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APPLICATIONS OF EPI-RETRACTABLE MODULES

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ABSTRACT. An R-module M is called *epi-retractable* if every submodule of M_R is a homomorphic image of M. It is shown that if R is a right perfect ring, then every projective slightly compressible module M_R is epi-retractable. If R is a Noetherian ring, then every epi-retractable right R-module has direct sum of uniform submodules. If endomorphism ring of a module M_R is von-Neumann regular, then M is semi-simple if and only if M is epi-retractable. If R is a quasi Frobenius ring, then R is a right hereditary ring if and only if every injective right R-module is semi-simple. A ring R is semi-simple if and only if R is right hereditary and every epi-retractable right R-module is projective. Moreover, a ring R is semi-simple if and only if R is pri and von-Neumann regular.

1. **Introduction**

All rings are associative with unit elements and all modules are unitary right modules. Let R be a ring. The ring R is said to be a principal right ideal (pri) ring if every right ideal of R is principal. Ghorbani and Vedadi [3] generalized this concept to modules. An R-module M is called epi-retractable if every submodule of M_R is a homomorphic image of M. Therefore, R is a pri ring if and only if R_R is epi-retractable. An

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R-module N is called *M-cyclic* if it is isomorphic to M/L, for some submodule L of M (see [10]). Note that M_R is epi-retractable if and only if every submodule of M is M-cyclic. Here, we shall investigate epi-retractable modules in terms of M-cyclic submodules and also provide those properties of epi-retractable modules which have not been studied earlier.

By [2, 6.9.3], an R-module M is called *compressible* if for every nonzero submodule N of M there exists a monomorphism from M to N. The concept of epi-retractable modules is dual to the concept of compressible modules. There exist some epi-retractable modules which are not compressible. For example, semi-simple modules are epi-retractable but not compressible.

In Section 2, we study two important properties of epi-retractable modules. We observe that every epi-retractable module is a *slightly compressible module* (see [6]), but the converse need not be true. In Theorem 2.2, we provide a sufficient condition for slightly compressible modules to be epi-retractable. We show that if R is a right perfect ring, then every projective slightly compressible module M_R is epi-retractable. This is a well known problem in the theory of rings and modules when a module has direct sum of uniform submodules. In Theorem 2.3, we show that if R is a Noetherian ring, then every epi-retractable right R-module has direct sum of uniform submodules.

In Section 3, we study the semi-simplicity of epi-retractable modules and pri rings. Note that every semi-simple module is epi-retractable, but the converse need not be true. In some results of that section, we provide sufficient conditions for the epi-retractable modules to be semi-simple by injective modules, projective modules, right hereditary rings, von-Neumann regular rings. We show that if endomorphism ring of a module M is von-Neumann regular, then M is semi-simple if and only if M is an epi-retractable module. If R is a quasi Frobenius ring, then R is a right hereditary ring if and only if every injective R-module is semi-simple. We characterize semi-simple rings by epi-retractable modules so that a ring R is semi-simple if and only if R is right hereditary and every epi-retractable R-module is projective. We end up with a result that states: A ring R is semi-simple if and only if R is pri and von-Neumann regular.

We refer to [10] and [1] for all undefined notions used in the text.

2. **Epi-retractable modules**

Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a ring. Then, $M_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $N_R = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$ and $P_R = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ are right R-modules. It is clear that R_R, N_R, P_R and $(M/P)_R$ are epi-retractable R-modules. But, M_R is not an epi-retractable module. Moreover, submodules of an epi-retractable module need not be epi-retractable and also factors of an epi-retractable module need not be epi-retractable. We begin with the observation that the class of epi-retractable modules is closed under direct sums.

Proposition 2.1. *Let $\{M_i\}_{i \in I}$ be a family of epi-retractable modules. Then, $M = \oplus_{i \in I} M_i$ is an epi-retractable module.*

Proof. Let K be a submodule of M. Then, $K \cap M_i$ is a submodule of M_i, for each $i \in I$. Since each M_i is an epi-retractable module, there exists an epimorphism $\alpha_i : M_i \to K \cap M_i$. Define $\alpha = \sum_{i \in I} \alpha_i : M \to K$. Then, clearly α is a surjective homomorphism. Hence, $\oplus_{i \in I} M_i$ is an epi-retractable module. □

A projective R-module P together with a small epimorphism $\pi : P \to M$ is called a *projective cover* of M. A ring R is said to be *right perfect* if every R-module has a projective cover. In [6], Smith calls an R-module M *slightly compressible* if, for every non-zero submodule N of M, there exists a non-zero homomorphism from M to N. An R-module M is called to be *self-generator* if, for each submodule N of M, there exists an index set J and an epimorphism $\theta : M^{(J)} \to N$. It is clear that every epi-retractable module is self-generator. Moreover, every self-generator is slightly compressible. Then, epi-retractable modules are slightly compressible. In general, every slightly compressible module is not a self-generator (see [6, Proposition 3.1]). Therefore, every slightly compressible module need not be epi-retractable.

The following result shows a sufficient condition for slightly compressible modules to be epi-retractable.

Theorem 2.2. *Let R be a right perfect ring. Then, every projective slightly compressible R-module is epi-retractable.*

Proof. Assume that M is a projective and slightly compressible module. Let K be a submodule of M. Since R is right perfect, there is a projective cover P of K with a small $Ker(\pi)$, where $\pi : P \to K$ is an epimorphism.

Then, there exists a non-zero homomorphism $f : M \to K$. Consider the following diagram:

$$\begin{array}{ccc} & & M \\ & h \swarrow & \downarrow f \\ P & \xrightarrow[\pi]{} & K \end{array}$$

Since M is projective, f can be lifted to a homomorphism h from M to P such that the above diagram is commutative, that is, $f = \pi h$. It follows that $P = Im(h) + Ker(\pi)$. Then, $P = Im(h)$, because $Ker(\pi)$ is small. This implies that h is surjective. Therefore, f is also surjective, and hence M is an epi-retractable module. □

A ring R is called a *right V-ring* if every simple R-module is injective. Moreover, if R is a right V-ring, then every projective R-module is slightly compressible (see [6, Theorem 1.5]). Theorem 2.2 has the following consequence.

Corollary 2.3. *Let R be a right perfect and right V-ring. Then, every projective R-module is epi-retractable.*

Proof. This follows from [6, Theorem 1.5] and Theorem 2.2. □

Following [9], an R-module M is called *quasi-polysimple* if every non-zero submodule of M contains a uniform submodule of M. Note that over a Noetherian ring R, every R-module is quasi-polysimple (see [5, Theorem 2.2]).

We shall now investigate when a epi-retractable module has direct sum of uniform submodules.

Theorem 2.4. *Let R be a Noetherian ring. If M is an epi-retractable R-module, then M has direct sum of uniform submodules of M.*

Proof. It is clear that M is quasi-polysimple. Therefore, M is an essential extension of the direct sum $\oplus_{i \in J} K_i$, where each K_i is the uniform submodule of M and J is some index set (see [5, Lemma 2.1]). Since M is epi-retractable, there exists an endomorphism $f \in S$ such that $f(M) = \oplus_{i \in J} K_i$. □

3. **Semi-simplicity of epi-retractable modules**

A ring R is called *right hereditary* if every right ideal is projective. Moreover, R is right hereditary if and only if every submodule of every projective R-module is projective and if and only if quotients of injective, R-modules are injective (see [4, Corollary 2.26] and [4, Theorem 3.22]). There are some modules which are injective, but not epi-retractable. For example, the set of rational numbers Q_Z is an injective module, but is not epi-retractable. Note that every semi-simple module is epi-retractable, but in general the converse is not true.

In the following, we investigate when an epi-retractable module is semi-simple.

Proposition 3.1. *Let R be a right hereditary ring. Then, the followings hold:*

(1) *Every injective epi-retractable R-module is semi-simple.*
(2) *Every projective epi-retractable R-module is semi-simple.*

Proof. (1). Assume that R is a right hereditary ring and K is submodule of an epi-retractable injective R-module M. Since M is epi-retractable, $K \cong M/L$, for some submodule L of M. It follows that K is injective. Suppose I is the identity map from K to K. Therefore, I can be extended to a homomorphism from M to K. Hence, K is a direct summand of M. This implies that M is semi-simple.

(2). This is clear. □

Recall that a ring R is said to be a quasi Frobenius ring if it is a (left) right self injective Noetherian ring. Note that if R is a ring such that every injective R-module is epi-retractable, then R is a quasi Frobenius ring (see [3, Proposition 3.2]). In the following, we characterize right hereditary rings.

Proposition 3.2. *Let R be a quasi Frobenius ring. Then, R is a right hereditary ring if and only if every injective R-module is semi-simple.*

Proof. Assume R is a right hereditary ring. Let M be an injective R-module. By [3, Proposition 3.2], M is an epi-retractable module. By Proposition 3.1, it is clear that M is a semi-simple module.

Conversely, assume that every injective R-module is semi-simple. Suppose that K is the homomorphic image of an injective R-module M. Then, K is a direct summand of M, because M is semi-simple. Therefore, K is also injective. This implies that quotients of injective R-modules are injective. This proves that R is a right hereditary ring. □

Theorem 3.3. *If the endomorphism ring S of a module M is von-Neumann regular, then M is semi-simple if and only if M is an epi-retractable module.*

Proof. Suppose M is an epi-retractable module and K is a submodule of M. Then, there is an epimorphism f from M to K. Since $S = End(M)$ is von-Neumann regular, $f(M) = K$ is a direct summand of M. Hence, M is a semi-simple module. The converse is obvious. □

Let R be a ring and M be an R-module. We denote $r(x) = \{s \in R : xs = 0\}$, for some $x \in M$. Note that $r(x)$ is a right ideal of R and $R/r(x) \cong xR$, for all $x \in M$. In the following, we characterize semi-simple ring.

Theorem 3.4. *A ring R is semi-simple if and only if R is right hereditary and every epi-retractable R-module is projective.*

Proof. Assume that R is a right hereditary ring and every epi-retractable R-module is projective. Let M be a simple R-module. It follows that M is epi-retractable and projective. For any $x \in M$, $xR \cong R/r(x)$. Then, xR (and hence $R/r(x)$) is projective, because R is a right hereditary ring. Therefore, the exact sequence $0 \to r(x) \to R \to R/r(x) \to 0$ splits. This implies that $r(x)$ is a direct summand of M. Since $r(x)$ is a maximal right ideal, R is a semi-simple ring. The converse is obvious. □

An R-module M is said to satisfy $(**)$*-property* if every non-zero endomorphism of M is an epimorphism (see [11]). In general, epi-retractable modules do not satisfy $(**)$-property. For example, Z as Z-module is epi-retractable, but it does not satisfy $(**)$-property. The following result shows that epi-retractable module with $(**)$-property is simple.

Proposition 3.5. *An R-module M is simple if and only if M is epi-retractable with $(**)$-property.*

Proof. Assume that M is epi-retractable with $(**)$-property. Let K be a proper submodule of M. Then, there is an epimorphism $f : M \to K$. This implies that f is a non-zero endomorphism from M to M. Since M satisfies $(**)$-property, $f(M) = M = K$. Hence, M is simple. The converse is obvious. □

Corollary 3.6. *If an R-module M is epi-retractable with $(**)$-property, then $End(M_R)$ is a division ring.*

An R-module M is said to satisfy (*)*-property* if every non-zero endomorphism of M is a monomorphism (see [7]). This is dual to the concept of (**)-property defined earliar.

Proposition 3.7. *Every epi-retractable module with (*)-property is a co-Hopfian module.*

Proof. Straightforward. □

Theorem 3.8. *A ring R is semi-simple if and only if R is a pri and von-Neumann regular ring.*

Proof. Assume that R is a pri ring. Then, every right ideal of ring R is a principal right ideal. This implies that every right ideal is a direct summand of R, because R is von-Neumann regular. It follows by [10, 20.7] that R is a semi-simple ring. □

Proposition 3.9. *Let R be a ring such that every slightly compressible R-module is pseudo-projective. Then, R is a right V-ring if and only if R is a semi-simple ring.*

Proof. Let M be a slightly compressible R-module. Suppose there is a free R-module F with an epimorphism $g : F \to M$. By [6, Theorem 1.5], F is a slightly compressible module. Then, $F \oplus M$ is a slightly compressible module by [6, Proposition 1.4]. Consider the exact sequence $0 \to Ker(g) \xrightarrow{i} F \xrightarrow{g} M \to 0$. This sequence splits by [8, Lemma 1.3]. Therefore, M is a direct summand of F. Hence, M is projective. In particular, every simple R-module is projective. It follows by [10, 20.7] that R is a semi-simple ring. □

Corollary 3.10. *Over a right V-ring R, if every slightly compressible R-module is pseudo-projective, then every R-module is epi-retractable.*

A ring R is called *right semi-artinian* if every non-zero R-module has non-zero socle.

Proposition 3.11. *Let R be a right semi-artinian right V-ring. Then, R is semi-simple if and only if every R-module is pseudo-projective.*

Proof. Assume that over a right semi-artinian right V-ring R, every R-module is pseudo-projective. By [6, Proposition 1.18], every right R-module is slightly compressible. It follows by Proposition 3.9 that R is semi-simple. □

Corollary 3.12. *Over a right semi-artinian right V-ring, every pseudo-projective module is an epi-retractable module.*

Recall that a ring R is *right PP-ring* if every cyclic right ideal of R is projective. A ring R is called a *regular* if for any $a \in R$ there is an element $b \in R$ with $aba = a$. Note that R is regular if and only if every right principal ideal is a direct summand in R (see [10, 3.10]).

Proposition 3.13. *The followings are equivalent for a pri ring R.*

(1) *R is a right PP-ring.*
(2) *R is a right hereditary ring.*
(3) *R is a von-Neumann regular ring.*

Proof. (1) $\Rightarrow$ (2). Straightforward.

(2) $\Rightarrow$ (3). Assume the condition (2). Let L be a principal right ideal of R. Then, L is projective, because R is a right hereditary ring. Suppose $\pi : R \to L$ is an epimorphism and $I : L \to L$ is the identity map. This implies that I can be lifted to a homomorphism f from L to R, that is, $I = \pi f$. It follows that L is a direct summand of R. Hence, R is a von-Neumann regular ring.

(3) $\Rightarrow$ (1). Obvious. □

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