

SOME VECTOR FIELDS ON A RIEMANNIAN MANIFOLD WITH SEMI-SYMMETRIC METRIC CONNECTION

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ABSTRACT. In the first part of our work, some results are given for a Riemannian manifold with semi-symmetric metric connection. In the second part, some special vector fields, such as torse-forming vector fields, recurrent vector fields and concurrent vector fields are examined in this manifold. We obtain some properties of this manifold having the vectors mentioned above.

1. Introduction

Let M be an n -dimensional Riemannian manifold with metric g . We denote the Levi-Civita connection ∇ and another linear connection by $\bar{\nabla}$. The torsion tensor T of M is given by [15]:

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y].$$

If the torsion tensor vanishes, then $\bar{\nabla}$ is said to be symmetric; otherwise, it is non-symmetric. If g is the metric tensor of M such that $\bar{\nabla}g = 0$, then the connection $\bar{\nabla}$ is said to be a metric connection; otherwise, it is non-metric.

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In 1924, Friedman and Schouten [9] introduced the notion of semi-symmetric linear connection on a differentiable manifold. Then, in 1932 Hayden [10] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of semi-symmetric metric connection on a Riemannian manifold was given by Yano [16].

Semi-symmetric metric connection plays an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the earth always facing one definite point, say Jarusalam or Mekka or the North Pole, then this displacement is semi-symmetric and metric [13]. During the mathematical congress in Moscow in 1934, one evening mathematicians invented the “Moscow displacement.” The streets of Moscow are approximately straight lines through the Kremlin and concentric circles around it. If a person walks in the street always facing the Kremlin, then this displacement is semi-symmetric and metric [13].

Let (M, g) be an n -dimensional Riemannian manifold of class C^∞ with the metric tensor g and let ∇ be the Riemannian connection of this manifold. A linear connection $\bar{\nabla}$ on (M, g) is said to be semi-symmetric [9] if the torsion tensor T of the connection $\bar{\nabla}$ satisfies

$$T(X, Y) = w(Y)X - w(X)Y$$

for any vector fields X, Y on M and w is a 1-form associated with the torsion tensor T of the connection $\bar{\nabla}$ given by

$$w(X) = g(X, U),$$

where U is the vector field associated with the 1-form w . The 1-form w is called the associated 1-form of the semi-symmetric connection and the vector field U is called the associated vector field of the connection.

A semi-symmetric connection $\bar{\nabla}$ is called semi-symmetric metric connection [10] if, in addition, it satisfies

$$\bar{\nabla}g = 0.$$

The relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Riemannian connection ∇ of (M, g) is given by [15]

$$\bar{\nabla}_X Y = \nabla_X Y + w(Y)X - g(X, Y)U.$$

In particular, if the 1-form w vanishes identically, then a semi-symmetric metric connection reduces to the Riemannian connection. Riemannian

manifolds with a semi-symmetric metric connection have been also studied by some authors in [16]-[12].

If R and \bar{R} denote the curvature tensors with respect to the connections ∇ and $\bar{\nabla}$, respectively, then we have [16]

$$(1.1) \quad \begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z - \pi(Y, Z)X + \pi(X, Z)Y \\ &\quad - g(Y, Z)AX + g(X, Z)AY, \end{aligned}$$

where π is a tensor field of type (0,2) defined by

$$\pi(X, Y) = (\nabla_X w)(Y) - w(X)w(Y) + \frac{1}{2}w(\rho)g(X, Y)$$

and A is a tensor field of type (1, 1) defined by

$$g(AX, Y) = \pi(X, Y),$$

for any vector fields X and Y .

If the torsion tensor T of the connection $\bar{\nabla}$ on M is

$$T_{ij}^h = \delta_i^h w_j - \delta_j^h w_i,$$

then we have

$$(1.2) \quad \Gamma_{ij}^h = \left\{ \begin{matrix} h \\ ij \end{matrix} \right\} + \delta_i^h w_j - g_{ij}w^h,$$

where $w^h = w_t g^{th}$, with w^h being the contravariant components of the generating vector w_h and

$$(1.3) \quad \bar{\nabla}_j w_i = \nabla_j w_i - w_i w_j + g_{ij}w, \quad w = w^h w_h.$$

By the aid of (1.2) and (1.3), the curvature tensor of M is defined by [9]

$$(1.4) \quad \bar{R}_{ijkh} = R_{ijkh} - g_{ih}\pi_{jk} + g_{jh}\pi_{ik} - g_{jk}\pi_{ih} + g_{ik}\pi_{jh},$$

where

$$(1.5) \quad \pi_{kj} = \nabla_k w_j - w_k w_j + \frac{1}{2}g_{kj}w.$$

Multiplying (1.4) by g^{ih} , it is obtained:

$$\bar{R}_{jk} = R_{jk} - (n - 2)\pi_{jk} - \pi g_{jk}, \quad \pi = \pi_{ih}g^{ih}.$$

Thus, we find the scalar curvature as

$$\bar{R} = R - 2(n - 1)\pi.$$

The purpose of our work here is to introduce some properties of the torsion-forming vector fields and their special cases, the recurrent and concurrent vector fields on a Riemannian manifold with a semi-symmetric metric connection.

The reminder of our work is organized as follows.

In Section 2, we will give some necessary notations and terminologies. Assuming that the associated vectors of these manifolds are torsion-free, recurrent and concurrent, some properties of these manifolds are obtained.

Now, we give some definitions and theorems to be used in the next section.

Definition 1.1. *A Riemannian manifold is said to be a manifold with quasi-constant curvature if its curvature tensor satisfies the condition*

$$(1.6) \quad \begin{aligned} R_{ijkl} &= \alpha(g_{jk}g_{ih} - g_{ik}g_{jh}) \\ &+ \beta(g_{ih}a_ja_k - g_{ik}a_ja_h + g_{jk}a_i a_h - g_{jh}a_i a_k), \end{aligned}$$

where α, β are certain non-zero scalars and the a_k are non-zero covariant vectors [3].

A Riemannian or semi-Riemannian manifold (M, g) ($n = \dim M \geq 2$) is said to be an Einstein manifold if the condition

$$R_{ij} = \frac{R}{n}g_{ij}$$

holds on M , where R_{ij} and R denote the Ricci tensor and the scalar curvature of (M, g) , respectively. Einstein manifolds play important roles in Riemannian Geometry as well as in general theory of relativity. Also, Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensors. For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M, g) realizing the relation

$$(1.7) \quad R_{jk} = ag_{jk} + bv_jv_j,$$

where a and b are certain non-zero scalars [4].

A non-flat Riemannian manifold (M, g) ($n > 2$) is defined to be a quasi-Einstein manifold [4] if its Ricci tensor R_{ij} is not identically zero and satisfies the condition (1.7).

Definition 1.2. *A Riemannian manifold is said to be semisymmetric [18] if its curvature tensor R_{ijk}^h of type (1,3) satisfies the condition*

$$(1.8) \quad \nabla_m \nabla_l R_{ijk}^h - \nabla_l \nabla_m R_{ijk}^h = 0.$$

Contracting h and k in (1.8), we obtain:

$$(1.9) \quad \nabla_m \nabla_l R_{ij} - \nabla_l \nabla_m R_{ij} = 0.$$

A Riemannian manifold is said to be Ricci semisymmetric if the Ricci tensor R_{ij} satisfies the condition (1.9). Again, the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla_l R_{ij} = 0$) as a proper subset. It is clear that every semisymmetric manifold is Ricci semisymmetric.

Definition 1.3. A vector field v in a Riemannian manifold is called torse-forming if it satisfies $\nabla_X v = \phi(X)v + \rho X$, where $X \in TM$, $\phi(X)$, is a linear form and ρ is a function [17]. In local coordinates, it reads as

$$(1.10) \quad \nabla_k v^h = \phi_k v^h + \rho \delta_k^h,$$

where v^h and ϕ_k are the components of v and ϕ , respectively, and δ_k^h is the Kronecker symbol.

A torse forming vector field is called recurrent [13] if $\rho = 0$. We can characterize concurrent vector fields \vec{v} in the following form:

$$\nabla_k v^h = \rho \delta_k^h.$$

Theorem 1.4. The Ricci tensor of the semi-symmetric metric connection $\bar{\nabla}$ is symmetric if and only if the curvature tensor with respect to the connection $\bar{\nabla}$ satisfies one of the following conditions [7]:

$$(i) \bar{R}_{ikjm} = \bar{R}_{jmik} \quad (ii) \bar{R}_{ikjm} + \bar{R}_{kjim} + \bar{R}_{jikm} = 0.$$

Theorem 1.5. The Ricci tensor of a semi-symmetric metric connection $\bar{\nabla}$ with the associated 1-form w is symmetric if and only if w is closed [7].

Theorem 1.6. In order for a Riemannian manifold to admit a semi-symmetric metric connection whose curvature tensor vanishes, it is necessary and sufficient that the Riemannian manifold be conformally flat [16].

Theorem 1.7. A conformally flat quasi-Einstein manifold $(QE)_n$ ($n > 3$) is semisymmetric if and only if the generating vector v_i of the manifold $(QE)_n$ satisfies the condition $\nabla_m \nabla_l v_i = \nabla_l \nabla_m v_i$ [5].

Theorem 1.8. A conformally flat quasi-Einstein manifold ($n > 3$) is semi-symmetric if and only if the sum of the associated scalars is zero [5].

2. Some vector fields on a Riemannian manifold admitting semi-symmetric metric connection

In this section, we shall consider the torse-forming vector field and its special cases, the recurrent and concurrent vector fields, on a Riemannian manifold with a semi-symmetric metric connection.

2.1. Torse-forming vector fields on M . Assume that w is a torse-forming vector field. Then, by the aid of Definition 1.3, we can write

$$(2.1) \quad \nabla_k w_j = \phi_k w_j + \rho g_{jk}.$$

By changing the indices j and k in (2.1) and subtracting (2.1) from the last equation, we find

$$\nabla_k w_j - \nabla_j w_k = \phi_k w_j - \phi_j w_k.$$

Thus, we have the following result.

Theorem 2.1. *Assume that a Riemannian manifold with semi-symmetric metric connection admits a torse-forming vector field w . Then, w is closed if and only if ϕ_k and w_k are collinear.*

Theorem 2.2. *If the generating vector of a conformally flat Riemannian manifold M admitting a semi-symmetric metric connection is a torse-forming vector field with respect to the Levi-Civita connection, then M is either of quasi-constant or of constant curvature ($n > 2$).*

Proof. If we assume that the manifold is conformally flat, then we have $\bar{R}_{ijkh} = 0$. Thus, from (1.4), we get

$$(2.2) \quad R_{ijkh} = g_{ih}\pi_{jk} - g_{jh}\pi_{ik} + g_{jk}\pi_{ih} - g_{ik}\pi_{jh},$$

where $\pi_{jk} = \nabla_j w_k - w_j w_k + \frac{1}{2}g_{jk}w$.

By using (1.5) and (2.1), we find

$$(2.3) \quad \pi_{jk} = \phi_j w_k - w_j w_k + \left(\rho + \frac{1}{2}w\right)g_{jk}.$$

If we put (2.3) in (2.2), we obtain:

$$(2.4) \quad \begin{aligned} R_{ijkh} = & (w + 2\rho)(g_{ih}g_{jk} - g_{jh}g_{ik}) + g_{ih}(\phi_j w_k - w_j w_k) \\ & - g_{jh}(\phi_i w_k - w_i w_k) + g_{jk}(\phi_i w_h - w_i w_h) \\ & - g_{ik}(\phi_j w_h - w_j w_h). \end{aligned}$$

By using the relation $R_{ijkh} = R_{khij}$, as an immediate consequence of (2.4), we get

$$\phi_k w_j - \phi_j w_k = 0.$$

If we take

$$(2.5) \quad \phi_j = \beta w_j,$$

where β is a non-zero scalar function, then from (2.4) and (2.5), we find

$$(2.6) \quad \begin{aligned} R_{ijkh} = & (w + 2\rho)(g_{ih}g_{jk} - g_{jh}g_{ik}) \\ & + (\beta - 1)(g_{ih}w_jw_k - g_{jh}w_iw_k + g_{jk}w_iw_h - g_{ik}w_jw_h), \end{aligned}$$

where $w \neq -2\rho$. By using Definition 1.1, if we take $\beta \neq 1$, in (2.6), it is seen that this manifold is of quasi constant curvature. If $\beta = 1$, then this manifold is of constant curvature. The proof is now complete. \square

Theorem 2.3. *If the generating vector of a conformally flat Riemannian manifold M admitting a semi-symmetric metric connection is a torse-forming vector field with respect to the Levi-Civita connection, then M is either a quasi-Einstein or an Einstein manifold.*

Proof. Multiplying (2.4) by g^{ih} , we get

$$(2.7) \quad R_{jk} = \theta g_{jk} + (n - 2)(\beta - 1)w_jw_k,$$

where $\theta = (n - 2 + \beta)w + 2\rho(n - 1)$. If $\beta = 1$ then from (2.7), it is clear that this manifold is an Einstein manifold. If $\beta \neq 1$ then from (1.7), we can say that this manifold is a quasi-Einstein manifold. \square

2.2. Recurrent vector fields on M . We consider the case of the generating vector field w on M that is recurrent. Thus, by using (1.10), we have

$$(2.8) \quad \nabla_k w_j = \phi_k w_j.$$

By putting (2.8) in (1.5), we find

$$(2.9) \quad \pi_{jk} = \phi_j w_k - w_j w_k + \frac{1}{2}g_{jk}w.$$

Thus, by the aid of (1.4) and (2.9), we obtain the relation between the curvature tensors as

$$(2.10) \quad \begin{aligned} \bar{R}_{ijkh} = & R_{ijkh} - g_{ih}(\phi_j w_k - w_j w_k) + g_{jh}(\phi_i w_k - w_i w_k) \\ & - g_{jk}(\phi_i w_h - w_i w_h) + g_{ik}(\phi_j w_h - w_j w_h) + w(g_{ik}g_{jh} - g_{jk}g_{ih}). \end{aligned}$$

Theorem 2.4. *The curvature tensor of a Riemannian manifold with semi-symmetric metric connection satisfies the following algebraic properties.*

$$\bar{R}_{ijkh} = -\bar{R}_{jikh} = -\bar{R}_{ijhk},$$

Proof. By changing the indices i and j in (2.10), we find

$$\begin{aligned} \bar{R}_{jikh} &= R_{jikh} - g_{jh}(\phi_i w_k - w_i w_k) + g_{ih}(\phi_j w_k - w_j w_k) \\ &\quad - g_{ik}(\phi_j w_h - w_j w_h) + g_{jk}(\phi_i w_h - w_i w_h) + w(g_{jk}g_{ih} - g_{jh}g_{ik}) \\ (2.11) \quad &= -\bar{R}_{ijkh}. \end{aligned}$$

The same result is obtained for the indices h and k as

$$\bar{R}_{ijkh} = -\bar{R}_{ijhk}.$$

Finally, we have

$$\bar{R}_{ijkh} = -\bar{R}_{jikh} = -\bar{R}_{ijhk}.$$

□

Theorem 2.5. *The curvature tensor of a Riemannian manifold with semi-symmetric metric connection satisfies one of the following conditions*

- (i) $\bar{R}_{ijkh} = \bar{R}_{khij}$ or
 - (ii) $\bar{R}_{ijkh} + \bar{R}_{jkih} + \bar{R}_{kijh} = 0$,
- if and only if ϕ_k and w_k are collinear.

Proof. By using the equation (2.8) and remembering that ϕ_k and w_k are collinear, it is clear that w_k is the gradient. Applying Theorem 1.1 and Theorem 1.2 the proof is completed. □

Theorem 2.6. *If a conformally flat Riemannian manifold admits a semi-symmetric metric connection whose non-null generating vector field is recurrent, then the manifold is Ricci semisymmetric ($n > 3$).*

Proof. By using the equation (2.11) and Theorem 1.3, we find

$$\begin{aligned} R_{khij} &= g_{jk}(\phi_h w_i - w_h w_i) - g_{hj}(\phi_k w_i - w_k w_i) + g_{hi}(\phi_k w_j - w_k w_j) \\ (2.12) \quad &\quad - g_{ki}(\phi_h w_j - w_h w_j) - w(g_{ik}g_{hj} - g_{ih}g_{jk}). \end{aligned}$$

Remembering $R_{khij} = R_{ijkh}$ and using (2.12), we obtain

$$(2.13) \quad (n-2)(\phi_j w_k - \phi_k w_j) = 0.$$

Since $n > 3$, we get from (2.13),

$$(2.14) \quad \phi_j = \beta w_j,$$

where β is a non-zero scalar function and $\beta \neq 1$. Multiplying the equation (2.12) by g^{jk} and using (2.14), we obtain:

$$(2.15) \quad R_{ih} = (n + \beta - 2)w g_{ih} + (n - 2)(\beta - 1)w_h w_i.$$

In the case $n + \beta \neq 2$, from (2.15), we can say that a conformally flat Riemannian manifold admitting a semi-symmetric metric connection with its generating vector field being recurrent is a quasi-Einstein manifold.

On the other hand, taking the covariant derivative of (2.8) and using (2.14), we find

$$(2.16) \quad \nabla_k \nabla_j w_i = w_i \nabla_k \phi_j + \phi_j \phi_k w_i.$$

By changing the indices j and k in (2.16) and subtracting the last equation from (2.16), we get

$$(2.17) \quad \nabla_k \nabla_j w_i - \nabla_j \nabla_k w_i = (\nabla_k \phi_j - \nabla_j \phi_k) w_i.$$

Applying the Ricci identity to (2.17), we have

$$(2.18) \quad R_{kjih} w^h = (\nabla_j \phi_k - \nabla_k \phi_j) w^i.$$

Transvecting (2.18) with w^i , we obtain:

$$(2.19) \quad (\nabla_j \phi_k - \nabla_k \phi_j) w = 0.$$

If w is non-null, then $w \neq 0$. Hence, from (2.19), we get

$$(2.20) \quad (\nabla_j \phi_k - \nabla_k \phi_j) = 0,$$

which means that ϕ_j is a gradient. With the help of (2.20), (2.17) reduces to

$$\nabla_k \nabla_j w_i = \nabla_j \nabla_k w_i.$$

By the aid of Theorem 1.7, the proof is complete. \square

Remark 2.7. *In the case $\beta = 1$ in (2.14), from (2.12), it is clear that a conformally flat Riemannian manifold admitting semi-symmetric metric connection with its generating vector being recurrent is of a constant curvature. Moreover, if we remember that the scalar curvature of this manifold being constant for $n \geq 3$, then we get $w = w_m w^m \equiv \text{const} \neq 0$. Taking the covariant derivative of the last equation and using (2.8), for $\phi_k \neq 0$, we obtain that $w = 0$. Since this is not possible, thus it must be $\beta \neq 1$.*

2.3. Concurrent vector fields on M . Here, we consider that the dimension of M is greater than 3 and assume that 1-form w is concurrent. Thus, from Definition 1.3, we get

$$(2.21) \quad \nabla_k w_j = \rho g_{jk}.$$

In this case, with the help of (1.5) and (2.21), we find

$$(2.22) \quad \pi_{jk} = Ag_{jk} - w_j w_k, \quad A = \rho + \frac{1}{2}w.$$

By putting (2.22) in (1.4), we get

$$(2.23) \quad \begin{aligned} \bar{R}_{ijkh} &= R_{ijkh} - g_{ih}(Ag_{jk} - w_j w_k) + g_{jh}(Ag_{ik} - w_i w_k) \\ &\quad - g_{jk}(Ag_{ih} - w_i w_h) + g_{ik}(Ag_{jh} - w_j w_h). \end{aligned}$$

Thus, multiplying (2.23) by g^{ih} , we find

$$(2.24) \quad \bar{R}_{jk} = R_{jk} - (2(n-1)\rho + (n-2)w)g_{jk} + (n-2)w_j w_k,$$

and multiplying (2.24) by g^{jk} , the scalar curvature can be found to be

$$(2.25) \quad \bar{R} = R - 2n(n-1)\rho - (n-1)(n-2)w.$$

Theorem 2.8. *A conformally flat Riemannian manifold with semi-symmetric metric connection and with positive definite metric cannot admit a concurrent generating vector field.*

Proof. If we assume that M is conformally flat, then from Theorem 1.6, we have $\bar{R}_{jk} = 0$. In this case, if we put this result in (2.25), we find

$$(2.26) \quad R_{jk} = ag_{jk} + bw_j w_k,$$

where

$$(2.27) \quad a = 2(n-1)\rho + (n-2)w \neq 0, \quad b = 2 - n.$$

By using (2.26), we can see that M is a quasi-Einstein manifold. From (2.21), we have

$$(2.28) \quad \nabla_m \nabla_k w_j = \nabla_k \nabla_m w_j.$$

With the help of (2.28) and Theorem 1.7, it is clear that a conformally flat Riemannian manifold with semi-symmetric metric connection is semisymmetric.

From Theorem 1.8 and the equations (2.26) and (2.27), we find

$$(2.29) \quad 2(n-1)\rho + (n-2)(w-1) = 0.$$

By taking the covariant derivative of (2.29) and remembering that ρ is a constant and $w = w_h w^h$, we find $w_k = 0$. Thus, a Riemannian manifold with semi-symmetric metric connection reduces to a Riemannian manifold. \square

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