

FINITE p -GROUPS WITH FEW NON-LINEAR IRREDUCIBLE CHARACTER KERNELS

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ABSTRACT. We classify all finite p -groups with at most three non-linear irreducible character kernels.

1. Introduction

We classify finite p -groups with at most three non-linear irreducible character kernels. This solves the first half of a question posed by Berkovich [1, Research Problem 23]. The second part of this problem involves the quasikernels and is still open. Throughout the paper, G is a finite non-abelian p -group for a fixed prime p . Denote by $\text{Kern}(G)$ the set of non-linear irreducible character kernels of G . If $|\text{Kern}(G)| = 1$, then $|G'| = p$ and $Z(G)$ is cyclic and vice versa (see Lemma 2.1 and Lemma 2.2 below). Also, the main theorem of [8] implies that if $|\text{Kern}(G)| > 1$, then G is of maximal class if and only if $\text{Kern}(G)$ is a chain with respect to inclusion. Our main theorem is the following.

Theorem 1.1. *Let G be a finite non-abelian p -group. Let $t \leq 3$ be the number of non-linear irreducible character kernels of G . Then,*

- (1) $t = 1$ if and only if $|G'| = p$ and $Z(G)$ is cyclic.

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(2) $t = 2$ if and only if one of the following cases occurs:

- (a) G is of order p^4 and class 3.
- (b) $|G'| = 2$, $Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^r}$ ($r \geq 1$) and $r > 1$ implies that $G' \subseteq \Phi(Z(G))$.

(3) $t = 3$ if and only if one of the following cases occurs:

- (a) G is of order p^5 and class 4.
- (b) G is of order 32 and class 3 and $Z(G)$ is cyclic.
- (c) $|G'| = 3$, $Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_{3^r}$ ($r \geq 1$) and $r > 1$ implies that $G' \subseteq \Phi(Z(G))$.
- (d) $G' = Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $N \leq Z(G)$, for each $N \triangleleft G$ not containing G' .
- (e) $|G'| = 2$, $Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ and $G' \neq \Phi(Z(G))$.
- (f) G is of class 2, $|G'| = 4$, $Z(G)$ is cyclic and $|NZ(G) : Z(G)| \leq 2$, for each $N \triangleleft G$ not containing G' .

Throughout this paper, all characters are complex. By $c(G)$, we mean the nilpotency class of the group and $Z_i(G)$ is the i th term of the upper central series of G . We denote the set of non-linear irreducible characters of G and the set of irreducible character degrees of G by $\text{Irr}_1(G)$ and $\text{cd}(G)$, respectively. The Frattini subgroup of G is denoted by $\Phi(G)$.

2. Preliminary results

In this section, we state some known facts, mainly about p -groups and their characters. First we need the following easy result.

Lemma 2.1. *Let H be a non-abelian finite group. Then,*

$$\bigcap_{K \in \text{Kern}(H)} K = 1.$$

Proof. Consider the group H/N , where $N = \bigcap_{K \in \text{Kern}(H)} K$ and note that N is contained in all non-linear irreducible character kernels of H . Since non-linear irreducible characters of H/N are the same as those of H , the equality $|H| = |H : H'| + \sum_{\chi \in \text{Irr}_1(H)} \chi(1)^2$ yields:

$$|H| - |H : H'| = |H : N| - |H : H'N|,$$

which implies:

$$(2.1) \quad (|H'| - 1) |N : H' \cap N| = |H' : H' \cap N| - 1.$$

Thus, $|H'| - 1$ divides $|H' : H' \cap N| - 1$. That is, $H' \cap N = 1$. Now, using (2.1) again, one deduces that $N \leq H'$, whence $N = 1$. \square

Lemma 1 of [2] implies that $|G'| = p$ and $Z(G)$ is cyclic if and only if all non-linear irreducible characters of G are faithful and $\text{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. By the next lemma we show that the condition $\text{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$ is indeed superfluous.

Lemma 2.2. *All non-linear irreducible characters of G are faithful if and only if $|G'| = p$ and $Z(G)$ is cyclic.*

Proof. Assume that all of the non-linear irreducible characters of G are faithful. Since G has some faithful irreducible characters, then by [7, Lemma 2.32], $Z(G)$ is cyclic. Now, assume that N is a normal subgroup of G , not containing G' . Then N is contained in some non-linear irreducible characters of G . But, all non-linear irreducible characters of G are faithful. Hence, we get $N = 1$. That is, G' is the unique minimal normal subgroup of G . Equivalently, $|G'| = p$. The result now follows by [2, Lemma 1] \square

To prove Theorem 1.1, we need the notion of the strong and weak conditions on normal subgroups, introduced by Fernández-Alcober and Moretó [3]. Assume that $\mathcal{N}(G)$ is the set of all normal subgroups of G , not containing G' . Then, we say that G satisfies the strong condition if each element of $\mathcal{N}(G)$ is central. Also, G satisfies the weak condition if for each $N \in \mathcal{N}(G)$, $|NZ(G) : Z(G)| \leq p$. Throughout this paper, we use these notions frequently. It is clear that we may relax to consider only the maximal elements of $\mathcal{N}(G)$ which are among the members of $\text{Kern}(G)$. So, we have the following equivalent definition.

Definition 2.3. *We say that G satisfies the strong condition if for each $K \in \text{Kern}(G)$, $K \leq Z(G)$. Similarly, G satisfies the weak condition if for each $K \in \text{Kern}(G)$, the $|KZ(G) : Z(G)| \leq p$.*

Proposition 2.4. *Assume that $|G'| = p$. Then, G satisfies the strong condition. Moreover, $Z(G/N) = Z(G)/N$, for each normal subgroup N of G , not containing G' .*

Proof. Let N be a normal subgroup of G , not containing G' . Then, $N \cap G' = 1$ and consequently, $N \leq Z(G)$. Hence, G satisfies the strong condition. Now, assume that $xN \in Z(G/N)$. Then, for each $y \in G$, $[x, y] \in N \cap G' = 1$. That is, $x \in Z(G)$ and the result follows. \square

Proposition 2.5. *Assume that G satisfies the weak condition. Then, $c(G) \leq 4$ and the following statements hold:*

- (1) *if $c(G) = 4$, then $|G : Z(G)| = p^4$,*
- (2) *if $c(G) = 3$ and $|Z_2(G) : Z(G)| = p$, then $|G : Z(G)| = p^3$.*

Proof. See [3, Propositions 4.1, 6.1 and 6.5]. \square

Lemma 2.6. *Assume that $|(G/Z(G))'| = p$. Then, $|G : Z_2(G)| = p^2$.*

Proof. It suffices to apply [3, Lemma 2.5] to $G/Z(G)$. \square

Proposition 2.7. [8] *Kern(G) is a chain with respect to inclusion if and only if G satisfies one of the following conditions:*

- (1) *G' is the unique minimal normal subgroup of G .*
- (2) *G is of maximal class.*

Now, we can prove the following useful corollary. For the preliminary facts about the p -groups of maximal class, see, for example, [5, Chapter III, Section 14].

Corollary 2.8. *If Kern(G) is a chain with respect to inclusion, then $|\text{Kern}(G)| = c(G) - 1$.*

Proof. By Proposition 2.7, G' is the unique minimal normal subgroup of G or G is of maximal class. In the former case, G is of class 2. Furthermore, [7, Lemma 12.3] implies that $|\text{Kern}(G)| = 1$. So, we may assume that G is of maximal class. In particular, $|Z(G)| = p$. Use induction on $c(G)$. If $c(G) = 2$, then G is a non-abelian group of order p^3 . Thus, we have $|\text{Kern}(G)| = 1$. So, assume that $c(G) > 2$. Obviously, $G/Z(G)$ satisfies the hypothesis of the induction. Hence, $|\text{Kern}(G/Z(G))| = c(G/Z(G)) - 1$. To complete the proof, it suffices to show that $|\text{Kern}(G/Z(G))| = |\text{Kern}(G)| - 1$. As $Z(G)$ is contained all the non-trivial non-linear irreducible character kernels of G , we conclude that there exists a one to one correspondence between the non-linear irreducible character kernels of G , which are non-trivial, and the non-linear irreducible character kernels of $G/Z(G)$. In particular, $|\text{Kern}(G/Z(G))| = |\text{Kern}(G)| - 1$, and the result follows. \square

3. Main results

In this section, we prove Theorem 1.1. First we deal with the case that G has just two non-linear irreducible characters.

Lemma 3.1. *Suppose G has no faithful irreducible characters. Then, $|\text{Kern}(G)| = 2$ if and only if $|G'| = 2$, $Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^r}$ ($r \geq 1$) and either $G' \subseteq \Phi(Z(G))$ or $r = 1$.*

Proof. Assume that $\text{Kern}(G) = \{K_1, K_2\}$. Let $N \leq K_1 \cap Z(G)$ and N be of order p . By Lemma 2.1, $K_1 \cap K_2 = 1$. Thus, G/N has only one non-linear irreducible character kernel. As a result of Lemma 2.1, all non-linear irreducible characters of G/N are faithful. Hence, $K_1 = N$ and we get $|K_1| = p$. A similar argument shows that $|G'| = |K_2| = p$. Noting that, under our assumption, every subgroup of $Z(G)$ having order p , except G' , is in $\text{Kern}(G)$, we have that $Z(G)$ contains exactly three subgroups of order p . So, we get $p = 2$ and $Z(G) \cong \mathbb{Z}_{2^r} \times \mathbb{Z}_{2^s}$ for some positive integers r, s . On the other hand, by Proposition 2.4, $Z(G/K_i) = Z(G)/K_i$, ($1 \leq i \leq 2$). However, the former is cyclic, by Lemma 2.2. So, we conclude that either $r = 1$ or $s = 1$. If $G' \not\subseteq \Phi(Z(G))$, then we may find a maximal subgroup M of $Z(G)$ such that $G' \not\subseteq M$. Now, G/M is a non-abelian group and consequently contained in some character kernels of G , while the character kernels of G are of order 2. Therefore, $|M| = 2$ and we conclude that $r = s = 1$. Conversely, assume that $Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^r}$ and $|G'| = 2$. Then, all non-linear irreducible character kernels of G are central. If $r = 1$, then $Z(G)$ has exactly two proper normal subgroups, apart from G' . Both of these subgroups are easily seen to be character kernels of G . Therefore, in this case we have $|\text{Kern}(G)| = 2$. Now, assume that $r > 1$. Then by our assumption, $G' \subseteq \Phi(Z(G))$ and consequently, G' is contained in all normal subgroups of G of order greater than 2. So, the non-linear irreducible character kernels of G , not containing G' are just the normal subgroups of order 2, except G' . This completes the proof. □

Proposition 3.2. *Let G be a p -group and assume that G satisfies one of the followings:*

- (a) G is of order p^5 and class 4,
- (b) G is of order 32 and class 3 and $Z(G)$ is cyclic,

- (c) $|G'| = 3$, $Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_{3^r}$ ($r \geq 1$) and $r > 1$ implies that $G' \subseteq \Phi(Z(G))$,
- (d) $G' = Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and G satisfies the strong condition.
- (e) $|G'| = 2$, $Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ and $G' \neq \Phi(Z(G))$,
- (f) G is of class 2, $|G'| = 4$, $Z(G)$ is cyclic and G satisfies the weak condition.

Then, $|\text{Kern}(G)| = 3$.

Proof. If G is of type (a), then the result follows by Proposition 2.7 and Corollary 2.8. Also, we may use GAP [4] to verify that the groups of type (b) have precisely three non-linear irreducible character kernels. Now, let G be of type (c). Then, G satisfies the strong condition and consequently the non-linear character kernels of G are just the non-trivial proper subgroups of $Z(G)$, apart from G' , while we know that $Z(G)$ has exactly four subgroups of order 3, one of which is G' . A similar argument shows that the result holds for groups of type (d). Next, assume that G is of type (e). Then, $Z(G)$ has three subgroups of order 2, and three subgroups of order 4. Since G satisfies the strong condition, the normal subgroups of G not containing G' , are just the subgroups of $Z(G)$. Let $F = \Phi(Z(G))$ and note that $|F| = 2$. Also by definition, F is contained in all subgroups of $Z(G)$ of order 4. On the other hand, among 6 subgroups of $Z(G)$ of order 2 or 4, four subgroups do not contain G' . Also F can not be a non-linear character kernel of G . So, G has precisely one character kernel of order 2 and two character kernels of order 4. Now, let G be of type (f). Let K be a non-linear irreducible character kernel of G . Since $Z(G)$ is cyclic, then G' is contained in all central subgroups of G of order greater than 2. Therefore, $|K \cap Z(G)| \leq 2$. As G satisfies the weak condition, we conclude that $|K| = 4$. Let N be the unique minimal normal subgroup of G . Then, all non-linear irreducible character kernels of G/N are of order 2. Also, $|(G/N)'| = 2$. We claim that G/N has exactly three normal subgroups of order 2. Otherwise, G'/N is not contained in the product of at least two subgroups of order 2. This forces G/N to contain some non-linear irreducible character kernels of order greater than 2, which is a contradiction. The proof is now completed. □

Lemma 3.3. *Let G be a 2-group of class exceeding 2 with $|\text{Kern}(G)| = 3$. Then, $|G| = 32$ or 64.*

Proof. If G is of maximal class, then Proposition 2.7 and Corollary 2.8 imply that $|G| = 32$. Assume that G is not of maximal class. Set $\text{Kern}(G) = \{K_1, K_2, K_3\}$. First, suppose $1 \notin \text{Kern}(G)$. If all elements of $\text{Kern}(G)$ are mutually disjoint, then it is easy to see that all members of $\text{Kern}(G)$ are of order 2. Thus, by Proposition 2.5(1), $|G| = 16$, which is a contradiction. So, we may assume that $K_1 \cap K_2 = N \neq 1$. Lemma 3.1 yields that $|K_1| = |K_2| = |G'| = 4$. We show that $|K_3| \leq 4$. Indeed, if $|K_3| \geq 4$, then it contains a normal subgroup N of G with $|N| = 2$. By Lemma 2.1, N contains exactly two members of $\text{Kern}(G)$. Hence, by Lemma 3.1, $|K_3| = 4$, as wanted. If $|Z(G)| = 4$, then the result follows by Proposition 2.5(2). So, assume that $|Z(G)| > 4$. Then, $Z(G)$ has at least 3 subgroups of order 4. Since G' is not contained in $Z(G)$, we conclude that $Z(G)$ has exactly three subgroups of order 4. Consequently, K_3 is also central and G satisfies the strong condition. The result now follows by Proposition 2.5(1). Next, we assume that $1 \in \text{Kern}(G)$. Then $Z(G)$ is cyclic and consequently, G has only one minimal normal subgroup. Let $K_3 = 1$ and note that by Proposition 2.7, $|K_1| > 2$ and $|K_2| > 2$. Thus, $K_1 \cap K_2 = N \neq 1$ and G/N is a group with two non-linear irreducible character kernels. Note that by Lemma 2.1, N is the unique minimal normal subgroup of G . Now, apply Lemma 3.1 to G/N and obtain $|G'| = |K_1| = |K_2| = 4$. In particular, G satisfies the weak condition. Since $Z(G)$ is cyclic, it contains only one subgroup of order 4. Thus, $Z(G)$ does not contain both K_1 and K_2 . On the other hand, by our hypothesis, G is of class greater than 2. So, $Z(G)$ contains either K_1 or K_2 . Also Lemma 2.1 implies that $Z(G)$ coincides to one of these subgroups. Hence, $|Z(G)| = 4$ and the result follows by Proposition 2.5(2). □

Remark 3.4. *Using GAP, one may check that groups of order 32 of class exceeding 2 with a cyclic center have precisely three non-linear irreducible character kernels. Furthermore, among the groups of order 32 or 64, these are the only groups of class greater than 2 that satisfy this property.*

Example 3.5. Let G be a group of order 32 such that $|G : G'| = 8$ and $\text{cd}(G) = \{1, 2, 4\}$ (see [6, Example 6.11]). Then $|\text{Kern}(G)| = 3$.

Proof of Theorem 1.1. Part (1) follows from Lemma 2.1 and Lemma 2.2. Now, assume that $t = 2$. If G has no faithful irreducible characters, then

by Lemma 3.1, it is of type (b). So, let $1 \in \text{Kern}(G)$. Since $|\text{Kern}(G)| = 2$, the non-linear irreducible character kernels of G constitute a chain with respect to inclusion. Thus by Proposition 2.7, G is of maximal class. Also, by Corollary 2.8, $c(G) = 3$. Hence, $|G| = p^4$ and G is of type (a).

To complete the proof, it remains to prove the third part of the theorem. If G satisfies one of the conditions (a)–(f) of (3), then by Lemma 3.2, $|\text{Kern}(G)| = 3$. So, assume that $|\text{Kern}(G)| = 3$. If G is of maximal class, then by Proposition 2.7 and Corollary 2.8 G is of type (a). So, assume that G is not of maximal class. Our goal is to prove that G is of type (b), (c), (d), (e) or (f). The proof is divided into several steps.

Step 1. If $|K| > p$, for some $K \in \text{Kern}(G)$, then $p = 2$.

First, we suppose that $1 \in \text{Kern}(G)$ and let N be the (unique) central subgroup of G of order p . Since G is not of maximal class, we get $N \notin \text{Kern}(G)$. So, by Lemma 3.1, G/N is a 2-group. Next, assume that $1 \notin \text{Kern}(G)$. Then, $Z(G)$ is not cyclic and in particular, contains at least $p + 1$ subgroups of order p . If $p > 2$, then we may choose a subgroup N of $Z(G)$ of order p with $N \neq G'$ and $N \notin \text{Kern}(G)$. Now, if $|\text{Kern}(G/N)| = 1$, then Lemma 2.1 implies that $N \in \text{Kern}(G)$ which is a contradiction. Also, if $|\text{Kern}(G/N)| = 3$, then $N \leq K$, for each $K \in \text{Kern}(G/N)$. Hence, $N = 1$ by Lemma 2.1 which is again a contradiction. Thus, we conclude that G/N has exactly two non-linear irreducible character kernels. Since $N \notin \text{Kern}(G)$, we conclude that G/N has no faithful irreducible characters. Thus by Lemma 3.1, G is a 2-group. This is the final contradiction.

Step 2. We have $p \leq 3$.

By Step 1, we may assume that all of the elements of $\text{Kern}(G)$ are of order p . Then, $Z(G)$ contains at least $p + 1$ subgroups of order p . On the other hand, each central subgroup N of order 2, except G' , is in $\text{Kern}(G)$. Thus, the number of the subgroups of $Z(G)$ of order 2 does not exceed 4. This gives us $p \leq 3$.

Step 3. If $p = 3$, then G' and all elements of $\text{Kern}(G)$ are of order 3, $Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_{3^r}$ ($r \geq 1$) and $r > 1$ implies that $G' \subseteq \Phi(Z(G))$.

By Step 1, all elements of $\text{Kern}(G)$ are of order 3. Since $Z(G)$ has at least four subgroups of order 3, then $|G'| = 3$ and $Z(G) \cong \mathbb{Z}_{3^r} \times \mathbb{Z}_{3^s}$, for some positive integers r and s . On the other hand, for each central subgroup $L (\neq G')$ of order 3, $Z(G/L)$ is cyclic. Hence $r = 1$ or $s = 1$. The reminder of the proof is almost the same as that of Lemma 3.1.

Step 4. If $p = 2$ and all elements of $\text{Kern}(G)$ are of order 2, then $G' = Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Certainly, $|G'| > 2$. On the other hand, if N is a central subgroup of G of order 4, then G/N does not admit any non-linear irreducible character. Hence, G' is the unique subgroup of $Z(G)$ of order 4. Since $Z(G)$ is not cyclic, we have $G' = Z(G)$.

Step 5. If $|K| > p$, for some $K \in \text{Kern}(G)$ and $c(G) = 2$, then G is in type (e) or (f).

Assume that $|K_1| > 2$. Let $N \leq Z(G) \cap K_1$ and $|N| = 2$. Then by Lemma 3.1, N contains two non-linear irreducible character kernels of G . So, we may assume that $N \leq K_2$ and $|K_1| = |K_2| = 4$. First, assume that $Z(G)$ is not cyclic. Then, we may choose three distinct normal subgroups of order 2. So, we must have $|G'| = |K_3| = 2$ and $Z(G)$ has exactly three subgroups of order 2. Also G' is not contained in $\Phi(Z(G))$. If $|Z(G)| > 8$ then it contains a subgroup of order 8, not containing G' . This is clearly a contradiction. So, $|Z(G)| \leq 8$. As G satisfies the strong condition, the equality holds. So, G is of type (e). Next, assume that $Z(G)$ is cyclic. Then, G satisfies the weak condition. Also by Lemma 3.1, $|G'/N| = 2$. Hence, $|G'| = 4$ and we obtain G is of type (f).

To complete the proof, we use the assertions of the above steps. If $p > 2$, then by Steps 2 and 3, G is of type (c). So, assume that $p = 2$. If $c(G) > 2$, then as a result of Lemma 3.3 and Remark 3.4, we conclude that the group G is of type (b). Also, if $c(G) = 2$, then by Step 1 and Step 5, G is of type (d), (e) or (f).

□

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