FINITE $p$-GROUPS WITH FEW NON-LINEAR IRREDUCIBLE CHARACTER KERNESELS

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ABSTRACT. We classify all finite $p$-groups with at most three non-linear irreducible character kernels.

1. Introduction

We classify finite $p$-groups with at most three non-linear irreducible character kernels. This solves the first half of a question posed by Berkovich [1, Research Problem 23]. The second part of this problem involves the quasikernels and is still open. Throughout the paper, $G$ is a finite non-abelian $p$-group for a fixed prime $p$. Denote by $\text{Kern}(G)$ the set of non-linear irreducible character kernels of $G$. If $|\text{Kern}(G)| = 1$, then $|G'| = p$ and $Z(G)$ is cyclic and vice versa (see Lemma 2.1 and Lemma 2.2 below). Also, the main theorem of [8] implies that if $|\text{Kern}(G)| > 1$, then $G$ is of maximal class if and only if $\text{Kern}(G)$ is a chain with respect to inclusion. Our main theorem is the following.

Theorem 1.1. Let $G$ be a finite non-abelian $p$-group. Let $t \leq 3$ be the number of non-linear irreducible character kernels of $G$. Then,

1. $t = 1$ if and only if $|G'| = p$ and $Z(G)$ is cyclic.
(2) \( t = 2 \) if and only if one of the following cases occurs:

(a) \( G \) is of order \( p^4 \) and class 3.
(b) \(|G'| = 2\), \( Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^r} \) \((r \geq 1)\) and \( r > 1 \) implies that \( G' \subseteq \Phi(Z(G)). \)

(3) \( t = 3 \) if and only if one of the following cases occurs:

(a) \( G \) is of order \( p^5 \) and class 4.
(b) \( G \) is of order 32 and class 3 and \( Z(G) \) is cyclic.
(c) \(|G'| = 3\), \( Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_{3^r} \) \((r \geq 1)\) and \( r > 1 \) implies that \( G' \subseteq \Phi(Z(G)). \)
(d) \( G' = Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( N \leq Z(G) \), for each \( N \vartriangleleft G \) not containing \( G' \).
(e) \(|G'| = 2\), \( Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \) and \( G' \neq \Phi(Z(G)). \)
(f) \( G \) is of class 2, \(|G'| = 4\), \( Z(G) \) is cyclic and \(|NZ(G) : Z(G)| \leq 2\), for each \( N \vartriangleleft G \) not containing \( G' \).

Throughout this paper, all characters are complex. By \( c(G) \), we mean the nilpotency class of the group and \( Z_i(G) \) is the \( i \)th term of the upper central series of \( G \). We denote the set of non-linear irreducible characters of \( G \) and the set of irreducible character degrees of \( G \) by \( \text{Irr}_1(G) \) and \( \text{cd}(G) \), respectively. The Frattini subgroup of \( G \) is denoted by \( \Phi(G) \).

2. Preliminary results

In this section, we state some known facts, mainly about \( p \)-groups and their characters. First we need the following easy result.

**Lemma 2.1.** Let \( H \) be a non-abelian finite group. Then,
\[
\bigcap_{K \in \text{Kern}(H)} K = 1.
\]

**Proof.** Consider the group \( H/N \), where \( N = \bigcap_{K \in \text{Kern}(H)} K \) and note that \( N \) is contained in all non-linear irreducible character kernels of \( H \). Since non-linear irreducible characters of \( H/N \) are the same as those of \( H \), the equality \(|H| = |H : H'| + \sum_{\chi \in \text{Irr}_1(H)} \chi(1)^2 \) yields:
\[
|H| - |H : H'| = |H : N| - |H : H'N|,
\]
which implies:
Thus, $|H'| - 1$ divides $|H' : H' \cap N| - 1$. That is, $H' \cap N = 1$. Now, using (2.1) again, one deduces that $N \leq H'$, whence $N = 1$. 

Lemma 1 of [2] implies that $|G'| = p$ and $Z(G)$ is cyclic if and only if all non-linear irreducible characters of $G$ are faithful and $\text{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. By the next lemma we show that the condition $\text{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$ is indeed superfluous.

**Lemma 2.2.** All non-linear irreducible characters of $G$ are faithful if and only if $|G'| = p$ and $Z(G)$ is cyclic.

**Proof.** Assume that all of the non-linear irreducible characters of $G$ are faithful. Since $G$ has some faithful irreducible characters, then by [7, Lemma 2.32], $Z(G)$ is cyclic. Now, assume that $N$ is a normal subgroup of $G$, not containing $G'$. Then $N$ is contained in some non-linear irreducible characters of $G$. But, all non-linear irreducible characters of $G$ are faithful. Hence, we get $N = 1$. That is, $G'$ is the unique minimal normal subgroup of $G$. Equivalently, $|G'| = p$. The result now follows by [2, Lemma 1].

To prove Theorem 1.1, we need the notion of the strong and weak conditions on normal subgroups, introduced by Fernández-Alcober and Moretó [3]. Assume that $\mathcal{N}(G)$ is the set of all normal subgroups of $G$, not containing $G'$. Then, we say that $G$ satisfies the strong condition if each element of $\mathcal{N}(G)$ is central. Also, $G$ satisfies the weak condition if for each $N \in \mathcal{N}(G)$, $|NZ(G) : Z(G)| \leq p$. Throughout this paper, we use these notions frequently. It is clear that we may relax to consider only the maximal elements of $\mathcal{N}(G)$ which are among the members of $\text{Kern}(G)$. So, we have the following equivalent definition.

**Definition 2.3.** We say that $G$ satisfies the strong condition if for each $K \in \text{Kern}(G)$, $K \leq Z(G)$. Similarly, $G$ satisfies the weak condition if for each $K \in \text{Kern}(G)$, the $|KZ(G) : Z(G)| \leq p$.

**Proposition 2.4.** Assume that $|G'| = p$. Then, $G$ satisfies the strong condition. Moreover, $Z(G/N) = Z(G)/N$, for each normal subgroup $N$ of $G$, not containing $G'$.
Proof. Let $N$ be a normal subgroup of $G$, not containing $G'$. Then, $N \cap G' = 1$ and consequently, $N \leq Z(G)$. Hence, $G$ satisfies the strong condition. Now, assume that $xN \in Z(G/N)$. Then, for each $y \in G$, $[x, y] \in N \cap G' = 1$. That is, $x \in Z(G)$ and the result follows.

Proposition 2.5. Assume that $G$ satisfies the weak condition. Then, $c(G) \leq 4$ and the following statements hold:

1. if $c(G) = 4$, then $|G : Z(G)| = p^4$,
2. if $c(G) = 3$ and $|Z_2(G) : Z(G)| = p$, then $|G : Z(G)| = p^3$.

Proof. See [3, Propositions 4.1, 6.1 and 6.5].

Lemma 2.6. Assume that $|(G/Z(G))'| = p$. Then, $|G : Z_2(G)| = p^2$.

Proof. It suffices to apply [3, Lemma 2.5] to $G/Z(G)$.

Proposition 2.7. [8] $\text{Kern}(G)$ is a chain with respect to inclusion if and only if $G$ satisfies one of the following conditions:

1. $G'$ is the unique minimal normal subgroup of $G$.
2. $G$ is of maximal class.

Now, we can prove the following useful corollary. For the preliminary facts about the $p$-groups of maximal class, see, for example, [5, Chapter III, Section 14].

Corollary 2.8. If $\text{Kern}(G)$ is a chain with respect to inclusion, then $|\text{Kern}(G)| = c(G) - 1$.

Proof. By Proposition 2.7, $G'$ is the unique minimal normal subgroup of $G$ or $G$ is of maximal class. In the former case, $G$ is of class 2. Furthermore, [7, Lemma 12.3] implies that $|\text{Kern}(G)| = 1$. So, we may assume that $G$ is of maximal class. In particular, $|Z(G)| = p$. Use induction on $c(G)$. If $c(G) = 2$, then $G$ is a non-abelian group of order $p^3$. Thus, we have $|\text{Kern}(G)| = 1$. So, assume that $c(G) > 2$. Obviously, $G/Z(G)$ satisfies the hypothesis of the induction. Hence, $|\text{Kern}(G/Z(G))| = c(G/Z(G)) - 1$. To complete the proof, it suffices to show that $|\text{Kern}(G/Z(G))| = |\text{Kern}(G)| - 1$. As $Z(G)$ is contained all the non-trivial non-linear irreducible character kernels of $G$, we conclude that there exists a one to one correspondence between the non-linear irreducible character kernels of $G$, which are non-trivial, and the non-linear irreducible character kernels of $G/Z(G)$. In particular, $|\text{Kern}(G/Z(G))| = |\text{Kern}(G)| - 1$, and the result follows.
3. Main results

In this section, we prove Theorem 1.1. First we deal with the case that $G$ has just two non-linear irreducible characters.

Lemma 3.1. Suppose $G$ has no faithful irreducible characters. Then, $|\text{Kern}(G)| = 2$ if and only if $|G'| = 2$, $Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^r} (r \geq 1)$ and either $G' \subseteq \Phi(Z(G))$ or $r = 1$.

Proof. Assume that $\text{Kern}(G) = \{K_1, K_2\}$. Let $N \leq K_1 \cap Z(G)$ and $N$ be of order $p$. By Lemma 2.1, $K_1 \cap K_2 = 1$. Thus, $G/N$ has only one non-linear irreducible character kernel. As a result of Lemma 2.1, all non-linear irreducible characters of $G/N$ are faithful. Hence, $K_1 \cap N = N$ and we get $|K_1| = p$. A similar argument shows that $|G'| = |K_2| = p$.

Noting that, under our assumption, every subgroup of $Z(G)$ having order $p$, except $G'$, is in $\text{Kern}(G)$, we have that $Z(G)$ contains exactly three subgroups of order $p$. So, we get $p = 2$ and $Z(G) \cong \mathbb{Z}_{2^r} \times \mathbb{Z}_{2^s}$ for some positive integers $r, s$. On the other hand, by Proposition 2.4, $Z(G/K_i) = Z(G)/K_i, (1 \leq i \leq 2)$. However, the former is cyclic, by Lemma 2.2. So, we conclude that either $r = 1$ or $s = 1$. If $G' \not\subseteq \Phi(Z(G))$, then we may find a maximal subgroup $M$ of $Z(G)$ such that $G' \not\subseteq M$. Now, $G/M$ is a non-abelian group and consequently contained in some character kernels of $G$, while the character kernels of $G$ are of order 2. Therefore, $|M| = 2$ and we conclude that $r = s = 1$. Conversely, assume that $Z(G) \cong \mathbb{Z}_{2^r} \times \mathbb{Z}_{2^s}$ and $|G'| = 2$. Then, all non-linear irreducible character kernels of $G$ are central. If $r = 1$, then $Z(G)$ has exactly two proper normal subgroups, apart from $G'$. Both of these subgroups are easily seen to be character kernels of $G$. Therefore, in this case we have $|\text{Kern}(G)| = 2$. Now, assume that $r > 1$. Then by our assumption, $G' \subseteq \Phi(Z(G))$ and consequently, $G'$ is contained in all normal subgroups of $G$ of order greater than 2. So, the non-linear irreducible character kernels of $G$, not containing $G'$ are just the normal subgroups of order 2, except $G'$. This completes the proof.

Proposition 3.2. Let $G$ be a $p$-group and assume that $G$ satisfies one of the followings:

(a) $G$ is of order $p^5$ and class 4,
(b) $G$ is of order 32 and class 3 and $Z(G)$ is cyclic,
(c) \(|G'| = 3, Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_{2^r}(r \geq 1)\) and \(r > 1\) implies that \(G' \subseteq \Phi(Z(G))\).

(d) \(G' = Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2\) and \(G\) satisfies the strong condition.

(e) \(|G'| = 2, Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_4\) and \(G' \neq \Phi(Z(G))\).

(f) \(G\) is of class 2, \(|G'| = 4, Z(G)\) is cyclic and \(G\) satisfies the weak condition.

Then, \(|\text{Kern}(G)| = 3\).

Proof. If \(G\) is of type (a), then the result follows by Proposition 2.7 and Corollary 2.8. Also, we may use GAP [4] to verify that the groups of type (b) have precisely three non-linear irreducible character kernels. Now, let \(G\) be of type (c). Then, \(G\) satisfies the strong condition and consequently the non-linear character kernels of \(G\) are just the non-trivial proper subgroups of \(Z(G)\), apart from \(G'\), while we know that \(Z(G)\) has exactly four subgroups of order 3, one of which is \(G'\). A similar argument shows that the result holds for groups of type (d). Next, assume that \(G\) is of type (e). Then, \(Z(G)\) has three subgroups of order 2, and three subgroups of order 4. Since \(G\) satisfies the strong condition, the normal subgroups of \(G\) not containing \(G'\), are just the subgroups of \(Z(G)\). Let \(F = \Phi(Z(G))\) and note that \(|F| = 2\). Also by definition, \(F\) is contained in all subgroups of \(Z(G)\) of order 4. On the other hand, among 6 subgroups of \(Z(G)\) of order 2 or 4, four subgroups do not contain \(G'\). Also \(F\) can not be a non-linear character kernel of \(G\). So, \(G\) has precisely one character kernel of order 2 and two character kernels of order 4. Now, let \(G\) be of type (f). Let \(K\) be a non-linear irreducible character kernel of \(G\). Since \(Z(G)\) is cyclic, then \(G'\) is contained in all central subgroups of \(G\) of order greater than 2. Therefore, \(|K \cap Z(G)| \leq 2\). As \(G\) satisfies the weak condition, we conclude that \(|K| = 4\). Let \(N\) be the unique minimal normal subgroup of \(G\). Then, all non-linear irreducible character kernels of \(G/N\) are of order 2. Also, \(|(G/N)'| = 2\). We claim that \(G/N\) has exactly three normal subgroups of order 2. Otherwise, \(G'/N\) is not contained in the product of at least two subgroups of order 2. This forces \(G/N\) to contain some non-linear irreducible character kernels of order greater than 2, which is a contradiction. The proof is now completed.

\(\square\)

Lemma 3.3. Let \(G\) be a 2-group of class exceeding 2 with \(|\text{Kern}(G)| = 3\).
Then, \(|G| = 32\) or 64.
Proof. If $G$ is of maximal class, then Proposition 2.7 and Corollary 2.8 imply that $|G| = 32$. Assume that $G$ is not of maximal class. Set $\text{Kern}(G) = \{K_1, K_2, K_3\}$. First, suppose $1 \notin \text{Kern}(G)$. If all elements of $\text{Kern}(G)$ are mutually disjoint, then it is easy to see that all members of $\text{Kern}(G)$ are of order 2. Thus, by Proposition 2.5(1), $|G| = 16$, which is a contradiction. So, we may assume that $K_1 \cap K_2 = N \neq 1$. Lemma 3.1 yields that $|K_1| = |K_2| = |G'| = 4$. We show that $|K_3| \leq 4$. Indeed, if $|K_3| \geq 4$, then it contains a normal subgroup $N$ of $G$ with $|N| = 2$. By Lemma 2.1, $N$ contains exactly two members of $\text{Kern}(G)$. Hence, by Lemma 3.1, $|K_3| = 4$, as wanted. If $|Z(G)| = 4$, then the result follows by Proposition 2.5(2). So, assume that $|Z(G)| > 4$. Then, $Z(G)$ has at least 3 subgroups of order 4. Since $G'$ is not contained in $Z(G)$, we conclude that $Z(G)$ has exactly three subgroups of order 4. Consequently, $K_3$ is also central and $G$ satisfies the strong condition. The result now follows by Proposition 2.5(1). Next, we assume that $1 \in \text{Kern}(G)$. Then $Z(G)$ is cyclic and consequently, $G$ has only one minimal normal subgroup. Let $K_3 = 1$ and note that by Proposition 2.7, $|K_1| > 2$ and $|K_2| > 2$. Thus, $K_1 \cap K_2 = N \neq 1$ and $G/N$ is a group with two non-linear irreducible character kernels. Note that by Lemma 2.1, $N$ is the unique minimal normal subgroup of $G$. Now, apply Lemma 3.1 to $G/N$ and obtain $|G'| = |K_1| = |K_2| = 4$. In particular, $G$ satisfies the weak condition. Since $Z(G)$ is cyclic, it contains only one subgroup of order 4. Thus, $Z(G)$ does not contain both $K_1$ and $K_2$. On the other hand, by our hypothesis, $G$ is of class greater that 2. So, $Z(G)$ contains either $K_1$ or $K_2$. Also Lemma 2.1 implies that $Z(G)$ coincides to one of these subgroups. Hence, $|Z(G)| = 4$ and the result follows by Proposition 2.5(2).

Remark 3.4. Using GAP, one may check that groups of order 32 of class exceeding 2 with a cyclic center have precisely three non-linear irreducible character kernels. Furthermore, among the groups of order 32 or 64, these are the only groups of class greater than 2 that satisfy this property.

Example 3.5. Let $G$ be a group of order 32 such that $|G : G'| = 8$ and $\text{cd}(G) = \{1, 2, 4\}$ (see [6, Example 6.11]). Then $|\text{Kern}(G)| = 3$.

Proof of Theorem 1.1. Part (1) follows from Lemma 2.1 and Lemma 2.2. Now, assume that $t = 2$. If $G$ has no faithful irreducible characters, then
by Lemma 3.1, it is of type (b). So, let $1 \in \text{Kern}(G)$. Since $|\text{Kern}(G)| = 2$, the non-linear irreducible character kernels of $G$ constitute a chain with respect to inclusion. Thus by Proposition 2.7, $G$ is of maximal class. Also, by Corollary 2.8, $c(G) = 3$. Hence, $|G| = p^4$ and $G$ is of type (a).

To complete the proof, it remains to prove the third part of the theorem. If $G$ satisfies one of the conditions (a)−(f) of (3), then by Lemma 3.2, $|\text{Kern}(G)| = 3$. So, assume that $|\text{Kern}(G)| = 3$. If $G$ is of maximal class, then by Proposition 2.7 and Corollary 2.8 $G$ is of type (a). So, assume that $G$ is not of maximal class. Our goal is to prove that $G$ is of type (b), (c), (d), (e) or (f). The proof is divided into several steps.

**Step 1.** If $|K| > p$, for some $K \in \text{Kern}(G)$, then $p = 2$.

First, we suppose that $1 \in \text{Kern}(G)$ and let $N$ be the (unique) central subgroup of $G$ of order $p$. Since $G$ is not of maximal class, we get $N \not\in \text{Kern}(G)$. So, by Lemma 3.1, $G/N$ is a 2-group. Next, assume that $1 \not\in \text{Kern}(G/N)$. Then, $Z(G)$ is not cyclic and in particular, contains at least $p + 1$ subgroups of order $p$. If $p > 2$, then we may choose a subgroup $N$ of $Z(G)$ of order $p$ with $N \neq G'$ and $N \not\in \text{Kern}(G)$. Now, if $|\text{Kern}(G/N)| = 1$, then Lemma 2.1 implies that $N \in \text{Kern}(G)$ which is a contradiction. Also, if $|\text{Kern}(G/N)| = 3$, then $N \leq K$, for each $K \in \text{Kern}(G/N)$. Hence, $N = 1$ by Lemma 2.1 which is again a contradiction. Thus, we conclude that $G/N$ has exactly two non-linear irreducible character kernels. Since $N \not\in \text{Kern}(G)$, we conclude that $G/N$ has no faithful irreducible characters. Thus by Lemma 3.1, $G$ is a 2-group. This is the final contradiction.

**Step 2.** We have $p \leq 3$.

By Step 1, we may assume that all of the elements of $\text{Kern}(G)$ are of order $p$. Then, $Z(G)$ contains at least $p + 1$ subgroups of order $p$. On the other hand, each central subgroup $N$ of order 2, except $G'$, is in $\text{Kern}(G)$. Thus, the number of the subgroups of $Z(G)$ of order 2 does not exceed 4. This gives us $p \leq 3$. 

Step 3. If \( p = 3 \), then \( G' \) and all elements of \( \text{Kern}(G) \) are of order 3, \( Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) \( (r \geq 1) \) and \( r > 1 \) implies that \( G' \subseteq \Phi(Z(G)) \).

By Step 1, all elements of \( \text{Kern}(G) \) are of order 3. Since \( Z(G) \) has at least four subgroups of order 3, then \( |G'| = 3 \) and \( Z(G) \cong \mathbb{Z}_{3r} \times \mathbb{Z}_{3s} \), for some positive integers \( r \) and \( s \). On the other hand, for each central subgroup \( L \neq G' \) of order 3, \( Z(G/L) \) is cyclic. Hence \( r = 1 \) or \( s = 1 \).

The reminder of the proof is almost the same as that of Lemma 3.1.

Step 4. If \( p = 2 \) and all elements of \( \text{Kern}(G) \) are of order 2, then \( G' = Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \).

Certainly, \( |G'| > 2 \). On the other hand, if \( N \) is a central subgroup of \( G \) of order 4, then \( G/N \) does not admit any non-linear irreducible character. Hence, \( G' \) is the unique subgroup of \( Z(G) \) of order 4. Since \( Z(G) \) is not cyclic, we have \( G' = Z(G) \).

Step 5. If \( |K| > p \), for some \( K \in \text{Kern}(G) \) and \( c(G) = 2 \), then \( G \) is in type (e) or (f).

Assume that \( |K_1| > 2 \). Let \( N \leq Z(G) \cap K_1 \) and \( |N| = 2 \). Then by Lemma 3.1, \( N \) contains two non-linear irreducible character kernels of \( G \). So, we may assume that \( N \leq K_2 \) and \( |K_1| = |K_2| = 4 \). First, assume that \( Z(G) \) is not cyclic. Then, we may choose three distinct normal subgroups of order 2. So, we must have \( |G'| = |K_3| = 2 \) and \( Z(G) \) has exactly three subgroups of order 2. Also \( G' \) is not contained in \( \Phi(Z(G)) \). If \( |Z(G)| > 8 \) then it contains a subgroup of order 8, not containing \( G' \). This is clearly a contradiction. So, \( |Z(G)| \leq 8 \). As \( G \) satisfies the strong condition, the equality holds. So, \( G \) is of type (e).

Next, assume that \( Z(G) \) is cyclic. Then, \( G \) satisfies the weak condition. Also by Lemma 3.1, \( |G'/N| = 2 \). Hence, \( |G'| = 4 \) and we obtain \( G \) is of type (f).

To complete the proof, we use the assertions of the above steps. If \( p > 2 \), then by Steps 2 and 3, \( G \) is of type (c). So, assume that \( p = 2 \). If \( c(G) > 2 \), then as a result of Lemma 3.3 and Remark 3.4, we conclude that the group \( G \) is of type (b). Also, if \( c(G) = 2 \), then by Step 1 and Step 5, \( G \) is of type (d), (e) or (f).

\( \square \)
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