results**

In order to give our results, we need the following lemma.

Lemma 2.1. *(see [7]). Let Ω be a set in the complex plane C and let b be a complex number such that $Reb > 0$. Suppose that the function $\psi : C^2 \times U \to C$ satisfies the condition*

$$\psi(ix, y; z) \notin \Omega,$$

for all real $x, y \leq -|b - ix|^2/(2Reb)$ and all $z \in U$. If the function $p(z)$, defined by $p(z) = b + b_1 z + b_2 z^2 + \cdots$, is analytic in U and if

$$\psi(p(z), zp'(z); z) \in \Omega,$$

then $Rep(z) > 0$ in U.

We now obtain some properties of the operator $Q^{\alpha}_{\beta,p}$.

Theorem 2.2. *Let $\alpha \geq 0$, $\beta > -1$, $\sigma \geq 1$ and $\gamma > 0$. Let $f(z) \in A(p)$. Then,*

$$Re\left\{\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha+1}_{\beta,p}f(z)}\right\} < \frac{\alpha+\beta+p+\gamma}{\alpha+\beta+p} \quad (z \in U) \tag{2.1}$$

implies

$$Re\left\{\left(\frac{Q^{\alpha+1}_{\beta,p}f(z)}{z^p}\right)^{-1/2\sigma\gamma}\right\} > 2^{-1/\sigma} \quad (z \in U). \tag{2.2}$$

The bound $2^{-1/\sigma}$ is the best possible.

Proof. From (1.5) and (2.1), we have

$$Re\left\{\frac{z(Q^{\alpha+1}_{\beta,p}f(z))'}{Q^{\alpha+1}_{\beta,p}f(z)}\right\} < p+\gamma \quad (z \in U).$$

That is,

$$\frac{1}{2\gamma}\left(\frac{z(Q^{\alpha+1}_{\beta,p}f(z))'}{Q^{\alpha+1}_{\beta,p}f(z)} - p\right) \prec \frac{z}{1-z}. \tag{2.3}$$

Let

$$p(z) = \left(\frac{Q^{\alpha+1}_{\beta,p}f(z)}{z^p}\right)^{-1/2\gamma}.$$

Then, (2.3) may be written as

$$z(logp(z))' \prec z\left(log\frac{1}{1-z}\right)'. \tag{2.4}$$

By applying a well-known result [9] to (2.4), we obtain that:

$$p(z) \prec \frac{1}{1-z},$$

that is,

$$\left(\frac{Q^{\alpha+1}_{\beta,p}f(z)}{z^p}\right)^{-1/2\sigma\gamma} = \left(\frac{1}{1-w(z)}\right)^{1/\sigma}, \tag{2.5}$$

where $w(z)$ is analytic in U, $w(0) = 0$ and $|w(z)| < 1$, for $z \in U$.

According to $Re\left(t^{1/\sigma}\right) \geq (Ret)^{1/\sigma}$ for $Ret > 0$ and $\sigma \geq 1$, (2.5) yields (2.2) as follows:

$$\begin{aligned} Re\left\{\left(\frac{Q_{\beta,p}^{\alpha+1}f(z)}{z^p}\right)^{-1/2\sigma\gamma}\right\} &\geq \left(Re\left(\frac{1}{1-w(z)}\right)\right)^{1/\sigma} \\ &> 2^{-1/\sigma} \quad (z \in U). \end{aligned}$$

To see that the bound $2^{-1/\sigma}$ cannot be increased, we consider the function

$$g(z) = z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(k+\beta)\Gamma(p+\alpha+\beta)} \cdot \frac{2\gamma(2\gamma-1)\cdots(2\gamma-k+p+1)}{(k-p)!} z^k.$$

Since $g(z)$ satisfies

$$\frac{Q_{\beta,p}^{\alpha+1}g(z)}{z^p} = (1+z)^{2\gamma},$$

we easily have that $g(z)$ satisfies (2.1) and

$$Re\left\{\left(\frac{Q_{\beta,p}^{\alpha+1}g(z)}{z^p}\right)^{-1/2\sigma\gamma}\right\} \to 2^{-1/\sigma},$$

as $Rez \to 1^-$. The proof of the theorem is now complete. □

Theorem 2.3. *Let $\alpha \geq 1$, $\beta > -1$, $\lambda \geq 0$ and $\gamma > 1$. Suppose that $f(z) \in A(p)$. Then,*

$$Re\left\{(1-\lambda)\frac{Q_{\beta,p}^{\alpha}f(z)}{Q_{\beta,p}^{\alpha+1}f(z)} + \lambda\frac{Q_{\beta,p}^{\alpha-1}f(z)}{Q_{\beta,p}^{\alpha}f(z)}\right\} < \gamma \quad (z \in U) \tag{2.6}$$

implies

$$Re\left\{\frac{Q_{\beta,p}^{\alpha}f(z)}{Q_{\beta,p}^{\alpha+1}f(z)}\right\} < M \quad (z \in U), \tag{2.7}$$

where $M \in (1, +\infty)$ is the positive root of the equation

$$2(\alpha+\beta+p+\lambda-1)x^2 - [\lambda + 2\gamma(\alpha+\beta+p-1)]x - \lambda = 0. \tag{2.8}$$

Proof. Let

$$p(z) = \frac{1}{M-1}\left[M - \frac{Q_{\beta,p}^{\alpha}f(z)}{Q_{\beta,p}^{\alpha+1}f(z)}\right]. \tag{2.9}$$

Then, $p(z)$ is analytic in U and $p(0) = 1$. Differentiating (2.9) with respect to z and using (1.5), we obtain

$$\begin{aligned}
&(1-\lambda)\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha+1}_{\beta,p}f(z)} + \lambda\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)} \\
&= M + \frac{\lambda(M-1)}{\alpha+\beta+p-1} - \frac{(M-1)(\alpha+\beta+p+\lambda-1)}{\alpha+\beta+p-1}p(z) \\
&\quad - \frac{\lambda(M-1)}{\alpha+\beta+p-1}\cdot\frac{zp'(z)}{M-(M-1)p(z)} \\
&= \psi(p(z), zp'(z)),
\end{aligned}$$

where

$$\begin{aligned}
\psi(r,s) &= M + \frac{\lambda(M-1)}{\alpha+\beta+p-1} - \frac{(M-1)(\alpha+\beta+p+\lambda-1)}{\alpha+\beta+p-1}r \\
&- \frac{\lambda(M-1)}{\alpha+\beta+p-1}\cdot\frac{s}{M-(M-1)r}. \qquad (2.10)
\end{aligned}$$

Using (2.6) and (2.10), we have

$$\{\psi(p(z), zp'(z)) : z \in U\} \subset \Omega = \{w \in C : Re w < \gamma\}.$$

Now, for all real $x, y \le -(1+x^2)/2$, we have

$$\begin{aligned}
&Re\{\psi(ix,y)\} \\
&= M + \frac{\lambda(M-1)}{\alpha+\beta+p-1} - \frac{\lambda(M-1)}{\alpha+\beta+p-1}\cdot\frac{My}{M^2+(M-1)^2x^2} \\
&\ge M + \frac{\lambda(M-1)}{\alpha+\beta+p-1} + \frac{\lambda M(M-1)}{2(\alpha+\beta+p-1)}\cdot\frac{1+x^2}{M^2+(M-1)^2x^2} \\
&\ge M + \frac{\lambda(M-1)}{\alpha+\beta+p-1} + \frac{\lambda(M-1)}{2M(\alpha+\beta+p-1)} \\
&= M + \frac{\lambda(M-1)(2M+1)}{2M(\alpha+\beta+p-1)} = \gamma,
\end{aligned}$$

where M is the positive root of the equation (2.8).

Note that $\alpha \ge 1$, $\beta > -1$, $\lambda \ge 0$, $\gamma > 1$ and let

$$g(x) = 2(\alpha+\beta+p+\lambda-1)x^2 - [\lambda + 2\gamma(\alpha+\beta+p-1)]x - \lambda.$$

Then, $g(0) = -\lambda \le 0$, and $g(1) = -2(\alpha+\beta+p-1)(\gamma-1) < 0$. This shows $M \in (1, +\infty)$. Hence, for each $z \in U$, $\psi(ix, y) \notin \Omega$. By Lemma 2.1, we get $Rep(z) > 0$. This proves (2.7). □

Finally, we prove the following result.

Theorem 2.4. *Let* $\alpha \ge 0$, $\beta > -1$, $\lambda \ge 0$, $\gamma > 1$ *and* $0 \le \delta < 1$. *Let* $g(z) \in A(p)$ *satisfy*

$$Re\left\{\frac{Q^{\alpha+1}_{\beta,p}g(z)}{Q^{\alpha}_{\beta,p}g(z)}\right\} > \delta \quad (z \in U). \tag{2.11}$$

If $f(z) \in A(p)$ *satisfies*

$$Re\left\{(1-\lambda)\frac{Q^{\alpha+1}_{\beta,p}f(z)}{Q^{\alpha+1}_{\beta,p}g(z)} + \lambda\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right\} < \gamma \quad (z \in U), \tag{2.12}$$

then

$$Re\left\{\frac{Q^{\alpha+1}_{\beta,p}f(z)}{Q^{\alpha+1}_{\beta,p}g(z)}\right\} < \frac{2\gamma(\alpha+\beta+p)+\lambda\delta}{2(\alpha+\beta+p)+\lambda\delta} \quad (z \in U). \tag{2.13}$$

Proof. Let $M = \frac{2\gamma(\alpha+\beta+p)+\lambda\delta}{2(\alpha+\beta+p)+\lambda\delta}$ $(M > 1)$ and consider the function

$$p(z) = \frac{1}{M-1}\left[M - \frac{Q^{\alpha+1}_{\beta,p}f(z)}{Q^{\alpha+1}_{\beta,p}g(z)}\right]. \tag{2.14}$$

The function $p(z)$ is analytic in U and $p(0) = 1$. Setting

$$B(z) = \frac{Q^{\alpha+1}_{\beta,p}g(z)}{Q^{\alpha}_{\beta,p}g(z)},$$

we have $Re\{B(z)\} > \delta$ $(z \in U)$. Differentiating (2.14) with respect to z and using (1.5), we have

$$\begin{aligned}&(1-\lambda)\frac{Q^{\alpha+1}_{\beta,p}f(z)}{Q^{\alpha+1}_{\beta,p}g(z)} + \lambda\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\\ &= M - (M-1)p(z) - \frac{\lambda(M-1)}{\alpha+\beta+p}B(z)\cdot zp'(z).\end{aligned}$$

Let

$$\psi(r,s) = M - (M-1)r - \frac{\lambda(M-1)}{\alpha+\beta+p}B(z)\cdot s.$$

Then, from (2.11) and (2.12), we deduce that

$$\{\psi(p(z), zp'(z)) : z \in U\} \subset \Omega = \{w \in C : Rew < \gamma\}.$$

Now, for all real $x, y \leq -(1+x^2)/2$, we have

$$\begin{aligned} Re\{\psi(ix, y)\} &= M - \frac{\lambda(M-1)y}{\alpha+\beta+p} Re\{B(z)\} \\ &\geq M + \frac{\lambda\delta(M-1)}{2(\alpha+\beta+p)}(1+x^2) \\ &\geq M + \frac{\lambda\delta(M-1)}{2(\alpha+\beta+p)} = \gamma. \end{aligned}$$

Hence, for each $z \in U$, $\psi(ix, y) \notin \Omega$. Thus, by Lemma 2.1, $Rep(z) > 0$ in U. This proves (2.13). The proof of the theorem is now complete. $\square$

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