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AN ITERATIVE METHOD FOR AMENABLE SEMIGROUP AND INFINITE FAMILY OF NON EXPANSIVE MAPPINGS IN HILBERT SPACES

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Abstract. We introduce an iterative method for amenable semigroup of non expansive mappings and infinite family of non expansive mappings in the frame work of Hilbert spaces. We prove the strong convergence of the proposed iterative algorithm to the unique solution of a variational inequality, which is the optimality condition for a minimization problem. The results presented here mainly extend the corresponding results announced by Qin et al. and several others.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Recall that a mapping T of C into itself is called non-expansive if $\| Tx - Ty \| \leq \| x - y \|$, for all $x, y \in C$. By $Fix(T)$, we denote the set of fixed points of T, i.e., $Fix(T) = \{x \in H : Tx = x\}$. It is well known that $Fix(T)$ is closed convex. Recall also that a self-mapping $f\colon C \to C$ is a contraction on C if there is a constant $\alpha \in (0,1)$ such that

$$\| f(x) - f(y) \| \leq \alpha \| x - y \|, \qquad \forall x, y \in C.$$

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Assume A is strongly positive, that is, there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma} \parallel x \parallel^2, \qquad \forall x \in H.$$

Iterative methods for non-expansive mappings have recently been applied to solve convex minimization problems (see [6, 12, 19, 22, 28] and the references therein). A typical problem is to minimize a quadratic function over the set of fixed points of a non-expansive mapping on a Hilbert space H:

$$\min_{x\in D} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where D is the fixed point set of a non-expansive mapping T and b is a given point in H.

Mann [11] introduced an iteration procedure for approximation of fixed points of a non-expansive mapping T on a Hilbert space as follows. Let $x_0 \in H$ and

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n x_n, \qquad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. See also Halpern [7].
On the other hand, Moudafi [13] introduced the viscosity approximation method for fixed points of non-expansive mappings (see [25] for further developments in both Hilbert and Banach spaces). Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n), \qquad n \geq 0, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. It is proved in [13, 25] that, under appropriate conditions imposed on α_n, the sequence $\{x_n\}$, generated by (1.1), converges strongly to the unique solution x^* in $Fix(T)$ of the variational inequality

$$\langle (I - f)x^*, x - x^*\rangle \geq 0, \qquad x \in Fix(T).$$

In [25] (see also [26]), it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \qquad n \geq 0, \tag{1.2}$$

converges strongly to the unique solution of the minimization problem

$$\min_{x\in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle,$$

provided that the sequence $\{\alpha_n\}$ satisfies certain conditions. Marino and Xu [12] combined the iterative (1.2) with the viscosity approximation method (1.1) and considered the following general iterative methods:

$$(1.3) \qquad x_0 \in H, \qquad x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \qquad n \geq 0,$$

where $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. They proved that if $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying certain conditions, then the sequence $\{x_n\}$ generated by (1.3) converges strongly, as $n \to \infty$, to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \qquad \forall x \in Fix(T),$$

which is the optimality condition for the minimization problem

$$(1.4) \qquad \min_{x\in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$, for all $x \in H$).

In 2005, Kim and Xu [8] introduced the following iterative process:

$$(1.5) \qquad \begin{cases} x_0 \in C, \quad \textit{arbitrarily chosen}, \\ y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \qquad n \geq 0, \end{cases}$$

where C is a nonempty closed convex subset of H, T is a non-expansive mapping of C into itself, and $u \in C$ is a given point. They proved that the sequence $\{x_n\}$ defined by (1.5) converges strongly to a fixed point of T provided that the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

A family of non-expansive mappings has been considered by many authors (see [2, 4, 5, 6, 7, 17, 19, 22, 27, 28] and references therein). Recently, Shang et al. [19] improved the results of Kim and Xu [8] from a single mapping to a finite family of mappings in the framework of Hilbert spaces. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of non-expansive mappings of C into itself and let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence of nonnegative real numbers in $[0,1]$. For each $n \geq 1$, define a mapping W_n of C into itself as follows:

$$
\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
&U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
&.\\
&.\\
&.\\
&U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
&U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&.\\
&.\\
&.\\
&U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
&W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{1.6}
$$

Such a mapping W_n is called the $W-$mapping generated by $T_1, T_2, \cdots, T_n$ and $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then, Qin et al. [16] proved the following strong convergence theorem.

Theorem 1.1. *Let H be a real Hilbert space and f be a contraction on H with coefficient $\alpha \in (0,1)$. Let A be a strongly positive linear bounded self-adjoint operator on H with coefficient $\overline{\gamma} > 0$ and $\{T_i\}_{i=1}^{\infty}$ be a sequence of non-expansive mappings of H into itself. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$ and $\mathcal{F} = \bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the composite iteration process:*

$$
\begin{cases}
x_1 = x \in H, \quad \textit{arbitrarily chosen},\\
z_n = \gamma_n x_n + (1-\gamma_n) W_n x_n,\\
y_n = \beta_n \gamma f(z_n) + (I - \beta_n A) z_n,\\
x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \qquad n \geq 1,
\end{cases} \tag{1.7}
$$

where $\{W_n\}$ is a sequence defined by (1.6), $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0,1]$. If the conditions

(A_1) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,
(A_2) $\sum_{n=0}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$,
(A_3) $\lim_{n\to\infty} \mid \gamma_{n+1} - \gamma_n \mid = 0$,
(A_4) *there exists a constant $\lambda \in [0,1)$ such that $\gamma_n \leq \lambda$ for all $n \geq 1$,*

are satisfied, then the sequence $\{x_n\}$ *converges strongly to* $x^* \in \mathcal{F}$*, which also uniquely solves the variational inequality*

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \mathcal{F}.$$

Here, motivated and inspired by Kim and Xu [8], Qin et al. [16], Atsushiha and Takahashi [1], Lau et al. [9], Marino and Xu [12] and Saeidi [18], we introduce a composite iteration scheme as follows:

$$\text{(1.8)} \qquad \begin{cases} x_1 = x \in H, \quad \textit{arbitrarily chosen}, \\ z_n = \gamma_n x_n + (1-\gamma_n) T_{\mu_n} W_n x_n, \\ y_n = \beta_n \gamma f(z_n) + (I - \beta_n A) z_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \qquad n \geq 1, \end{cases}$$

where $\{W_n\}$ is a sequence defined by (1.6), f is a contraction on H with coefficient $\alpha \in (0,1)$, A is a strongly positive linear bounded self-adjoint operator on H with coefficient $\overline{\gamma} > 0$, γ is a positive real number such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$, $\varphi = \{T_t : t \in S\}$ is a non-expansive semigroup on H such that $Fix(\varphi) = \bigcap_{t \in S} Fix(T_t) \neq \emptyset$, X is a subspace of the space of all bounded real valued functions defined on S such that $1 \in X$ the mapping $t \to \langle T_t(x), y \rangle$ is an element of X for each $x, y \in H$, and $\{\mu_n\}$ is a sequence of means on X. Our purpose here is to introduce this general iterative algorithm for approximating the common fixed points of left amenable semigroup of non-expansive mappings and an infinite family of non-expansive mappings which also solve some variational inequalities, while being the optimality conditions for the convex minimization problem (1.4). Our results improve and extend the corresponding ones announced by Qin et al. [16] and many others.

2. **Preliminaries**

Let S be a semigroup and let $B(S)$ be the space of all bounded real valued functions defined on S with the supremum norm. For $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ in $B(S)$ by

$$(l_s f)(t) = f(st), \qquad (r_s f)(t) = f(ts), \qquad \forall t \in S.$$

Let X be a subspace of $B(S)$ containing 1 and let X^* be its dual. An element μ in X^* is said to be a mean on X if $\| \mu \| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$, for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp., right invariant), i.e., $l_s(X) \subset X$ (resp., $r_s(X) \subset X$), for each $s \in S$. A mean μ on X is said to be left invariant (resp., right invariant) if $\mu(l_s f) = \mu(f)$ (resp., $\mu(r_s f) = \mu(f)$), for each $s \in S$ and $f \in X$. X is said to be left (resp., right) amenable if X has a left (resp., right) invariant mean. X is amenable if X is both left and right amenable. As is well known, $B(S)$ is amenable when S is a commutative semigroup; see [10]. A net $\{\mu_\alpha\}$ of means on X is said to be strongly left regular if

$$\lim_{\alpha} \| l_s{}^*\mu_\alpha - \mu_\alpha \| = 0,$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let C be a nonempty closed and convex subset of a reflexive Banach space E. A family $\varphi = \{T_t : t \in S\}$ of mappings from C into itself is said to be a non-expansive semigroup on C if T_t is non-expansive and $T_{ts} = T_t T_s$, for each $t, s \in S$. We denote by $Fix(\varphi)$ the set of common fixed points of φ, i.e.,

$$Fix(\varphi) = \bigcap_{t \in S} \{x \in C : T_t(x) = x\}.$$

Lemma 2.1. [10, 15] *Let f be a function of semigroup S into a reflexive Banach space E such that the weak closure of $\{f(t) : t \in E\}$ is weakly compact and let X be a subspace of $B(S)$ containing all functions $t \to \langle f(t), x^* \rangle$ with $x^* \in E^*$. Then, for any $\mu \in X^*$, there exists a unique element f_μ in E such that*

$$\langle f_\mu, x^* \rangle = \mu_t \langle f(t), x^* \rangle,$$

for all $x^* \in E^*$. Moreover, if μ is a mean on X, then

$$\int f(t) d\mu(t) \in \overline{co}\{f(t) : t \in S\}.$$

We can write f_μ by $\int f(t) d\mu(t)$.

Lemma 2.2. [10, 15] *Let C be a closed convex subset of a Hilbert space H, $\varphi = \{T_t : t \in S\}$ be semigroup from C into C such that $F(\varphi) \neq \emptyset$ and the mapping $t \to \langle T_t(x), y \rangle$ be an element of X for each $x \in C$ and $y \in H$, and μ be a mean on X. If we write $T_\mu(x)$, instead of $\int T_t(x) d\mu(t)$, then the followings hold:*

(i) *T_μ is non-expansive mapping from C into C.*
(ii) *$T_\mu(x) = x$, for each $x \in Fix(\varphi)$.*
(iii) *$T_\mu(x) \in \overline{co}\{T_t(x) : t \in S\}$, for each $x \in C$.*
(iv) *If μ is left invariant, then T_μ is a non-expansive retraction from C onto $Fix(\varphi)$.*

Let C be a nonempty subset of a Hilbert space H and $T\colon C \to H$ be a mapping. T is said to be demiclosed at $v \in H$ if, for any sequence $\{x_n\}$ in C, the following implication holds:

$$x_n \rightharpoonup u \in C \qquad \textit{and} \qquad Tx_n \to v \qquad \textit{imply} \qquad Tu = v,$$

where $\to$ (respectively $\rightharpoonup$) denotes strong (respectively, weak) convergence.

Lemma 2.3. [15] *Let C be a nonempty closed convex subset of a Hilbert space H and suppose that $T\colon C \to H$ is non-expansive. Then, the mapping $I - T$ is semiclosed at zero.*

Let C be a nonempty subset of a normed space E and let $x \in E$. An element $y_0 \in C$ is said to be the best approximation to x if

$$\| x - y_0 \| = d(x, C),$$

where $d(x, C) = \inf_{y \in C} \| x - y \|$. The number $d(x, C)$ is called the distance from x to C or the error in approximating x by C. The (possibly empty) set of all best approximations from x to C is denoted by

$$P_C(x) = \{y \in C : \| x - y \| = d(x, C)\}.$$

This defines a mapping P_C from X into 2^C and is called metric (nearest point) projection onto C. It is well-known that P_C is a non-expansive mapping of H onto C.

Lemma 2.4. [27] *Let C be a nonempty convex subset of a Hilbert space H and P_C be the metric projection mapping from H onto C. Let $x \in H$ and $y \in C$. Then, the followings are equivalent:*

(i) $y = P_C(x)$.
(ii) $\langle x - y, y - z)\rangle \geq 0, \qquad \forall z \in C.$

Lemma 2.5. [21] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\alpha_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Suppose $x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n$, for all integers $n \geq 0$, and*

$$\limsup_{n\to\infty} (\| y_{n+1} - y_n \| - \| x_{n+1} - x_n \|) \leq 0.$$

Then, $\lim_{n\to\infty} \| y_n - x_n \| = 0.$

The following lemma is well known.

Lemma 2.6. *Let H be a real Hilbert space. Then, for all $x, y \in H$, $\| x - y \|^2 \leq \| x \|^2 + 2\langle y, x + y\rangle$.*

Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of non-expansive mappings of C into itself, where C is a nonempty closed convex subset of a real Hilbert space H. Given a sequence $\{\lambda_i\}_{i=1}^{\infty}$ in $[0,1]$, we define a sequence $\{W_n\}_{n=1}^{\infty}$ of self mappings on C by (1.6). Then, we have the following results.

Lemma 2.7. [20] *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i : C \to C\}$ be an infinite family of non-expansive mappings with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$, $\{\lambda_i\}$ be a real sequence such that $0 < \lambda_i \leq b < 1, \forall i \geq 1$. Then,*

(1) *W_n is non-expansive and $Fix(W_n) = \bigcap_{i=1}^{n} Fix(T_i)$ for each $n \geq 1$,*
(2) *for each $x \in C$ and for each positive integer j, $\lim_{n\to\infty} U_{n,j}x$ exists.*
(3) *The mapping $W : C \to C$, defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C,$$

is a non-expansive mapping satisfying $Fix(W) = \bigcap_{i=1}^{\infty} Fix(T_i)$ and is called the W-mapping generated by $T_1, T_2, \cdots$ and $\lambda_1, \lambda_2, \cdots$.

Lemma 2.8. [29] *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i : C \to C\}$ be a countable family of non-expansive mappings with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$, $\{\lambda_i\}$ be a real sequence such that $0 < \lambda_i \leq b < 1, \forall i \geq 1$. If D is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in D} \| Wx - W_n x \| = 0.$$

Lemma 2.9. [23] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n)a_n + b_n c_n, \qquad n \geq 0,$$

where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $\{b_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} b_n = \infty$,
(ii) *either* $\limsup_{n\to\infty} c_n \leq 0$ *or* $\sum_{n=0}^{\infty} |b_n c_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10. [12] *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with the coefficient $\overline{\gamma} > 0$ and* $0 < \rho < \| A \|^{-1}$. *Then,* $\| I - \rho A \| \leq 1 - \rho\overline{\gamma}$.

Lemma 2.11. [14] *Let $(E, \langle ., . \rangle)$ be an inner product space. Then, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ such that $\alpha + \beta + \gamma = 1$, we have*

$$\begin{aligned} &\| \alpha x + \beta y + \gamma z \|^2 \\ &= \alpha \| x \|^2 + \beta \| y \|^2 + \gamma \| z \|^2 \\ &\quad - \alpha\beta \| x - y \|^2 - \alpha\gamma \| x - z \|^2 - \beta\gamma \| y - z \|^2 . \end{aligned}$$

Notation Throughout the rest of this paper, the open ball of radius r centered at 0 is denoted by B_r. For a subset A of H, we denote by $\overline{co}A$ the closed convex hull of A. For $\epsilon > 0$ and a mapping $T\colon D \to H$, we let $F_\epsilon(T; D)$ be the set of $\epsilon-$approximate fixed points of T, i.e., $F_\epsilon(T; D) = \{x \in D : \| x - Tx \| \leq \epsilon\}$. Weak convergence is denoted by $\rightharpoonup$ and strong convergence is denoted by $\to$.

3. Strong convergence

Now, we are ready to give our main results.

Theorem 3.1. *Let H be a real Hilbert space and f be a contraction on H with coefficient $\alpha \in (0, 1)$. Let A be a strongly positive linear bounded self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Let S be a semigroup and $\varphi = \{T_t : t \in S\}$ be a non-expansive semigroup from H into H such that $Fix(\varphi) = \bigcap_{t \in S} Fix(T_t) \neq \emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \to \langle T_t x, y \rangle$ be an element of X for each $x, y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on X such that $\lim_{n\to\infty} \| \mu_{n+1} - \mu_n \| = 0$. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of non-expansive mappings of H into itself such that $\mathcal{F} = \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap Fix(\varphi) \neq \emptyset$ and $T_i(Fix(\varphi)) \subset Fix(\varphi)$, for all $i \in \mathbb{N}$. Let $\{x_n\}$ be a sequence generated by the composite iteration process*

$$\left\{ \begin{array}{l} x_1 = x \in H, \quad \textit{arbitrarily chosen}, \\ z_n = \gamma_n x_n + (1 - \gamma_n) T_{\mu_n} W_n x_n, \\ y_n = \beta_n \gamma f(z_n) + (I - \beta_n A) z_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n, \qquad n \geq 1, \end{array} \right. \tag{3.1}$$

where $\{W_n\}$ is a sequence defined by (1.6), $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$. If the conditions

(A_1) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

(A_2) $\sum_{n=0}^{\infty}\beta_n=\infty,\ \lim_{n\to\infty}\beta_n=0,$
(A_3) $\lim_{n\to\infty}\mid\gamma_{n+1}-\gamma_n\mid=0,$
(A_4) *there exists a constant* $\lambda\in[0,1)$ *such that* $\gamma_n\leq\lambda$, *for all* $n\geq1$,

are satisfied, then the sequence $\{x_n\}$ *converges strongly to* $x^*\in\mathcal{F}$, *which also uniquely solves the variational inequality*

$$\langle(\gamma f-A)x^*,x-x^*\rangle\leq0,\quad\forall x\in\mathcal{F}. \tag{3.2}$$

Proof. We shall divide the proof into several steps.
Step 1. The sequence $\{x_n\}$ is bounded.
Proof of Step 1. Let $x^*\in\mathcal{F}$. Using Lemma 2.2, we have

$$\begin{aligned}\parallel z_n-x^*\parallel&\leq\gamma_n\parallel x_n-x^*\parallel+(1-\gamma_n)\parallel T_{\mu_n}W_nx_n-x^*\parallel\\&\leq\gamma_n\parallel x_n-x^*\parallel+(1-\gamma_n)\parallel x_n-x^*\parallel=\parallel x_n-x^*\parallel.\end{aligned}\tag{3.3}$$

From the condition (A_1), we may assume, with no loss of generality, that $\beta_n<\parallel A\parallel^{-1}$, for all $n\geq1$. From Lemma 2.10, we know that

$$\begin{aligned}&\parallel y_n-x^*\parallel\\&\quad=\parallel\beta_n(\gamma f(z_n)-Ax^*)+(I-\beta_nA)(z_n-x^*)\parallel\\&\quad\leq\beta_n\parallel\gamma f(z_n)-Ax^*\parallel+\parallel I-\beta_nA\parallel\parallel z_n-x^*\parallel\\&\quad\leq\beta_n\gamma\parallel f(z_n)-f(x^*)\parallel+\beta_n\parallel\gamma f(x^*)-Ax^*\parallel\\&\qquad+(1-\beta_n\overline{\gamma})\parallel z_n-x^*\parallel\\&\quad\leq[1-\beta_n(\overline{\gamma}-\gamma\alpha)]\parallel x_n-x^*\parallel+\beta_n\parallel\gamma f(x^*)-Ax^*\parallel.\end{aligned}$$

It follows that

$$\begin{aligned}&\parallel x_{n+1}-x^*\parallel\\&\leq\alpha_n\parallel x_n-x^*\parallel+(1-\alpha_n)\parallel y_n-x^*\parallel\\&\leq\alpha_n\parallel x_n-x^*\parallel+(1-\alpha_n)[(1-\beta_n(\overline{\gamma}-\gamma\alpha))\parallel x_n-x^*\parallel\\&\quad+\beta_n\parallel\gamma f(x^*)-Ax^*\parallel]\\&=[1-\beta_n(\overline{\gamma}-\gamma\alpha)(1-\alpha_n)]\parallel x_n-x^*\parallel+\beta_n(1-\alpha_n)\parallel\gamma f(x^*)-Ax^*\parallel.\end{aligned}$$

By a simple induction, we have

$$\parallel x_n-x^*\parallel\leq\max\left\{\parallel x_1-x^*\parallel,\frac{1}{\overline{\gamma}-\gamma\alpha}\parallel\gamma f(x^*)-Ax^*\parallel\right\}=M_0,$$

which gives that the sequence $\{x_n\}$ is bounded and also that $\{z_n\}$ and $\{y_n\}$ are bounded.

Step 2. $\lim_{n\to\infty} \| x_n - T_{\mu_n} W_n x_n \| = 0$ and $\lim_{n\to\infty} \| y_n - T_{\mu_n} W_n y_n \| = 0$.
Proof of Step 2. It follows from (3.1) that

$$\begin{aligned} &z_{n+1} - z_n \\ &\quad = \gamma_{n+1}[x_{n+1} - x_n] + (1-\gamma_{n+1})[T_{\mu_{n+1}} W_{n+1} x_{n+1} - T_{\mu_n} W_n x_n] \\ &\quad\quad + (\gamma_{n+1} - \gamma_n)[x_n - T_{\mu_n} W_n x_n]. \end{aligned}$$

This implies that

$$\begin{aligned} &\| z_{n+1} - z_n \| \\ &\quad = \gamma_{n+1} \| x_{n+1} - x_n \| + | \gamma_{n+1} - \gamma_n | \| x_n - T_{\mu_n} W_n x_n \| \\ &\quad\quad + (1-\gamma_{n+1}) \| T_{\mu_{n+1}} W_{n+1} x_{n+1} - T_{\mu_n} W_n x_n \| . \end{aligned} \tag{3.4}$$

Since W_n and T_{μ_n} are non-expansive, we have

$$\begin{aligned} &\| T_{\mu_{n+1}} W_{n+1} x_{n+1} - T_{\mu_n} W_n x_n \| \\ &\quad \leq \| T_{\mu_{n+1}} W_{n+1} x_{n+1} - T_{\mu_{n+1}} W_n x_n \| \\ &\quad\quad + \| T_{\mu_{n+1}} W_n x_n - T_{\mu_n} W_n x_n \| \\ &\quad \leq \| W_{n+1} x_{n+1} - W_n x_n \| + \| T_{\mu_{n+1}} W_n x_n - T_{\mu_n} W_n x_n \| \\ &\quad \leq \| W_{n+1} x_{n+1} - W_{n+1} x_n \| + \| W_{n+1} x_n - W_n x_n \| \\ &\quad\quad + \| T_{\mu_{n+1}} W_n x_n - T_{\mu_n} W_n x_n \| \\ &\quad \leq \| x_{n+1} - x_n \| + \| W_{n+1} x_n - W_n x_n \| \\ &\quad\quad + \| T_{\mu_{n+1}} W_n x_n - T_{\mu_n} W_n x_n \| . \end{aligned}$$

Since T_i and $U_{n,i}$ are non-expansive, from (1.6), we have

$$\begin{aligned} &\| W_{n+1} x_n - W_n x_n \| \\ &\quad = \| \lambda_1 T_1 U_{n+1,2} x_n + (1-\lambda_1) x_n - \lambda_1 T_1 U_{n,2} x_n - (1-\lambda_1) x_n \| \\ &\quad \leq \lambda_1 \| U_{n+1,2} x_n - U_{n,2} x_n \| \\ &\quad = \lambda_1 \| \lambda_2 T_2 U_{n+1,3} x_n + (1-\lambda_2) x_n - \lambda_2 T_2 U_{n,3} y_n - (1-\lambda_2) x_n \| \\ &\quad \leq \lambda_1 \lambda_2 \| U_{n+1,3} x_n - U_{n,3} x_n \| \\ &\quad \leq \lambda_1 \lambda_2 \cdots \lambda_n \| U_{n+1,n+1} x_n - U_{n,n+1} x_n \| \\ &\quad \leq M_1 \prod_{i=1}^{n} \lambda_i, \end{aligned}$$

where $M_1 \geq 0$ is an appropriate constant such that
$\| U_{n+1,n+1}x_n - U_{n,n+1}x_n \| \leq M_1$ for all $n \geq 1$. Therefore, we have

$$\begin{aligned} &\| T_{\mu_{n+1}} W_{n+1} x_{n+1} - T_{\mu_n} W_n x_n \| \\ &\leq \| x_{n+1} - x_n \| + M_1 \prod_{i=1}^{n} \lambda_i + \| T_{\mu_{n+1}} W_n x_n - T_{\mu_n} W_n x_n \| . \end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4), we have

$$\begin{aligned} &\| z_{n+1} - z_n \| \\ &\quad \leq \| x_{n+1} - x_n \| + | \gamma_{n+1} - \gamma_n | \| x_n - T_{\mu_n} W_n x_n \| \\ &\qquad + (1 - \gamma_{n+1})[M_1 \prod_{i=1}^{n} \lambda_i + \| T_{\mu_{n+1}} W_n x_n - T_{\mu_n} W_n x_n \|]. \end{aligned} \tag{3.6}$$

On other the hand, we have

$$\begin{aligned} &\| y_{n+1} - y_n \| \\ &\quad = \| \beta_{n+1}\gamma[f(z_{n+1}) - f(z_n)] + (I - \beta_{n+1}A)(z_{n+1} - z_n) \\ &\qquad + (\beta_{n+1} - \beta_n)[\gamma f(z_n) - Az_n] \| \\ &\quad \leq [1 - \beta_{n+1}(\overline{\gamma} - \gamma\alpha)] \| z_{n+1} - z_n \| \\ &\qquad + | \beta_{n+1} - \beta_n | [\gamma \| f(z_n) \| + \| Az_n \|] \\ &\quad \leq \| z_{n+1} - z_n \| + | \beta_{n+1} - \beta_n | M_2, \end{aligned} \tag{3.7}$$

where M_2 is an appropriate constant such that
$M_2 \geq \sup_{n \geq 1}[\gamma \| f(z_n) \| + \| Az_n \|]$. Substituting (3.7) into (3.6), yields:

$$\begin{aligned} &\| y_{n+1} - y_n \| - \| x_{n+1} - x_n \| \\ &\quad \leq | \beta_{n+1} - \beta_n | M_2 + | \gamma_{n+1} - \gamma_n | \| x_n - T_{\mu_n} W_n x_n \| \\ &\qquad + (1 - \gamma_{n+1})[M_1 \prod_{i=1}^{n} \lambda_i + \| T_{\mu_{n+1}} W_n x_n - T_{\mu_n} W_n x_n \|]. \end{aligned}$$

Using the conditions $(A_2), (A_3)$ and noting that $0 < \lambda_i \leq b < 1$, for all $i \geq 1$, we have

$$\limsup_{n \to \infty} [\| y_{n+1} - y_n \| - \| x_{n+1} - x_n \|] \leq 0.$$

By virtue of Lemma 2.5, we have

$$\lim_{n \to \infty} \| y_n - x_n \| = 0. \tag{3.8}$$

Noticing that

$$\begin{aligned}&\| x_n - z_n \| \\ &\quad\leq \| x_n - y_n \| + \| y_n - z_n \| \\ &\quad\leq \| x_n - y_n \| + \beta_n[\gamma \| f(z_n) \| + \| Az_n \|] \\ &\quad\leq \| x_n - y_n \| + \beta_n M_2,\end{aligned}$$

it follows from (3.8) and the condition (A_2) that

$$\lim_{n\to\infty} \| x_n - z_n \| = 0. \tag{3.9}$$

On the other hand, we have

$$\begin{aligned}&\| x_n - T_{\mu_n} W_n x_n \| \\ &\quad\leq \| x_n - z_n \| + \| z_n - T_{\mu_n} W_n x_n \| \\ &\quad= \| x_n - z_n \| + \gamma_n \| x_n - T_{\mu_n} W_n x_n \| .\end{aligned}$$

Therefore,

$$\| x_n - T_{\mu_n} W_n x_n \| \leq \frac{1}{1-\gamma_n} \| x_n - z_n \| . \tag{3.10}$$

It follows from (3.9) and condition (A_4) that

$$\lim_{n\to\infty} \| x_n - T_{\mu_n} W_n x_n \| = 0. \tag{3.11}$$

Using Lemma 2.2, we have

$$\begin{aligned}&\| y_n - T_{\mu_n} W_n y_n \| \\ &\quad\leq \| y_n - x_n \| + \| x_n - T_{\mu_n} W_n x_n \| + \| T_{\mu_n} W_n x_n - T_{\mu_n} W_n y_n \| \\ &\quad\leq 2 \| y_n - x_n \| + \| x_n - T_{\mu_n} W_n x_n \| .\end{aligned}$$

From (3.8) and (3.11), we get

$$\lim_{n\to\infty} \| y_n - T_{\mu_n} W_n y_n \| = 0.$$

Step 3. $\lim_{n\to\infty} \| y_n - T_t y_n \| = 0$, for all $t \in S$.
Proof of Step 3. Let $x^* \in \mathcal{F}$ and $D = \{y \in H : \| y - x^* \| \leq M_0\}$. We point out that D is a bounded closed convex set, $\{x_n\}, \{y_n\} \subset D$ being invariant under φ and W_n, for all $n \in \mathbb{N}$. We will show that

$$\limsup_{n\to\infty} \sup_{y\in D} \| T_{\mu_n} y - T_t T_{\mu_n} y \| = 0, \qquad \forall t \in S. \tag{3.12}$$

Let $\epsilon > 0$. By [3, Theorem 1.2], there exists $\delta > 0$ such that

$$\overline{co} F_\delta(T_t; D) + B_\delta \subset F_\epsilon(T_t; D), \qquad \forall t \in S. \tag{3.13}$$

Also by [3, Corollary 1.1], there exists a natural number N such that

$$(3.14)\qquad \| \frac{1}{N+1}\sum_{i=0}^{N} T_{t^is}y - T_t\left(\frac{1}{N+1}\sum_{i=0}^{N} T_{t^is}y\right) \|\leq \delta,$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$. Since $\{\mu_n\}$ is strongly left regular, there exists $N_0 \in \mathbb{N}$ such that $\| \mu_n - l^*_{t^i}\mu_n \|\leq \frac{\delta}{(M_0+\|x^*\|)}$, for $n \geq N_0$ and $i = 1, 2, \cdots, N$. Then, we have

$$\begin{aligned}
&\sup_{y\in D} \| T_{\mu_n}y - \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^is}yd\mu_n(s) \| \\
&\quad = \sup_{y\in D}\sup_{\|z\|=1} | \langle T_{\mu_n}y, z\rangle - \langle \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^is}yd\mu_n(s), z\rangle | \\
&\quad = \sup_{y\in D}\sup_{\|z\|=1} | \frac{1}{N+1}\sum_{i=0}^{N}(\mu_n)_s\langle T_sy, z\rangle - \frac{1}{N+1}\sum_{i=0}^{N}(\mu_n)_s\langle T_{t^is}y, z\rangle | \\
&\quad \leq \frac{1}{N+1}\sum_{i=0}^{N}\sup_{y\in D}\sup_{\|z\|=1} | (\mu_n)_s\langle T_sy, z\rangle - (l^*_{t^i}\mu_n)_s\langle T_sy, z\rangle | \\
(3.15)&\\
&\quad \leq \max_{i=1,2,\cdots,N} \| \mu_n - l^*_{t^i}\mu_n \| (M_0+ \| x^* \|) \leq \delta, \qquad \forall n \geq N_0.
\end{aligned}$$

By Lemma 2.2, we have

$$(3.16)\qquad \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^is}yd\mu_n(s) \in \overline{co}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i}(T_sy) : s \in S\right\}.$$

It follows from (3.13), (3.14), (3.15) and (3.16) that

$$\begin{aligned}
T_{\mu_n}y &\in \overline{co}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^is}y : s \in S\right\} + B_\delta \\
&\subset \overline{co}F_\delta(T_t; D) + B_\delta \subset F_\epsilon(T_t; D),
\end{aligned}$$

for all $y \in D$ and $n \geq N_0$. Therefore,

$$\limsup_{n\to\infty}\sup_{y\in D} \| T_t(T_{\mu_n}y) - T_{\mu_n}y \|\leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get (3.12).
Let $t \in S$ and $\epsilon > 0$. Then, there exists $\delta > 0$, which satisfies (3.13). From (3.8), (3.12), condition (A_2) and Step 2, there exists $N_1 \in \mathbb{N}$ such

that $\| x_n - z_n \| < \frac{\delta}{3}$, $T_{\mu_n} y \in F_\delta(T_t; D)$, for all $y \in D$, $\beta_n < \frac{\delta}{3M_2}$ and $\| x_n - T_{\mu_n} W_n x_n \| < \frac{\delta}{3}$, for all $n \geq N_1$. We note that

$$\begin{aligned}&\beta_n \| (1-\alpha_n)\gamma f(z_n) - (1-\alpha_n) A z_n \| \\ &\quad \leq \beta_n[(1-\alpha_n)\gamma \| f(z_n) \| + (1-\alpha_n) \| A z_n \|] \\ &\quad \leq \beta_n[\gamma \| f(z_n) \| + \| A z_n \|] \leq \beta_n M_2 \leq \frac{\delta}{3},\end{aligned}$$

for all $n \geq N_1$. Therefore, we have

$$\begin{aligned}x_{n+1} &= \alpha_n x_n + (1-\alpha_n)[\beta_n \gamma f(z_n) + (I - \beta_n A) z_n] \\ &= T_{\mu_n} W_n x_n + \beta_n[(1-\alpha_n)\gamma f(z_n) - (1-\alpha_n) A z_n] \\ &\quad + \alpha_n(x_n - z_n) + \gamma_n(x_n - T_{\mu_n} W_n x_n) \\ &\in F_\delta(T_t; D) + B_{\frac{\delta}{3}} + B_{\frac{\delta}{3}} + B_{\frac{\delta}{3}} \subset F_\delta(T_t; D) + B_\delta \subset F_\epsilon(T_t; D),\end{aligned}$$

for all $n \geq N_1$. This show that

$$\| x_n - T_t x_n \| \leq \epsilon, \qquad \forall n \geq N_1.$$

Since $\epsilon > 0$ is arbitrary, we get

$$\lim_{n\to\infty} \| x_n - T_t x_n \| = 0. \tag{3.17}$$

Observe that

$$\begin{aligned}\| y_n - T_t y_n \| &\leq \| y_n - x_n \| + \| x_n - T_t x_n \| + \| T_t x_n - T_t y_n \| \\ &\leq 2 \| y_n - x_n \| + \| x_n - T_t x_n \| .\end{aligned}$$

It follows from (3.8) and (3.17) that

$$\lim_{n\to\infty} \| y_n - T_t y_n \| = 0.$$

Step 4. There exists a unique $x^* \in \mathcal{F}$ such that
$\limsup_{n\to\infty} \langle \gamma f(x^*) - A x^*, y_n - x^* \rangle \leq 0$.
Proof of Step 4. $P_{\mathcal{F}}(I - A + \gamma f)$ is a contraction of H into itself. In fact, we see that

$$\begin{aligned}&\| P_{\mathcal{F}}(I - A + \gamma f)x - P_{\mathcal{F}}(I - A + \gamma f)y \| \\ &\quad \leq \| (I - A + \gamma f)x - (I - A + \gamma f)y \| \\ &\quad \leq \| I - A \| \| x - y \| + \gamma\alpha \| x - y \| \\ &\quad \leq (1 - (\overline{\gamma} - \gamma\alpha)) \| x - y \|,\end{aligned}$$

and hence $P_{\mathcal{F}}(I - A + \gamma f)$ is a contraction due to $(1 - (\overline{\gamma} - \gamma\alpha)) \in (0, 1)$. Therefore, by the Banach contraction principal, $P_{\mathcal{F}}(I - A + \gamma f)$ has a unique fixed point $x^* \in \mathcal{F}$. Then, using Lemma 2.4 , we have

$$(3.18)\qquad \langle \gamma f(x^*) - Ax^*, y - x^*\rangle \le 0, \qquad \forall y \in \mathcal{F}.$$

We can choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$n \to \infty\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle = \lim_{k\to\infty} \langle \gamma f(x^*) - Ax^*, y_{n_k} - x^*\rangle.$$

Since $\{y_n\}$ is bounded, without loss of generality, we may assume that $y_{n_k} \rightharpoonup z$. It follows from Step 3 and Lemma 2.3 that $z \in Fix(\varphi)$. Moreover, from Lemma 2.7 it follows that $\bigcap_{i=1}^{\infty} Fix(T_i) = Fix(W)$. Assume that $z \notin Fix(W)$. Then, $z \neq Wz$. Since $z \in Fix(\varphi)$, by our assumption, we have $T_i z \in Fix(\varphi)$, for all $i \in \mathbb{N}$ and then $W_n z \in Fix(\varphi)$, for all $n \in \mathbb{N}$. Hence, $T_{\mu_n} W_n z = W_n z$, $\forall n \in \mathbb{N}$. Now, by Step 2 and using Opial's property of a Hilbert space, we have

$$\begin{aligned}
\liminf_{n\to\infty} \| y_{n_k} - z \| &< \liminf_{n\to\infty} \| y_{n_k} - Wz \| \\
&\le \liminf_{n\to\infty} [\| y_{n_k} - T_{\mu_{n_k}} W_{n_k} y_{n_k} \| \\
&\quad + \| T_{\mu_{n_k}} W_{n_k} y_{n_k} - T_{\mu_{n_k}} W_{n_k} z \| \\
&\quad + \| T_{\mu_{n_k}} W_{n_k} z - Wz \|] \\
&\le \liminf_{n\to\infty} [\| y_{n_k} - T_{\mu_{n_k}} W_{n_k} y_{n_k} \| \\
&\quad + \| y_{n_k} - z \| + \| W_{n_k} z - Wz \|] \\
&= \liminf_{n\to\infty} \| y_{n_k} - z \| .
\end{aligned}$$

This is a contradiction. Therefore, z must belong to $\bigcap_{i=1}^{\infty} Fix(T_i) = Fix(W)$. It follows that $z \in \mathcal{F}$, and so noticing (3.18),

$$\limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle \le 0.$$

Step 5. The sequence $\{x_n\}$ converges strongly to x^*.
Proof of Step 5. From (3.3) and Lemma 2.6, we have

$$\begin{aligned}
&\| y_n - x^* \|^2 \\
&\quad = \| (I - \beta_n A)(z_n - x^*) + \beta_n(\gamma f(z_n) - Ax^*) \|^2 \\
&\quad \le \| (I - \beta_n A)(z_n - x^*) \|^2 + 2\beta_n \langle \gamma f(z_n) - Ax^*, y_n - x^*\rangle \\
&\quad \le (1 - \beta_n \overline{\gamma})^2 \| x_n - x^* \|^2 \\
&\qquad + \beta_n \gamma\alpha(\| x_n - x^* \|^2 + \| y_n - x^* \|^2) \\
&\qquad + 2\beta_n \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\| y_n - x^* \|^2 \\
&\quad \leq \frac{(1-\beta_n\overline{\gamma})^2 + \beta_n\gamma\alpha}{1-\beta_n\gamma\alpha} \| x_n - x^* \|^2 \\
&\qquad + \frac{2\beta_n}{1-\beta_n\gamma\alpha}\langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle \\
&\quad \leq [1 - \frac{2\beta_n(\overline{\gamma}-\gamma\alpha)}{1-\beta_n\gamma\alpha}] \| x_n - x^* \|^2 \\
&\qquad + \frac{2\beta_n(\overline{\gamma}-\gamma\alpha)}{1-\beta_n\gamma\alpha}[\frac{1}{\overline{\gamma}-\gamma\alpha}\langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle \\
&\qquad + \frac{\beta_n\overline{\gamma}^2}{2(\overline{\gamma}-\gamma\alpha)} M_4], \qquad (3.19)
\end{aligned}$$

where M_4 is an appropriate constant such that $M_4 \geq \sup_{n\geq 1} \| x_n - x^* \|$. On the other hand, using Lemma 2.11, we have

$$\| x_{n+1} - x^* \|^2 \leq \alpha_n \| x_n - x^* \|^2 + (1-\alpha_n) \| y_n - x^* \|^2 . \qquad (3.20)$$

Substituting (3.19) into (3.20) yields:

$$\begin{aligned}
&\| x_{n+1} - x^* \|^2 \\
&\quad \leq [1 - (1-\alpha_n)\frac{2\beta_n(\overline{\gamma}-\gamma\alpha)}{1-\beta_n\gamma\alpha}] \| x_n - x^* \|^2 \\
&\qquad + (1-\alpha_n)\frac{2\beta_n(\overline{\gamma}-\gamma\alpha)}{1-\beta_n\gamma\alpha}[\frac{1}{\overline{\gamma}-\gamma\alpha}\langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle \\
&\qquad + \frac{\beta_n\overline{\gamma}^2}{2(\overline{\gamma}-\gamma\alpha)} M_4]. \qquad (3.21)
\end{aligned}$$

Putting $\zeta_n = (1-\alpha_n)\frac{2\beta_n(\overline{\gamma}-\gamma\alpha)}{1-\beta_n\gamma\alpha}$ and noting

$$\eta_n = \frac{1}{\overline{\gamma}-\gamma\alpha}\langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle + \frac{\beta_n\overline{\gamma}^2}{2(\overline{\gamma}-\gamma\alpha)} M_4,$$

it follows from (3.21) that

$$\| x_{n+1} - x^* \|^2 \leq (1-\zeta_n) \| x_n - x^* \|^2 + \zeta_n\eta_n. \qquad (3.22)$$

From the condition (A_2) and Step 4, we have $\sum_{n=1}^{\infty} \zeta_n = \infty$ and $\limsup_{n\to\infty} \eta_n \leq 0$. Applying Lemma 2.9 to (3.22), we obtain $x_n \to x^*$, as $n \to \infty$. This completes the proof. □

Corollary 3.2. (*See Qin et al.* [16]) *Let H be a real Hilbert space and f be a contraction on H with coefficient $\alpha \in (0,1)$. Let A be a strongly positive linear bounded self-adjoint operator on H with coefficient $\overline{\gamma} > 0$ and $\{T_i\}_{i=1}^{\infty}$ be a sequence of non-expansive mappings of H into itself. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$ and $\mathcal{F} = \bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the composite iteration process*

$$\begin{cases} x_1 = x \in H, \quad \textit{arbitrarily chosen}, \\ z_n = \gamma_n x_n + (1-\gamma_n) W_n x_n, \\ y_n = \beta_n \gamma f(z_n) + (I - \beta_n A) z_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \qquad n \geq 1, \end{cases}$$

where $\{W_n\}$ is a sequence defined by (1.6), $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0,1]$. If the conditions

(A_1) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,
(A_2) $\sum_{n=0}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$,
(A_3) $\lim_{n\to\infty} \mid \gamma_{n+1} - \gamma_n \mid = 0$,
(A_4) *there exists a constant $\lambda \in [0,1)$ such that $\gamma_n \leq \lambda$ for all $n \geq 1$,*

are satisfied, then the sequence $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$, which also uniquely solves the variational inequality:

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \mathcal{F}.$$

Proof. Take $\varphi = \{I\}$ (the identity mapping) in Theorem 3.1. Then, we have $T_{\mu_n} = I$. So, from Theorem 3.1, the sequence $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$, which also uniquely solves the variational inequality

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \mathcal{F}.$$

□

Corollary 3.3. *Let H be a real Hilbert space and f be a contraction on H with coefficient $\alpha \in (0,1)$. Let A be a strongly positive linear bounded self-adjoint operator on H with coefficient $\overline{\gamma} > 0$ and T be a non-expansive mapping of H into itself. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$ and $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the composite iteration process*

$$\begin{cases} x_1 = x \in H, \quad \textit{arbitrarily chosen}, \\ z_n = \gamma_n x_n + (1-\gamma_n) T x_n, \\ y_n = \beta_n \gamma f(z_n) + (I - \beta_n A) z_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \qquad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0,1]$. If the conditions

(A_1) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

(A_2) $\sum_{n=0}^{\infty} \beta_n = \infty,\ \lim_{n\to\infty} \beta_n = 0$,
(A_3) $\lim_{n\to\infty} \mid \gamma_{n+1} - \gamma_n \mid = 0$,
(A_4) *there exists a constant* $\lambda \in [0,1)$ *such that* $\gamma_n \leq \lambda$ *for all* $n \geq 1$,

are satisfied, then the sequence $\{x_n\}$ *converges strongly to* $x^* \in Fix(T)$, *which also uniquely solves the variational inequality*

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \quad \forall x \in Fix(T).$$

Proof. Take $\varphi = \{I\}$ (the identity mapping) and $W_n = T$, for all $n \in \mathbb{N}$, in Theorem, 3.1. Then, we have $T_{\mu_n} = I$. So from Theorem 3.1, the sequence $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$, which also uniquely solves the variational inequality

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \mathcal{F}.$$

□

The following corollary is Theorem 3.1 of Kim and Xu [8] in the frame work of Hilbert spaces.

Corollary 3.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space,* $T\colon C \to C$ *be a non expansive mapping such that* $Fix(T) \neq \emptyset$. *Let* $f\colon C \to C$ *be a contraction with coefficient* $\alpha \in (0,1)$. *Let* $\{x_n\}$ *be a sequence generated by the composite iteration process*

$$\begin{cases} x_1 = x \in C, \quad \textit{arbitrarily chosen}, \\ y_n = \beta_n f(z_n) + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n, \qquad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$, *and the conditions*

(A_1) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,
(A_2) $\sum_{n=0}^{\infty} \beta_n = \infty,\ \lim_{n\to\infty} \beta_n = 0$,
(A_3) $\lim_{n\to\infty} \mid \gamma_{n+1} - \gamma_n \mid = 0$,

are satisfied, then the sequence $\{x_n\}$ *converges strongly to* $x^* = P_{Fix(T)}x^*$.

Proof. Take $\gamma_n = 0$, for all $n \in \mathbb{N}$, $\gamma = 1$ and $A = I$ in Corollary 3.3. □

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