# OPTIMAL ORDER FINITE ELEMENT APPROXIMATION FOR A HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATION 

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#### Abstract

Semidiscrete finite element approximation of a hyperbolic type integro-differential equation is studied. The model problem is treated as the wave equation which is perturbed with a memory term. Stability estimates are obtained for a slightly more general problem. These, based on energy method, are used to prove optimal order a priori error estimates.


## 1. Introduction

We consider a hyperbolic type integro-differential equation

$$
\begin{array}{ll}
\ddot{u}+A u-\int_{0}^{t} K(t-s) A u(s) \mathrm{d} s=f & \text { in } \Omega \times(0, T] \\
u=0 & \text { on } \partial \Omega \times(0, T],  \tag{1.1}\\
u(\cdot, 0)=u_{0}, \quad u_{t}(\cdot, 0)=v_{0} & \text { in } \Omega,
\end{array}
$$

where $\Omega$ is a bounded polygonal domain in $\mathbb{R}^{d}, d=2,3$, with boundary $\partial \Omega$. Here, $A$ is a self-adjoint, positive definite uniformly elliptic second order operator, and $K$ is the kernel. The kernel $K$ is considered to be

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either smooth (exponential), or weakly singular, in both cases with the properties that

$$
\begin{equation*}
K \geq 0, \quad K^{\prime}(t) \leq 0, \quad\|K\|_{L_{1}(0, \infty)}<1 \tag{1.2}
\end{equation*}
$$

Problems of this nature occur, e.g., in linear and fractional order viscoelasticity; see [1] and references therein. The model problem (1.1) is of hyperbolic type, and numerical analysis of such a problem is inherent from numerical analysis of the hyperbolic problems. Hence, we treat the spatial finite element discretization of the problem as a wave equation perturbed with a memory term.

We note that, for example, completely monotone functions satisfy (1.2). That is, functions $g \in L_{1}(0, \infty) \cap \mathcal{C}^{2}(0, \infty)$, such that

$$
(-1)^{k} \frac{\mathrm{~d}^{k}}{\mathrm{~d} t^{k}} g(t) \geq 0, \quad \forall t>0, k=0,1,2 .
$$

For an example of this type of kernels, see [1].
There is an extensive literature on theoretical and numerical analysis of integro-differential equations and their applications; see, e.g., $[1,2,3$, $4,6,8,10]$, and references therein. Existence, uniqueness and regularity of a similar problem have been studied in [1] in the framework of the semigroup of linear operators. Spatial finite element approximations of integro-differential equations similar to (1.1) have been studied in [2] and [7], where optimal order $L_{\infty}\left(L_{2}\right)$ and $L_{\infty}\left(H^{1}\right)$ a priori error estimates have been proved for the displacement solution $u$, and $L_{\infty}\left(L_{2}\right)$ estimates for the velocity $\dot{u}$. However, compared with the optimal order $L_{\infty}\left(L_{2}\right)$ error estimate for the solution $u$, they require two extra derivatives of regularity of the solution. Here, we improve those results by relaxing the regularity assumptions on the smoothness of the solution by requiring just one extra derivative. To this end, we prove stability estimates for a slightly more general problem. These are then used to prove optimal order a priori error estimates. The same argument has be applied for the linear wave equation in [5]. In [9], based on an explicit representation of the solution, it has been proved that the resulting regularity requirement, for the wave equation, is minimal. However, a similar proof as in [9] can not be directly applied for the finite element semidiscretization of the model problem (1.1), since an explicit representation of the solution can not be easily obtained.

In the next section, some preliminaries and finite element spatial discretization of the problem, based on a velocity-displacement weak formulation, are provided. Then, in Section 3 an energy identity for a
modified form of the discrete problem is obtained. Finally, these are used in Section 4 to prove optimal order a priori error estimates for the displacement and the velocity.

## 2. Preleminaries and finite element spatial discretization

We denote the standard Sobolev spaces by $H^{i}=H^{i}(\Omega)^{d}$ with the corresponding norms $\|\cdot\|_{i}$. We recall that $A$ is a self-adjoint, positive definite uniformly elliptic second order operator with $\mathcal{D}(A)=H^{2} \cap$ $V$, and we correspond the energy inner product $a(v, w)=(A v, w)$ for smooth functions $v, w \in V$. Here, $V=\left\{v \in H^{1}: v=0\right.$ on $\left.\partial \Omega\right\}$ is equiped with norm $\|\cdot\|_{V}=a(\cdot, \cdot)$, which is equivalent to the standard Sobolev norm $\|\cdot\|_{1}$ on $V$. We also define $H=L_{2}(\Omega)^{d}$ with its usual inner product and norm denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively.

We put $u_{1}=u$ and $u_{2}=\dot{u}$. Then, the velocity-displacement weak formulation of (1.1) is to find $u_{1}(t), u_{2}(t) \in V$ such that

$$
\begin{align*}
& a\left(\dot{u}_{1}(t), v_{1}\right)-a\left(u_{2}(t), v_{1}\right)=0, \\
& \begin{aligned}
\left(\dot{u}_{2}(t), v_{2}\right)+a\left(u_{1}(t), v_{2}\right)- & \int_{0}^{t} K(t-s) a\left(u_{1}(s), v_{2}\right) \mathrm{d} s
\end{aligned}  \tag{2.1}\\
& \quad=\left(f(t), v_{2}\right), \quad \forall v_{1}, v_{2} \in V, t \in(0, T], \\
& u_{1}(0)=u_{0}, \quad u_{2}(0)=v_{0} .
\end{align*}
$$

2.1. Finite element spatial discretization. Let $\left\{\mathcal{T}_{h}\right\}$ be a regular family of triangulations of $\Omega$ with corresponding family of finite element spaces $V_{h} \subset V$, consisting of continuous piecewise polynomials of degree at most $r-1$, that vanish on $\partial \Omega$ (so, the mesh is required to fit $\partial \Omega$ ). Here, $r \geq 2$ is an integer number. We define piecewise constant mesh function $h_{K}$ by

$$
h_{K}(x)=\operatorname{diam}(K), \quad \text { for } x \in K, K \in \mathcal{T}_{h} .
$$

For our error analysis, we denote $h=\max _{K \in \mathcal{T}_{h}} h_{K}$, and we note that the finite element spaces $V_{h}$ have the property

$$
\begin{equation*}
\min _{\chi \in V_{h}}\left\{\|v-\chi\|+h\|v-\chi\|_{1}\right\} \leq C h^{i}\|v\|_{i}, \quad \text { for } v \in H^{i} \cap V, 1 \leq i \leq r . \tag{2.2}
\end{equation*}
$$

Then, the spatial finite element discretization of (2.1) is to find $u_{h, 1}(t)$, $u_{h, 2}(t) \in V_{h}$, such that

$$
\begin{align*}
& a\left(\dot{u}_{h, 1}(t), \chi_{1}\right)-a\left(u_{h, 2}(t), \chi_{1}\right)=0, \\
& \left(\dot{u}_{h, 2}(t), \chi_{2}\right)+a\left(u_{h, 1}(t), \chi_{2}\right)-\int_{0}^{t} K(t-s) a\left(u_{h, 1}(s), \chi_{2}\right) \mathrm{d} s  \tag{2.3}\\
& \quad=\left(f(t), \chi_{2}\right), \quad \forall \chi_{1}, \chi_{2} \in V_{h}, t \in(0, T], \\
& u_{h, 1}(0)=u_{h, 0}, \quad u_{h, 2}(0)=v_{h, 0},
\end{align*}
$$

where $u_{h, 0}, v_{h, 0}$ are, respectively, suitable approximations of $u_{0}$ and $v_{0}$ in $V_{h}$.

Denoting the error by $\left(e_{1}, e_{2}\right)=\left(u_{h, 1}, u_{h, 2}\right)-\left(u_{1}, u_{2}\right)$, we can write the Galerkin orthogonality in the form

$$
\begin{align*}
& a\left(\dot{e}_{1}(t), \chi_{1}\right)-a\left(e_{2}(t), \chi_{1}\right)=0 \\
& \begin{aligned}
\left(\dot{e}_{2}(t), \chi_{2}\right)+a\left(e_{1}(t), \chi_{2}\right)- & \int_{0}^{t} K(t-s) a\left(e_{1}(s), \chi_{2}\right) \mathrm{d} s \\
& =0, \quad \forall \chi_{1}, \chi_{2} \in V_{h}, t \in(0, T] .
\end{aligned} \tag{2.4}
\end{align*}
$$

We recall the $L_{2}$-projection $\mathcal{P}_{h}: H \rightarrow V_{h}$ and the Ritz projection $\mathcal{R}_{h}: V \rightarrow V_{h}$, defined by

$$
a\left(\mathcal{R}_{h} v, \chi\right)=a(v, \chi) \quad \text { and } \quad\left(\mathcal{P}_{h} v, \chi\right)=(v, \chi), \quad \forall \chi \in V_{h} .
$$

We also recall the elliptic regularity estimate $\|v\|_{2} \leq C\|A v\|, \forall v \in$ $\mathcal{D}(A)=H^{2} \cap V$. Then, for the Ritz projection $\mathcal{R}_{h}$, the error estimates (2.2) hold; see [11]. That is,

$$
\begin{align*}
&\left\|\left(\mathcal{R}_{h}-I\right) v\right\|+h \|\left(\mathcal{R}_{h}-I\right) v \|_{1} \\
&  \tag{2.5}\\
& \leq C h^{i}\|v\|_{i}, \quad \text { for } v \in H^{i} \cap V, 1 \leq i \leq r .
\end{align*}
$$

We use the discrete norms

$$
\left\|v_{h}\right\|_{h, l}=\left\|A_{h}^{l / 2} v_{h}\right\|=\sqrt{\left(A_{h}^{l} v_{h}, v_{h}\right)}, \quad \forall v_{h} \in V_{h}, l \in \mathbb{R}
$$

where $A_{h}: V_{h} \rightarrow V_{h}$ is the discrete operator defined by

$$
\left(A_{h} v_{h}, w_{h}\right)=a\left(v_{h}, w_{h}\right), \quad \forall v_{h}, w_{h} \in V_{h}
$$

It is easy to see that, for $v_{h} \in V_{h}$,

$$
\begin{equation*}
\left\|v_{h}\right\|_{h, 0}=\left\|v_{h}\right\|, \quad\left\|v_{h}\right\|_{h, 1}=\left\|A^{1 / 2} v_{h}\right\|=\left\|v_{h}\right\|_{1} \tag{2.6}
\end{equation*}
$$

and, for $v \in V^{*}$,

$$
\begin{equation*}
\left\|\mathcal{P}_{h} v\right\|_{h,-1} \leq\|v\|_{-1}, \tag{2.7}
\end{equation*}
$$

where $V^{*}$ is the corresponding dual space of $V$.

## 3. Stability estimates

In our error analysis, we use stability estimates for a slightly more general form of (2.3), by putting an extra load term to the first equation. That is, we find stability estimates for the modified problem

$$
\begin{align*}
& a\left(\dot{u}_{h, 1}(t), \chi_{1}\right)-a\left(u_{h, 2}(t), \chi_{1}\right)=a\left(f_{1}(t), \chi_{1}\right), \\
& \left(\dot{u}_{h, 2}(t), \chi_{2}\right)+a\left(u_{h, 1}(t), \chi_{2}\right)-\int_{0}^{t} K(t-s) a\left(u_{h, 1}(s), \chi_{2}\right) \mathrm{d} s  \tag{3.1}\\
& =\left(f_{2}(t), \chi_{2}\right), \quad \forall \chi_{1}, \chi_{2} \in V_{h}, t \in(0, T], \\
& u_{h, 1}(0)=u_{h, 0}, \quad u_{h, 2}(0)=v_{h, 0} .
\end{align*}
$$

Theorem 3.1. Let $\left(u_{h, 1}, u_{h, 2}\right)$ be a solution of (3.1). Then, for any $l \in \mathbb{R}$ and $T>0$, we have the identity

$$
\begin{align*}
\left\|u_{h, 2}(T)\right\|_{h, l}^{2} & +\kappa(T)\left\|u_{h, 1}(T)\right\|_{h, l+1}^{2}+\int_{0}^{T} K\left\|u_{h, 1}\right\|_{h, l+1}^{2} \mathrm{~d} t  \tag{3.2}\\
& +\int_{0}^{T} K(s)\left\|w_{h, 1}(T, s)\right\|_{h, l+1} \mathrm{~d} s \\
& +\int_{0}^{T} \int_{0}^{t}(K(s)-K(t)) D_{s}\left\|w_{h, 1}(t, s)\right\|_{h, l+1} \mathrm{~d} s \mathrm{~d} t \\
= & \left\|v_{h, 0}\right\|_{h, l}^{2}+\left\|u_{h, 0}\right\|_{h, l+1}^{2} \\
& +2 \int_{0}^{T}\left(\mathcal{P}_{h} f_{2}, A_{h}^{l} u_{h, 2}\right) \mathrm{d} t+2 \int_{0}^{T} \kappa a\left(\mathcal{R}_{h} f_{1}, A_{h}^{l} u_{h, 1}\right) \mathrm{d} t \\
& +2 \int_{0}^{T} \int_{0}^{t} K(t-s) a\left(\mathcal{R}_{h} f_{1}, A_{h}^{l} w_{h, 1}(t, s)\right) \mathrm{d} s \mathrm{~d} t
\end{align*}
$$

where $w_{h, 1}(t, s)=u_{h, 1}(t)-u_{h, 1}(t-s)$ and

$$
\begin{equation*}
0<\kappa(t)=1-\int_{0}^{t} K(s) d s \leq 1, \quad t \in[0, T] . \tag{3.3}
\end{equation*}
$$

All terms on the left side are non-negative. Moreover, we have the stability estimate

$$
\begin{aligned}
& \left\|u_{h, 2}(T)\right\|_{h, l}+\left\|u_{h, 1}(T)\right\|_{h, l+1} \\
& \quad \leq C\left(\left\|v_{h, 0}\right\|_{h, l}+\left\|u_{h, 0}\right\|_{h, l+1}+\int_{0}^{T}\left\|\mathcal{R}_{h} f_{1}\right\|_{h, l+1}+\left\|\mathcal{P}_{h} f_{2}\right\|_{h, l} \mathrm{~d} t\right) .
\end{aligned}
$$

Proof. From the first equation of (3.1), we simply conclude

$$
\begin{equation*}
u_{h, 2}=\dot{u}_{h, 1}-\mathcal{R}_{h} f_{1} \tag{3.5}
\end{equation*}
$$

Now, using the new function $w_{h, 1}(t, s)$ and (3.3), we can write the second equation of (3.1) in the form

$$
\begin{align*}
\left(\dot{u}_{h, 2}(t), \chi_{2}\right)+\kappa(t) a\left(u_{h, 1}(t), \chi_{2}\right) & +\int_{0}^{t} K(s) a\left(w_{h, 1}(t, s), \chi_{2}\right) \mathrm{d} s  \tag{3.6}\\
& =\left(f_{2}(t), \chi_{2}\right), \quad \forall \chi_{2} \in V_{h} .
\end{align*}
$$

Then, we choose $\chi_{2}=A_{h}^{l} u_{h, 2}(t)$ and integrate to obtain

$$
\begin{align*}
\int_{0}^{T}\left(\dot{u}_{h, 2}, A_{h}^{l} u_{h, 2}\right) \mathrm{d} t & +\int_{0}^{T} \kappa a\left(u_{h, 1}, A_{h}^{l} u_{h, 2}\right) \mathrm{d} t \\
& +\int_{0}^{T} \int_{0}^{t} K(s) a\left(w_{h, 1}(t, s), A_{h}^{l} u_{h, 2}\right) \mathrm{d} s \mathrm{~d} t  \tag{3.7}\\
& =\int_{0}^{T}\left(\mathcal{P}_{h} f_{2}, A_{h}^{l} u_{h, 2}\right) \mathrm{d} t .
\end{align*}
$$

Now, we need to study the three terms in the left side of the above equation.

We can write the first term of the left side of (3.7), in a standard way, as

$$
\begin{align*}
\int_{0}^{T}\left(\dot{u}_{h, 2}, A_{h}^{l} u_{h, 2}\right) \mathrm{d} t & =\frac{1}{2} \int_{0}^{T} D_{t}\left\|u_{h, 2}\right\|_{h, l}^{2} \mathrm{~d} t  \tag{3.8}\\
& =\frac{1}{2}\left(\left\|u_{h, 2}(T)\right\|_{h, l}^{2}-\left\|v_{h, 0}\right\|_{h, l}^{2}\right) .
\end{align*}
$$

For the second one, using (3.5), we have

$$
\begin{aligned}
& \int_{0}^{T} \kappa a\left(u_{h, 1}, A_{h}^{l} u_{h, 2}\right) \mathrm{d} t \\
& \quad=\frac{1}{2} \int_{0}^{T} \kappa D_{t}\left\|u_{h, 1}\right\|_{h, l+1}^{2} \mathrm{~d} t-\int_{0}^{T} \kappa a\left(u_{h, 1}, A_{h}^{l} \mathcal{R}_{h} f_{1}\right) \mathrm{d} t
\end{aligned}
$$

and using integration by parts in the first term of the right side, and the fatcs that $\dot{\kappa}(t)=-K(t)$ and $\kappa(0)=1$, we have

$$
\begin{align*}
& \int_{0}^{T} \kappa a\left(u_{h, 1}, A_{h}^{l} u_{h, 2}\right) \mathrm{d} t \\
& =\frac{1}{2}\left(\kappa(T)\left\|u_{h, 1}(T)\right\|_{h, l+1}^{2}-\left\|u_{h, 0}\right\|_{h, l+1}^{2}\right)  \tag{3.9}\\
& \quad+\frac{1}{2} \int_{0}^{T} K\left\|u_{h, 1}\right\|_{h, l+1}^{2} \mathrm{~d} t-\int_{0}^{T} \kappa a\left(u_{h, 1}, A_{h}^{l} \mathcal{R}_{h} f_{1}\right) \mathrm{d} t .
\end{align*}
$$

For the third term of the left side of (3.7), recalling (3.5), we have

$$
\begin{align*}
& \int_{0}^{T} \int_{0}^{t} K(s) a\left(w_{h, 1}(t, s), A_{h}^{l} u_{h, 2}\right) \mathrm{d} s \mathrm{~d} t \\
&= \int_{0}^{T} \int_{0}^{t} K(s) a\left(w_{h, 1}(t, s), A_{h}^{l} \dot{u}_{h, 1}\right) \mathrm{d} s \mathrm{~d} t \\
&-\int_{0}^{T} \int_{0}^{t} K(s) a\left(w_{h, 1}(t, s), A_{h}^{l} \mathcal{R}_{h} f_{1}\right) \mathrm{d} s \mathrm{~d} t \\
&= \frac{1}{2} \int_{0}^{T} \int_{0}^{t} K(s) D_{t}\left\|w_{h, 1}(t, s)\right\|_{h, l+1}^{2} \mathrm{~d} s \mathrm{~d} t  \tag{3.10}\\
&+\frac{1}{2} \int_{0}^{T} \int_{0}^{t} K(s) D_{s}\left\|w_{h, 1}(t, s)\right\|_{h, l+1}^{2} \mathrm{~d} s \mathrm{~d} t \\
&-\int_{0}^{T} \int_{0}^{t} K(s) a\left(w_{h, 1}(t, s), A_{h}^{l} \mathcal{R}_{h} f_{1}\right) \mathrm{d} s \mathrm{~d} t
\end{align*}
$$

where for the last equality we used the fact that $\dot{u}_{h, 1}(t)=D_{t} w_{h, 1}(t, s)+$ $D_{s} w_{h, 1}(t, s)$. In the first term on the right side, we change the order of the integration, and we have

$$
\begin{aligned}
& \frac{1}{2} \int_{0}^{T} \int_{0}^{t} K(s) D_{t}\left\|w_{h, 1}(t, s)\right\|_{h, l+1}^{2} \mathrm{~d} s \mathrm{~d} t \\
& \quad=\frac{1}{2} \int_{0}^{T} K(s)\left\|w_{h, 1}(T, s)\right\|_{h, l+1}^{2} \mathrm{~d} s-\frac{1}{2} \int_{0}^{T} K(s)\left\|w_{h, 1}(s, s)\right\|_{h, l+1}^{2} \mathrm{~d} s
\end{aligned}
$$

Now, using

$$
\frac{1}{2} \int_{0}^{T} K(s)\left\|w_{h, 1}(s, s)\right\|_{h, l+1}^{2} \mathrm{~d} s=\frac{1}{2} \int_{0}^{T} \int_{0}^{t} K(t) D_{s}\left\|w_{h, 1}(t, s)\right\|_{h, l+1}^{2} \mathrm{~d} s \mathrm{~d} t
$$

we can write (3.10) as

$$
\begin{array}{rl}
\int_{0}^{T} \int_{0}^{t} K & K(s) a\left(w_{h, 1}(t, s), A_{h}^{l} u_{h, 2}\right) \mathrm{d} s \mathrm{~d} t \\
= & \frac{1}{2} \int_{0}^{T} K(s)\left\|w_{h, 1}(T, s)\right\|_{h, l+1}^{2} \mathrm{~d} s \\
& +\frac{1}{2} \int_{0}^{T} \int_{0}^{t}(K(s)-K(t)) D_{s}\left\|w_{h, 1}(t, s)\right\|_{h, l+1}^{2} \mathrm{~d} s \mathrm{~d} t  \tag{3.11}\\
& -\int_{0}^{T} \int_{0}^{t} K(s) a\left(w_{h, 1}(t, s), A_{h}^{l} \mathcal{R}_{h} f_{1}\right) \mathrm{d} s \mathrm{~d} t
\end{array}
$$

We show that the second term on the right side in non-negative. To this end, for $0<\epsilon<t$, we have

$$
\begin{aligned}
\int_{\epsilon}^{t} & (K(s)-K(t)) D_{s}\left\|w_{h, 1}(t, s)\right\|_{h, l+1}^{2} \mathrm{~d} s \\
& =-(K(\epsilon)-K(t))\left\|w_{h, 1}(t, \epsilon)\right\|_{h, l+1}^{2}-\int_{\epsilon}^{t} K^{\prime}(s)\left\|w_{h, 1}(t, s)\right\|_{h, l+1}^{2} \mathrm{~d} s \\
& \geq-K(\epsilon)\left\|w_{h, 1}(t, \epsilon)\right\|_{h, l+1}^{2},
\end{aligned}
$$

where we used the facts that $K^{\prime}(s) \leq 0$ and $K(t) \geq 0$ from (1.2). Using

$$
w_{h, 1}(t, \epsilon)=w_{h, 1}(t, 0)+\int_{0}^{\epsilon} D_{s} w_{h, 1}(t, s) \mathrm{d} s=\int_{0}^{\epsilon} D_{s} w_{h, 1}(t, s) \mathrm{d} s,
$$

and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\left\|w_{h, 1}(t, \epsilon)\right\|_{h, l+1}^{2} & \leq\left(\int_{0}^{\epsilon}\left\|D_{s} w_{h, 1}(t, s)\right\|_{h, l+1}\right)^{2} \\
& \leq \int_{0}^{\epsilon} \frac{\mathrm{d} s}{K(s)} \int_{0}^{\epsilon} K(s)\left\|D_{s} w_{h, 1}(t, s)\right\|_{h, l+1}^{2} \mathrm{~d} s
\end{aligned}
$$

and consequently,

$$
\begin{aligned}
\int_{\epsilon}^{t}(K(s)-K(t)) & D_{s}\left\|w_{h, 1}(t, s)\right\|_{h, l+1}^{2} \mathrm{~d} s \\
& \geq-\int_{0}^{\epsilon} \frac{K(\epsilon)}{K(s)} \mathrm{d} s \int_{0}^{\epsilon} K(s)\left\|D_{s} w_{h, 1}(t, s)\right\|_{h, l+1}^{2} \mathrm{~d} s .
\end{aligned}
$$

But, the kernel $K$ is an decreasing function, which implies $\int_{0}^{\epsilon} \frac{K(\epsilon)}{K(s)} \mathrm{d} s \leq$ $\int_{0}^{\epsilon} \mathrm{d} s=\epsilon$ such that

$$
\begin{aligned}
\int_{\epsilon}^{t}(K(s)-K(t)) & D_{s}\left\|w_{h, 1}(t, s)\right\|_{h, l+1}^{2} \mathrm{~d} s \\
& \geq-\epsilon \int_{0}^{\epsilon} K(s)\left\|D_{s} w_{h, 1}(t, s)\right\|_{h, l+1}^{2} \mathrm{~d} s
\end{aligned}
$$

From the framework presented in [1], provided that the data are sufficiently smooth, we have $\int_{0}^{\epsilon} K(s)\left\|D_{s} w_{h, 1}(t, s)\right\|_{h, l+1}^{2} \mathrm{~d} s<\infty$. Therefore, we let $\epsilon \rightarrow 0$ and we have

$$
\begin{equation*}
\int_{\epsilon}^{t}(K(s)-K(t)) D_{s}\left\|w_{h, 1}(t, s)\right\|_{h, l+1}^{2} \mathrm{~d} s \geq 0 \tag{3.12}
\end{equation*}
$$

We put (3.8), (3.9), and (3.11) in (3.7) to obtain the energy identity (3.2). We recall that all terms on the left side are non-negative.

Now, as the final step, we prove the stability estimate (3.4). To this end, from the identity (3.2), we have

$$
\begin{aligned}
\| u_{h, 2}(T) & \left\|_{h, l}^{2}+\kappa(T)\right\| u_{h, 1}(T) \|_{h, l+1}^{2} \\
\leq & \left\|v_{h, 0}\right\|_{h, l}^{2}+\left\|u_{h, 0}\right\|_{h, l+1}^{2} \\
& +2 \int_{0}^{T}\left(\mathcal{P}_{h} f_{2}, A_{h}^{l} u_{h, 2}\right) \mathrm{d} t+2 \int_{0}^{T} \kappa a\left(\mathcal{R}_{h} f_{1}, A_{h}^{l} u_{h, 1}\right) \mathrm{d} t \\
& +2 \int_{0}^{T} \int_{0}^{t} K(t-s) a\left(\mathcal{R}_{h} f_{1}, A_{h}^{l} w_{h, 1}(t, s)\right) \mathrm{d} s \mathrm{~d} t \\
\leq & \left\|v_{h, 0}\right\|_{h, l}^{2}+\left\|u_{h, 0}\right\|_{h, l+1}^{2} \\
& +2 \int_{0}^{T}\left\|\mathcal{P}_{h} f_{2}\right\|_{h, l}\left\|u_{h, 2}\right\|_{h, l} \mathrm{~d} t+2 \int_{0}^{T}\left\|\mathcal{R}_{h} f_{1}\right\|_{h, l+1}\left\|u_{h, 1}\right\|_{h, l+1} \mathrm{~d} t \\
& +2 \int_{0}^{T} \int_{0}^{t} K(t-s)\left\|\mathcal{R}_{h} f_{1}\right\|_{h, l+1}\left\|w_{h, 1}(t, s)\right\|_{h, l+1} \mathrm{~d} s \mathrm{~d} t \\
\leq & \left\|v_{h, 0}\right\|_{h, l}^{2}+\left\|u_{h, 0}\right\|_{h, l+1}^{2} \\
& +C_{1} \max _{0 \leq t \leq T}\left\|u_{h, 2}\right\|_{h, l}^{2}+\frac{1}{C_{1}}\left(\int_{0}^{T}\left\|\mathcal{P}_{h} f_{2}\right\|_{h, l} \mathrm{~d} t\right)^{2} \\
& +C_{2} \max _{0 \leq t \leq T}\left\|u_{h, 1}\right\|_{h, l+1}^{2}+\frac{1}{C_{2}}\left(\int_{0}^{T}\left\|\mathcal{R}_{h} f_{1}\right\|_{h, l+1} \mathrm{~d} t\right)^{2} \\
& +C_{3}(\kappa(T)) \max _{0 \leq t \leq T}\left\|u_{h, 1}\right\|_{h, l+1}^{2}+\frac{1}{C_{3}(\kappa(T))}\left(\int_{0}^{T}\left\|\mathcal{R}_{h} f_{1}\right\|_{h, l+1} \mathrm{~d} t\right)^{2} .
\end{aligned}
$$

Then, in a standard way, we conclude the stability estimate (3.4), and the proof is complete.

Remark 3.2. We used the axuiliary function $w_{h, 1}(t, s)=u_{h, 1}(t)-$ $u_{h, 1}(t-s)$ to obtain the stability estimate (3.4). Another definition, $w_{h, 1}(t, s)=u_{h, 1}(t)-u_{h, 1}(s)$, can also be used to obtain the same estimate. The proof can be adapted from [1] and we omit the details.

## 4. A priori error estimates

Now, by energy methods, we obtain $L_{\infty}([0, T], H)$ and $L_{\infty}([0, T], V)$, optimal order error estimates for the displacement $u_{1}$, and $L_{\infty}([0, T], H)$, optimal order error estimate for the velocity $u_{2}$.

Theorem 4.1. Assume that $\Omega$ is a convex polygonal domain. Let $\boldsymbol{u}=$ $\left(u_{1}, u_{2}\right)$ and $\boldsymbol{u}_{h}=\left(u_{h, 1}, u_{h, 2}\right)$ be, respectively, the solutions of (2.1) and (2.3). Then, for $0 \leq t \leq T$, we have

$$
\begin{align*}
\left\|u_{h, 1}(t)-u_{1}(t)\right\| \leq & C\left(\left\|u_{h, 0}-\mathcal{R}_{h} u_{0}\right\|+\left\|v_{h, 0}-\mathcal{P}_{h} v_{0}\right\|_{-1}\right) \\
& +C h^{r}\left(\left\|u_{1}(t)\right\|_{r}+\int_{0}^{t}\left\|u_{2}\right\|_{r} d \tau\right),  \tag{4.1}\\
\left\|u_{h, 1}(t)-u_{1}(t)\right\|_{1} \leq & C\left(\left\|u_{h, 0}-\mathcal{R}_{h} u_{0}\right\|_{1}+\left\|v_{h, 0}-\mathcal{R}_{h} v_{0}\right\|\right) \\
& +C h^{r-1}\left(\left\|u_{1}(t)\right\|_{r}+\int_{0}^{t}\left\|\dot{u}_{2}\right\|_{r-1} d \tau\right),  \tag{4.2}\\
\left\|u_{h, 2}(t)-u_{2}(t)\right\| \leq & C\left(\left\|u_{h, 0}-\mathcal{R}_{h} u_{0}\right\|_{1}+\left\|v_{h, 0}-\mathcal{R}_{h} v_{0}\right\|\right) \\
& +C h^{r}\left(\left\|u_{2}(t)\right\|_{r}+\int_{0}^{t}\left\|\dot{u}_{2}\right\|_{r} d \tau\right) . \tag{4.3}
\end{align*}
$$

Proof. We split the error $\boldsymbol{u}_{h}-\boldsymbol{u}$ as

$$
\begin{equation*}
\boldsymbol{u}_{h}-\boldsymbol{u}=\boldsymbol{\theta}+\boldsymbol{\rho}=\left(\boldsymbol{u}_{h}-\Pi_{h} \boldsymbol{u}\right)+\left(\Pi_{h} \boldsymbol{u}-\boldsymbol{u}\right), \tag{4.4}
\end{equation*}
$$

where the operator $\Pi_{h}$ is chosen properly in terms of the elliptic and $L_{2}$-projectors $\mathcal{R}_{h}$ and $\mathcal{P}_{h}$, respectively. Due to (2.5), we only need to estimate $\boldsymbol{\theta}$. To this end, using (4.4) and the Galerkin orthogonality
(2.4), we get

$$
\begin{aligned}
& a\left(\dot{\theta}_{1}(t), \chi_{1}\right)-a\left(\theta_{2}(t), \chi_{1}\right)=-a\left(\dot{\rho}_{1}(t), \chi_{1}\right)+a\left(\rho_{2}(t), \chi_{1}\right) \\
& \begin{array}{c}
\left(\dot{\theta}_{2}(t), \chi_{2}\right)+a\left(\theta_{1}(t), \chi_{2}\right)-\int_{0}^{t} K(t-s) a\left(\theta_{1}(s), \chi_{2}\right) \mathrm{d} s \\
=-\left(\dot{\rho}_{2}(t), \chi_{2}\right)-a\left(\rho_{1}(t), \chi_{2}\right)+\int_{0}^{t} K(t-s) a\left(\rho_{1}(s), \chi_{2}\right) \mathrm{d} s, \\
\forall \chi_{1}, \chi_{2} \in V_{h}, t \in(0, T] .
\end{array}
\end{aligned}
$$

To prove the first error estimate (4.1), we choose

$$
\begin{array}{ll}
\theta_{1}=u_{h, 1}-\mathcal{R}_{h} u_{1}, & \rho_{1}=\left(\mathcal{R}_{h}-I\right) u_{1}, \\
\theta_{2}=u_{h, 2}-\mathcal{P}_{h} u_{2}, & \rho_{2}=\left(\mathcal{P}_{h}-I\right) u_{2} .
\end{array}
$$

By the definitions of the operators $\mathcal{R}_{h}$ and $\mathcal{P}_{h}$, we have

$$
\begin{aligned}
& a\left(\dot{\theta}_{1}(t), \chi_{1}\right)-a\left(\theta_{2}(t), \chi_{1}\right)=a\left(\rho_{2}(t), \chi_{1}\right) \\
& \begin{aligned}
\left(\dot{\theta}_{2}(t), \chi_{2}\right)+a\left(\theta_{1}(t), \chi_{2}\right)- & \int_{0}^{t} K(t-s) a\left(\theta_{1}(s), \chi_{2}\right) \mathrm{d} s \\
& =0, \quad \forall \chi_{1}, \chi_{2} \in V_{h}, t \in(0, T]
\end{aligned}
\end{aligned}
$$

that is, $\theta_{1}$ and $\theta_{2}$ satisfy (3.1) with $f_{1}=\rho_{2}, f_{2}=0$. Therefore, we apply stability inequality (3.4) with $l=-1$ to obtain

$$
\left\|\theta_{1}(T)\right\|_{h, 0} \leq C\left(\left\|\theta_{1}(0)\right\|_{h, 0}+\left\|\theta_{2}(0)\right\|_{h,-1}+\int_{0}^{T}\left\|\mathcal{R}_{h} \rho_{2}\right\|_{h, 0} \mathrm{~d} t\right) .
$$

Using (2.6), (2.7), (4.4), and

$$
\left\|\mathcal{R}_{h} \rho_{2}\right\|=\left\|\mathcal{P}_{h}\left(I-\mathcal{R}_{h}\right) u_{2}\right\| \leq\left\|\left(\mathcal{R}_{h}-I\right) u_{2}\right\|,
$$

we have

$$
\begin{aligned}
\left\|u_{h, 1}(T)-u_{1}(T)\right\| \leq & \left\|\left(\mathcal{R}_{h}-I\right) u_{1}(T)\right\| \\
& +C\left(\left\|u_{h, 0}-\mathcal{R}_{h} u_{0}\right\|+\left\|v_{h, 0}-\mathcal{P}_{h} v_{0}\right\|_{-1}\right. \\
& \left.\left.+\int_{0}^{T} \|\left(\mathcal{R}_{h}-I\right) u_{2}\right) \| \mathrm{d} t\right) .
\end{aligned}
$$

This, using (2.5), proves the first estimate in (4.1).
Now, to prove the error estimates of (4.2) and (4.3), we choose

$$
\theta_{i}=u_{h, i}-\mathcal{R}_{h} u_{i}, \quad \rho_{i}=\left(\mathcal{R}_{h}-I\right) u_{i}, \quad i=1,2 .
$$

Then, similar to the previous case,

$$
\begin{aligned}
a\left(\dot{\theta}_{1}(t), \chi_{1}\right)-a\left(\theta_{2}(t), \chi_{1}\right) & =0 \\
\left(\dot{\theta}_{2}(t), \chi_{2}\right)+a\left(\theta_{1}(t), \chi_{2}\right)- & \int_{0}^{t} K(t-s) a\left(\theta_{1}(s), \chi_{2}\right) \mathrm{d} s \\
& =-\left(\dot{\rho}_{2}, \chi_{2}\right), \quad \forall \chi_{1}, \chi_{2} \in V_{h}, t \in(0, T],
\end{aligned}
$$

that is, $\theta_{1}, \theta_{2}$ satisfy (3.1) with $f_{1}=0, f_{2}=-\dot{\rho}_{2}$. Therefore, we apply stability inequality (3.4), but this time with $l=0$, and obtain

$$
\begin{aligned}
\left\|\theta_{1}(T)\right\|_{h, 1} & +\left\|\theta_{2}(T)\right\|_{h, 0} \\
\leq & C\left(\left\|\theta_{1}(0)\right\|_{h, 1}+\left\|\theta_{2}(0)\right\|_{h, 0}+\int_{0}^{T}\left\|\mathcal{P}_{h} \dot{\rho}_{2}\right\|_{h, 0} \mathrm{~d} t\right) .
\end{aligned}
$$

Now, using (2.6), (4.4), and

$$
\left\|\mathcal{P}_{h} \dot{\rho}_{2}\right\|=\left\|\mathcal{P}_{h}\left(\mathcal{R}_{h}-I\right) \dot{u}_{2}\right\| \leq\left\|\left(\mathcal{R}_{h}-I\right) \dot{u}_{2}\right\|,
$$

we have

$$
\begin{aligned}
\left\|u_{h, 1}(T)-u_{1}(T)\right\|_{1} \leq & \left\|\left(\mathcal{R}_{h}-I\right) u_{1}(T)\right\|_{1} \\
& +C\left(\left\|u_{h, 0}-\mathcal{R}_{h} u_{0}\right\|_{1}+\left\|v_{h, 0}-\mathcal{R}_{h} v_{0}\right\|\right. \\
& \left.+\int_{0}^{T}\left\|\left(\mathcal{R}_{h}-I\right) \dot{u}_{2}\right\| \mathrm{d} t\right), \\
\left\|u_{h, 2}(T)-u_{2}(T)\right\| \leq & \left\|\left(\mathcal{R}_{h}-I\right) u_{2}(T)\right\| \\
& +C\left(\left\|u_{h, 0}-\mathcal{R}_{h} u_{0}\right\|_{1}+\left\|v_{h, 0}-\mathcal{R}_{h} v_{0}\right\|\right. \\
& \left.+\int_{0}^{T}\left\|\left(\mathcal{R}_{h}-I\right) \dot{u}_{2}\right\| \mathrm{d} t\right) .
\end{aligned}
$$

Using (2.5), we conclude the error estimates (4.2) and (4.3). The proof is now complete.

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