

PROPERTIES OF MULTIVALENT FUNCTIONS ASSOCIATED WITH A CERTAIN INTEGRAL OPERATOR

J-L LIU

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ABSTRACT. Let $A(p)$ denote the class of functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ ($p \in N = \{1, 2, 3, \dots\}$), which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$. By making use of a certain integral operator, we obtain some interesting properties of multivalent analytic functions.

1. Introduction

Let $A(p)$ denote the class of functions of the form

$$(1.1) \quad f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in N = \{1, 2, 3, \dots\}),$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$.

Recently, Jung, et al. [2] introduced the following integral operator, $Q_{\beta,1}^{\alpha} : A(1) \rightarrow A(1)$:

$$(1.2) \quad Q_{\beta,1}^{\alpha} f(z) = \binom{\alpha + \beta}{\beta} \frac{\alpha}{z^{\beta}} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t) dt$$

$(\alpha > 0, \beta > -1; f \in A(1)).$

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Some interesting subclasses of analytic functions, associated with the operator $Q_{\beta,1}^\alpha$ and its many special cases, have been considered by Jung et al. [2], Auof et al. [1], Liu [3, 4, 5], Liu and Owa [6] and Patel and Rout [8].

Motivated by Jung, et al.'s work [2], we now introduce a linear operator, $Q_{\beta,p}^\alpha : A(p) \rightarrow A(p)$, as follows:

$$(1.3) \quad Q_{\beta,p}^\alpha f(z) = \binom{p + \alpha + \beta - 1}{p + \beta - 1} \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t) dt$$

$(\alpha \geq 0, \beta > -1; f \in A(p)).$

We note that

$$(1.4) \quad Q_{\beta,p}^\alpha f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma(k + \beta)\Gamma(p + \alpha + \beta)}{\Gamma(k + \alpha + \beta)\Gamma(p + \beta)} a_k z^k.$$

It is easily verified from the definition (1.1),(1.2),(1.3) and (1.4) that

$$(1.5) \quad z \left(Q_{\beta,p}^{\alpha+1} f(z) \right)' = (\alpha + \beta + p) Q_{\beta,p}^\alpha f(z) - (\alpha + \beta) Q_{\beta,p}^{\alpha+1} f(z).$$

When $p = 1$, the identity (1.5) was given by Jung et al. [2]. Here, we shall derive certain interesting properties of the linear operator $Q_{\beta,p}^\alpha$.

2. Main results

In order to give our results, we need the following lemma.

Lemma 2.1. (see [7]). *Let Ω be a set in the complex plane C and let b be a complex number such that $\text{Re} b > 0$. Suppose that the function $\psi : C^2 \times U \rightarrow C$ satisfies the condition*

$$\psi(ix, y; z) \notin \Omega,$$

for all real $x, y \leq -|b - ix|^2 / (2\text{Re} b)$ and all $z \in U$. If the function $p(z)$, defined by $p(z) = b + b_1 z + b_2 z^2 + \dots$, is analytic in U and if

$$\psi(p(z), zp'(z); z) \in \Omega,$$

then $\text{Re} p(z) > 0$ in U .

We now obtain some properties of the operator $Q_{\beta,p}^\alpha$.

Theorem 2.2. *Let $\alpha \geq 0$, $\beta > -1$, $\sigma \geq 1$ and $\gamma > 0$. Let $f(z) \in A(p)$. Then,*

$$(2.1) \quad \operatorname{Re} \left\{ \frac{Q_{\beta,p}^\alpha f(z)}{Q_{\beta,p}^{\alpha+1} f(z)} \right\} < \frac{\alpha + \beta + p + \gamma}{\alpha + \beta + p} \quad (z \in U)$$

implies

$$(2.2) \quad \operatorname{Re} \left\{ \left(\frac{Q_{\beta,p}^{\alpha+1} f(z)}{z^p} \right)^{-1/2\sigma\gamma} \right\} > 2^{-1/\sigma} \quad (z \in U).$$

The bound $2^{-1/\sigma}$ is the best possible.

Proof. From (1.5) and (2.1), we have

$$\operatorname{Re} \left\{ \frac{z(Q_{\beta,p}^{\alpha+1} f(z))'}{Q_{\beta,p}^{\alpha+1} f(z)} \right\} < p + \gamma \quad (z \in U).$$

That is,

$$(2.3) \quad \frac{1}{2\gamma} \left(\frac{z(Q_{\beta,p}^{\alpha+1} f(z))'}{Q_{\beta,p}^{\alpha+1} f(z)} - p \right) \prec \frac{z}{1-z}.$$

Let

$$p(z) = \left(\frac{Q_{\beta,p}^{\alpha+1} f(z)}{z^p} \right)^{-1/2\gamma}.$$

Then, (2.3) may be written as

$$(2.4) \quad z(\log p(z))' \prec z \left(\log \frac{1}{1-z} \right)'$$

By applying a well-known result [9] to (2.4), we obtain that:

$$p(z) \prec \frac{1}{1-z},$$

that is,

$$(2.5) \quad \left(\frac{Q_{\beta,p}^{\alpha+1} f(z)}{z^p} \right)^{-1/2\sigma\gamma} = \left(\frac{1}{1-w(z)} \right)^{1/\sigma},$$

where $w(z)$ is analytic in U , $w(0) = 0$ and $|w(z)| < 1$, for $z \in U$.

According to $Re(t^{1/\sigma}) \geq (Ret)^{1/\sigma}$ for $Ret > 0$ and $\sigma \geq 1$, (2.5) yields (2.2) as follows:

$$Re \left\{ \left(\frac{Q_{\beta,p}^{\alpha+1} f(z)}{z^p} \right)^{-1/2\sigma\gamma} \right\} \geq \left(Re \left(\frac{1}{1-w(z)} \right) \right)^{1/\sigma} > 2^{-1/\sigma} \quad (z \in U).$$

To see that the bound $2^{-1/\sigma}$ cannot be increased, we consider the function

$$g(z) = z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma(k + \alpha + \beta)\Gamma(p + \beta)}{\Gamma(k + \beta)\Gamma(p + \alpha + \beta)} \cdot \frac{2\gamma(2\gamma - 1) \cdots (2\gamma - k + p + 1)}{(k - p)!} z^k.$$

Since $g(z)$ satisfies

$$\frac{Q_{\beta,p}^{\alpha+1} g(z)}{z^p} = (1 + z)^{2\gamma},$$

we easily have that $g(z)$ satisfies (2.1) and

$$Re \left\{ \left(\frac{Q_{\beta,p}^{\alpha+1} g(z)}{z^p} \right)^{-1/2\sigma\gamma} \right\} \rightarrow 2^{-1/\sigma},$$

as $Rez \rightarrow 1^-$. The proof of the theorem is now complete. □

Theorem 2.3. *Let $\alpha \geq 1$, $\beta > -1$, $\lambda \geq 0$ and $\gamma > 1$. Suppose that $f(z) \in A(p)$. Then,*

$$(2.6) \quad Re \left\{ (1 - \lambda) \frac{Q_{\beta,p}^{\alpha} f(z)}{Q_{\beta,p}^{\alpha+1} f(z)} + \lambda \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha} f(z)} \right\} < \gamma \quad (z \in U)$$

implies

$$(2.7) \quad Re \left\{ \frac{Q_{\beta,p}^{\alpha} f(z)}{Q_{\beta,p}^{\alpha+1} f(z)} \right\} < M \quad (z \in U),$$

where $M \in (1, +\infty)$ is the positive root of the equation

$$(2.8) \quad 2(\alpha + \beta + p + \lambda - 1)x^2 - [\lambda + 2\gamma(\alpha + \beta + p - 1)]x - \lambda = 0.$$

Proof. Let

$$(2.9) \quad p(z) = \frac{1}{M - 1} \left[M - \frac{Q_{\beta,p}^{\alpha} f(z)}{Q_{\beta,p}^{\alpha+1} f(z)} \right].$$

Then, $p(z)$ is analytic in U and $p(0) = 1$. Differentiating (2.9) with respect to z and using (1.5), we obtain

$$\begin{aligned} & (1 - \lambda) \frac{Q_{\beta,p}^\alpha f(z)}{Q_{\beta,p}^{\alpha+1} f(z)} + \lambda \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^\alpha f(z)} \\ &= M + \frac{\lambda(M-1)}{\alpha + \beta + p - 1} - \frac{(M-1)(\alpha + \beta + p + \lambda - 1)}{\alpha + \beta + p - 1} p(z) \\ &\quad - \frac{\lambda(M-1)}{\alpha + \beta + p - 1} \cdot \frac{zp'(z)}{M - (M-1)p(z)} \\ &= \psi(p(z), zp'(z)), \end{aligned}$$

where

$$\begin{aligned} \psi(r, s) &= M + \frac{\lambda(M-1)}{\alpha + \beta + p - 1} - \frac{(M-1)(\alpha + \beta + p + \lambda - 1)}{\alpha + \beta + p - 1} r \\ (2.10) \quad &- \frac{\lambda(M-1)}{\alpha + \beta + p - 1} \cdot \frac{s}{M - (M-1)r}. \end{aligned}$$

Using (2.6) and (2.10), we have

$$\{\psi(p(z), zp'(z)) : z \in U\} \subset \Omega = \{w \in C : Rew < \gamma\}.$$

Now, for all real $x, y \leq -(1 + x^2)/2$, we have

$$\begin{aligned} & Re\{\psi(ix, y)\} \\ &= M + \frac{\lambda(M-1)}{\alpha + \beta + p - 1} - \frac{\lambda(M-1)}{\alpha + \beta + p - 1} \cdot \frac{My}{M^2 + (M-1)^2x^2} \\ &\geq M + \frac{\lambda(M-1)}{\alpha + \beta + p - 1} + \frac{\lambda M(M-1)}{2(\alpha + \beta + p - 1)} \cdot \frac{1 + x^2}{M^2 + (M-1)^2x^2} \\ &\geq M + \frac{\lambda(M-1)}{\alpha + \beta + p - 1} + \frac{\lambda(M-1)}{2M(\alpha + \beta + p - 1)} \\ &= M + \frac{\lambda(M-1)(2M+1)}{2M(\alpha + \beta + p - 1)} = \gamma, \end{aligned}$$

where M is the positive root of the equation (2.8).

Note that $\alpha \geq 1, \beta > -1, \lambda \geq 0, \gamma > 1$ and let

$$g(x) = 2(\alpha + \beta + p + \lambda - 1)x^2 - [\lambda + 2\gamma(\alpha + \beta + p - 1)]x - \lambda.$$

Then, $g(0) = -\lambda \leq 0$, and $g(1) = -2(\alpha + \beta + p - 1)(\gamma - 1) < 0$. This shows $M \in (1, +\infty)$. Hence, for each $z \in U$, $\psi(ix, y) \notin \Omega$. By Lemma 2.1, we get $Rep(z) > 0$. This proves (2.7). \square

Finally, we prove the following result.

Theorem 2.4. *Let $\alpha \geq 0$, $\beta > -1$, $\lambda \geq 0$, $\gamma > 1$ and $0 \leq \delta < 1$. Let $g(z) \in A(p)$ satisfy*

$$(2.11) \quad \operatorname{Re} \left\{ \frac{Q_{\beta,p}^{\alpha+1} g(z)}{Q_{\beta,p}^{\alpha} g(z)} \right\} > \delta \quad (z \in U).$$

If $f(z) \in A(p)$ satisfies

$$(2.12) \quad \operatorname{Re} \left\{ (1 - \lambda) \frac{Q_{\beta,p}^{\alpha+1} f(z)}{Q_{\beta,p}^{\alpha+1} g(z)} + \lambda \frac{Q_{\beta,p}^{\alpha} f(z)}{Q_{\beta,p}^{\alpha} g(z)} \right\} < \gamma \quad (z \in U),$$

then

$$(2.13) \quad \operatorname{Re} \left\{ \frac{Q_{\beta,p}^{\alpha+1} f(z)}{Q_{\beta,p}^{\alpha+1} g(z)} \right\} < \frac{2\gamma(\alpha + \beta + p) + \lambda\delta}{2(\alpha + \beta + p) + \lambda\delta} \quad (z \in U).$$

Proof. Let $M = \frac{2\gamma(\alpha + \beta + p) + \lambda\delta}{2(\alpha + \beta + p) + \lambda\delta}$ ($M > 1$) and consider the function

$$(2.14) \quad p(z) = \frac{1}{M-1} \left[M - \frac{Q_{\beta,p}^{\alpha+1} f(z)}{Q_{\beta,p}^{\alpha+1} g(z)} \right].$$

The function $p(z)$ is analytic in U and $p(0) = 1$. Setting

$$B(z) = \frac{Q_{\beta,p}^{\alpha+1} g(z)}{Q_{\beta,p}^{\alpha} g(z)},$$

we have $\operatorname{Re}\{B(z)\} > \delta$ ($z \in U$). Differentiating (2.14) with respect to z and using (1.5), we have

$$\begin{aligned} & (1 - \lambda) \frac{Q_{\beta,p}^{\alpha+1} f(z)}{Q_{\beta,p}^{\alpha+1} g(z)} + \lambda \frac{Q_{\beta,p}^{\alpha} f(z)}{Q_{\beta,p}^{\alpha} g(z)} \\ &= M - (M - 1)p(z) - \frac{\lambda(M - 1)}{\alpha + \beta + p} B(z) \cdot zp'(z). \end{aligned}$$

Let

$$\psi(r, s) = M - (M - 1)r - \frac{\lambda(M - 1)}{\alpha + \beta + p} B(z) \cdot s.$$

Then, from (2.11) and (2.12), we deduce that

$$\{\psi(p(z), zp'(z)) : z \in U\} \subset \Omega = \{w \in C : Rew < \gamma\}.$$

Now, for all real $x, y \leq -(1 + x^2)/2$, we have

$$\begin{aligned} \operatorname{Re}\{\psi(ix, y)\} &= M - \frac{\lambda(M-1)y}{\alpha + \beta + p} \operatorname{Re}\{B(z)\} \\ &\geq M + \frac{\lambda\delta(M-1)}{2(\alpha + \beta + p)}(1 + x^2) \\ &\geq M + \frac{\lambda\delta(M-1)}{2(\alpha + \beta + p)} = \gamma. \end{aligned}$$

Hence, for each $z \in U$, $\psi(ix, y) \notin \Omega$. Thus, by Lemma 2.1, $\operatorname{Rep}(z) > 0$ in U . This proves (2.13). The proof of the theorem is now complete. \square

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Jin-Lin Liu

Department of Mathematics, Yangzhou University, 225002, Yangzhou City, China

Email: jlliu@yzu.edu.cn