

Chebyshev Centers and Approximation in Pre-Hilbert C^* -Modules

A. NIKNAM AND S. SHADKAM*

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ABSTRACT. We extend the study of Chebyshev centers in pre-Hilbert C^* -modules by considering the C^* -algebra valued map defined by $|x| = \langle x, x \rangle^{1/2}$. We prove that if T is a remotal subset of a pre-Hilbert C^* -module M , and $F \subseteq M$ is star-shaped at a relative Chebyshev center c of T with respect to F , then $|x - q_T(x)|^2 \geq |x - c|^2 + |c - q_T(c)|^2$ ($x \in F$). The uniqueness of Chebyshev center follows from this inequality. This is a generalization of a well-known result on Hilbert spaces.

1. Introduction

A normed algebra is an algebra A with a norm $\|\cdot\|$ such that $\|xy\| \leq \|x\|\|y\|$, $x, y \in A$. A complete normed algebra A is called a Banach algebra. An involution $*$ on an algebra A is a mapping $x \rightarrow x^*$ from A onto A such that $(\lambda x + y)^* = \bar{\lambda}x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$, for all $x, y \in A, \lambda \in \mathbb{C}$. An involutive Banach algebra is called a Banach $*$ -algebra. A Banach $*$ -algebra A is said to be a C^* -algebra if $\|xx^*\| = \|x\|^2$. An element x in a C^* -algebra A with unit e is called positive if $\text{sp}(x) \subseteq [0, \infty)$, where $\text{sp}(x) = \{\lambda \in \mathbb{C}; \lambda e - x \text{ is not invertible}\}$; we write $x \geq 0$ if x is a positive element.

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*Corresponding author

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Suppose that A is a C^* -algebra and E is a linear space, which is a right A -module and the scalar multiplication satisfies $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for all $x \in E, a \in A, \lambda \in \mathbb{C}$. The space E is called a pre-Hilbert A -module if there exists an A -valued map $\langle \cdot, \cdot \rangle : E \rightarrow A$ with the following properties:

- (i) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$.
- (ii) $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$, $x, y, z \in E, \lambda \in \mathbb{C}$.
- (iii) $\langle x, ya \rangle = \langle x, y \rangle a$, $x, y \in E$ and $a \in A$.
- (iv) $\langle x, y \rangle^* = \langle y, x \rangle$, $x, y \in E$.

Such a map $\langle \cdot, \cdot \rangle : E \rightarrow A$ is called an A -valued inner product. E is called a (right) Hilbert A -module if it is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. We note that Hilbert C^* -modules contain both Hilbert spaces and C^* -algebras. In fact, every Hilbert space is a Hilbert \mathbb{C} -module and if A is a C^* -algebra, then A is a Hilbert A -module, whenever we define $\langle a, b \rangle = a^*b$, $a, b \in A$.

We define an A -valued map by $|x| = \langle x, x \rangle^{1/2}$. This is not actually an extension of a norm, in general, since it may happen that the triangle inequality does not hold [7].

The importance of our approach to the theory of approximation in pre-Hilbert C^* -modules is that we do not use the triangle inequality. This may motivate us to study the geometry in case the triangle inequality does not hold.

Hilbert C^* -modules were first introduced and investigated by I. Kaplansky [5], M. Rieffel [13] and W. Paschke [11]. They played an essential role in operator algebras [12], KK-Theory [3], operator spaces [2], quantum group theory [14], Morita equivalence [13] and so on. They are a generalization of Hilbert spaces, but there are some differences between the two classes. For example, each operator on a Hilbert space has an adjoint, but a bounded A -module map on a Hilbert A -module is not adjointable, in general, ([7], page 8). Throughout this paper, we assume that $(M, \langle \cdot, \cdot \rangle)$ is a pre-Hilbert C^* -module over a commutative C^* -algebra A . In particular, the commutative C^* -algebras which are boundedly complete lattices with respect to their natural order structures, i.e., those having the property that each set of functions that has an upper bound has a least upper bound, are of special interest. An easy example is the complex field \mathbb{C} . One however shows that if a commutative C^* -algebra $C(X)$ is a boundedly complete lattice with respect to the natural partial ordering of its real-linear subspace $C(X, \mathbb{R})$ of continuous real-valued functions on X , then X is extremely disconnected,

i.e., its open sets have open closures [4].

Let T be a non-empty subset of M . The mapping $Q_T : M \rightarrow 2^T$ defined by $Q_T(x) = \{y \in T : |x - y| = \max\{|x - t| : t \in T\}\}$ is called the farthest point map of T . We call T a remotal (uniquely remotal) set, if for each $x \in M$ the set $Q_T(x)$ is non-empty (is a singleton). The element of $Q_T(x)$ is denoted by $q_T(x)$ if it is a singleton. A subset F of M is said to be star-shaped at a vertex $s \in F$ if and only if for each $x \in F$ the line segment $[s, x] = \{\lambda s + (1 - \lambda)x : 0 \leq \lambda \leq 1\}$ lies in F .

A relative Chebyshev center of $T \subseteq M$ in $F \subseteq M$ is an element c in M that satisfies $|c - q_T(c)| = \min\{|x - q_T(x)| : x \in F\} := r_F(T)$, if the minimum exists. In the case that $F = M$, we call c the Chebyshev center of T and denote $r_F(T)$ by $r(T)$. We represent by $d(T)$, the A -valued diameter $\max\{|t - s| : t, s \in T\}$ of T , if it exists.

One outstanding open problem in the geometry of normed spaces is the Farthest Point Problem [9]. This problem asks whether every uniquely remotal set in a normed space is a singleton. There are some cases such as the finite dimensional spaces and the Banach spaces c_0 and c , in which the problem is solved affirmatively [1]. The problem is related to the problem of proving the convexity of Chebyshev sets in a Hilbert space [6] (recall that a subset T of a normed space X is called Chebyshev, if for every $x \in X$ there exists a unique best approximation of x in T). The reader is referred to [7],[8] and [12] for details on Hilbert C^* -modules, on commutative C^* -algebras.

2. Main results

Let $(M, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert C^* -module over a commutative C^* -algebra A . We now establish some interesting results similar to those in [10] about Hilbert C^* -modules. We start our work with an applicable example of a remotal set.

Example 1. Let $X = \{a, b\}$, $A = C(X)$ and $E = \{f \in C(X) : f(a) = 0\}$. Then, E is a maximal ideal of the C^* -algebra A and so can be regarded as a Hilbert A -module. Assume that $T = \{f_1, f_2\} \subseteq E$, where $f_1(b) = 1$ and $f_2(b) = 2$. Then, T is remotal, since for each $f \in E$ there exists a function $q_T(f) \in T$ such that $|f(b) - q_T(f)(b)| = \max\{|f(b) - 1|, |f(b) - 2|\}$. In fact, a straightforward verification shows that for each $f \in E$, if $Re f(b) > \frac{3}{2}$, then $q_T(f) = f_1$; if $Re f(b) = \frac{3}{2}$, then $q_T(f)$ can

be chosen to be f_1 or f_2 ; and if $Ref(b) < \frac{3}{2}$, then $q_T(f) = f_2$ and also $d(T) = |f_1(b) - f_2(b)| = 1$.

Lemma 2.1. *Suppose T is a uniquely remotal subset of M and F is a star-shaped subset of M at a vertex c such that c is a relative center of T with respect to F . Then, 0 is a relative center of $c - T$ with respect to $c - F$.*

Proof. We first prove the identity $c - q_T(x) = q_{c-T}(c - x)$, for all $x \in F$. We know

$$|c - x - q_{c-T}(c - x)| \geq |c - x - (c - q_T(x))|.$$

Since $c - q_{c-T}(c - x) \in T$, $|x - q_T(x)| \geq |x - (c - q_{c-T}(c - x))|$. Hence,

$$|c - x - q_{c-T}(c - x)| = |c - x - (c - q_T(x))|.$$

Therefore, $q_{c-T}(c - x) = c - q_T(x)$, since T is a uniquely remotal set.

We now show that $|0 - q_{c-T}(0)| \leq |c_1 - q_{c-T}(c_1)|$, for all $c_1 \in c - F$.

We know that $|c - q_T(c)| \leq |x_1 - q_T(x_1)|$ for all $x_1 \in F$. So,

$$|q_{c-T}(0)| = |c - (c - q_{c-T}(0))| \leq |c - x_1 - (c - q_T(x_1))|.$$

It follows therefore that $|0 - q_{c-T}(0)| \leq |c - x_1 - q_{c-T}(c - x_1)|$, and so $|0 - q_{c-T}(0)| \leq |c_1 - q_{c-T}(c_1)|$, for all $c_1 = c - x_1 \in c - F$. \square

Theorem 2.2. *Suppose T is a uniquely remotal subset of M and F is a star-shaped subset of M at a vertex c such that c is also a relative center of T with respect to F . Then,*

(i) $Re(\langle c - x, c - q_T(x) \rangle) \leq 0$, for all $x \in F$.

(ii) if $q_T(c) \in F$ is a cluster point of $\bigcup\{Q_T(x) : x \in [c, q_T(c)]\}$, then $T = \{c\}$.

Proof. (i) By lemma 2.2, we may assume, without loss of generality, that $c = 0$. Let $0 < \alpha < 1$. By the definition of the farthest point map q_T , we have

$$|x - q_T(x)|^2 \geq |x - q_T(\alpha x)|^2, |\alpha x - q_T(\alpha x)|^2 \geq |\alpha x - q_T(x)|^2.$$

Therefore,

$$\langle x - q_T(x), x - q_T(x) \rangle \geq \langle x - q_T(\alpha x), x - q_T(\alpha x) \rangle,$$

$$\langle \alpha x - q_T(\alpha x), \alpha x - q_T(\alpha x) \rangle \geq \langle \alpha x - q_T(x), \alpha x - q_T(x) \rangle.$$

By adding both sides of these inequalities, we obtain

$$(1 - \alpha)[\langle x, q_T(\alpha x) \rangle + \langle q_T(\alpha x), x \rangle] \geq (1 - \alpha)[\langle x, q_T(x) \rangle + \langle q_T(x), x \rangle].$$

Hence,

$$(2.1) \quad \operatorname{Re}(\langle x, q_T(\alpha x) \rangle) \geq \operatorname{Re}(\langle x, q_T(x) \rangle).$$

On the other hand, $|\alpha x - q_T(\alpha x)| \geq |0 - q_T(0)| \geq |q_T(\alpha x) - 0|$, for all $x \in F$, since 0 is the relative Chebyshev center with respect to F . Hence,

$$\begin{aligned} \langle \alpha x - q_T(\alpha x), \alpha x - q_T(\alpha x) \rangle &= |\alpha x - q_T(\alpha x)|^2 \geq |q_T(\alpha x)|^2 \\ &= \langle q_T(\alpha x), q_T(\alpha x) \rangle. \end{aligned}$$

We have

$$\begin{aligned} \langle \alpha x, \alpha x \rangle - \langle q_T(\alpha x), \alpha x \rangle - \langle \alpha x, q_T(\alpha x) \rangle + \langle q_T(\alpha x), q_T(\alpha x) \rangle &\geq \\ \langle q_T(\alpha x), q_T(\alpha x) \rangle. & \end{aligned}$$

Therefore,

$$\alpha^2|x|^2 - \langle q_T(\alpha x), \alpha x \rangle - \langle \alpha x, q_T(\alpha x) \rangle \geq 0.$$

Dividing by α , we have

$$\alpha|x|^2 \geq \langle q_T(\alpha x), x \rangle + \langle x, q_T(\alpha x) \rangle.$$

Then,

$$(2.2) \quad \alpha|x|^2 \geq 2\operatorname{Re}(\langle x, q_T(\alpha x) \rangle).$$

We have from (2.1) and (2.2) that

$$(2.3) \quad \alpha|x|^2 \geq 2\operatorname{Re}(\langle x, q_T(x) \rangle).$$

Since (2.3) holds for each α ($0 < \alpha < 1$) and by the Gelfand representation of A , we get

$$\operatorname{Re}(\langle x, q_T(x) \rangle) \leq 0.$$

(ii) Suppose that there exists a sequence $\{\lambda_n\}$ in $[0, 1]$ such that $y_n = q_T(x_n) \rightarrow q_T(c)$, where $x_n = \lambda_n c + (1 - \lambda_n)q_T(c)$. It follows from (i) that

$$\operatorname{Re}(\langle c - x_n, c - q_T(x_n) \rangle) \leq 0.$$

But, $\langle c - x_n, c - y_n \rangle = \langle c - (\lambda_n c + (1 - \lambda_n)q_T(c)), c - y_n \rangle = (1 - \lambda_n)\langle c - q_T(c), c - y_n \rangle$. Since $1 - \lambda_n \geq 0$, we infer that $\operatorname{Re}(\langle c - q_T(c), c - y_n \rangle) \leq 0$. Due to $c - y_n \rightarrow c - q_T(c)$ and the continuity of the inner product, we conclude that $\operatorname{Re}(\langle c - q_T(c), c - q_T(c) \rangle) \leq 0$. Hence, $|c - q_T(c)|^2 = \operatorname{Re}(\langle c - q_T(c), c - q_T(c) \rangle) = 0$. Thus, $|c - q_T(c)| = \max\{|c - t| : t \in T\} = 0$. It follows that $c - t = 0$, for all $t \in T$. So, $T = \{c\}$. \square

Theorem 2.3. *Suppose T is a remotal subset of M , $d(T)$ exists, $F \subseteq M$ and c is a relative center of T with respect to F . Then, the followings hold:*

- (i) $|x - q_T(x)|^2 \geq |x - c|^2 + r_F^2(T)$, for all $x \in F$.
(ii) c is unique and if $F \cap Q_T(c) \neq \phi$, then $d(T) \geq \sqrt{2}r_F(T)$.
(iii) If T is uniquely remotal and $\operatorname{Re}(\langle c - x_0, c - q_T(x_0) \rangle) = 0$, for some $x_0 \in F$, then $q_T(x_0) = q_T(c)$, and therefore, if $q_T(c) \in F$, then T is a singleton if and only if $\operatorname{Re}(\langle c - q_T(c), c - q_T(q_T(c)) \rangle) = 0$.

Proof. (i) By lemma 2.2, we can assume that $c = 0$. By Theorem 2.3(i), we have

$\operatorname{Re}(\langle x, q_T(x) \rangle) \leq 0$, for all $x \in F$. Since F is star-shaped, $\langle \alpha x, q_T(\alpha x) \rangle + \langle q_T(\alpha x), \alpha x \rangle \leq 0$, for all $x \in F, 0 \leq \alpha \leq 1$.

It follows that $\operatorname{Re}(\langle x, q_T(\alpha x) \rangle) \leq 0$. We thus obtain:

$$\begin{aligned} r_F^2(T) &\leq |\alpha x - q_T(\alpha x)|^2 \\ &= \langle \alpha x - q_T(\alpha x), \alpha x - q_T(\alpha x) \rangle \\ &= \langle \alpha x - x + x - q_T(\alpha x), \alpha x - x + x - q_T(\alpha x) \rangle \\ &= (\alpha - 1)^2 \langle x, x \rangle + \langle (\alpha - 1)x, x - q_T(\alpha x) \rangle \\ &\quad + \langle x - q_T(\alpha x), (\alpha - 1)x \rangle + \langle x - q_T(\alpha x), x - q_T(\alpha x) \rangle \\ &= (\alpha - 1)^2 |x|^2 + 2(\alpha - 1) \langle x, x \rangle + (1 - \alpha) [\langle x, q_T(\alpha x) \rangle \\ &\quad + \langle q_T(\alpha x), x \rangle] + |x - q_T(\alpha x)|^2 \\ &\leq (\alpha^2 - 1) |x|^2 + |x - q_T(\alpha x)|^2 \\ &\leq (\alpha^2 - 1) |x|^2 + |x - q_T(x)|^2. \end{aligned}$$

Therefore, we have $|x - q_T(x)|^2 \geq (1 - \alpha^2) |x|^2 + r_F^2(T)$, for all $\alpha \in [0, 1]$. Therefore, $|x - q_T(x)|^2 \geq |x|^2 + r_F^2(T)$.

(ii) If c' is another Chebyshev center with respect to F , then by (i),

$$|c - q_T(c)|^2 = |c' - q_T(c')|^2 \geq |c' - c|^2 + r_F^2(T).$$

Hence, $|c' - c| = 0$. So, $c' = c$. This proves the uniqueness assertion.

Let $x = q_T(c) \in F \cap Q_T(c)$. We have $|q_T(c) - q_T(q_T(c))|^2 \geq |q_T(c) - c|^2 + r_F^2(T)$, and so $|q_T(c) - q_T(q_T(c))|^2 \geq 2r_F^2(T)$. Hence, $d(T)^2 \geq |q_T(c) - q_T(q_T(c))|^2 \geq 2r_F^2(T)$.

(iii) By (i) with $x = x_0$, we have $|c - q_T(c)|^2 + |x_0 - c|^2 \leq |x_0 - q_T(x_0)|^2$. Hence,

$$(2.4) \quad |c - q_T(c)|^2 \leq |x_0 - q_T(x_0)|^2 - |x_0 - c|^2.$$

But,

$$\langle c - x_0 - (c - q_T(x_0)), c - x_0 - (c - q_T(x_0)) \rangle = |q_T(x_0) - x_0|^2.$$

Therefore,

$$\begin{aligned} \langle c - x_0, c - x_0 \rangle + \langle c - q_T(x_0), c - q_T(x_0) \rangle + \langle c - x_0, c - q_T(x_0) \rangle \\ + \langle c - q_T(x_0), c - x_0 \rangle = |x_0 - q_T(x_0)|^2. \end{aligned}$$

Using our assumption on x_0 , we obtain:

$$\langle c - x_0, c - x_0 \rangle + \langle c - q_T(x_0), c - q_T(x_0) \rangle = |x_0 - q_T(x_0)|^2.$$

Therefore, $|c - q_T(x_0)|^2 + |x_0 - c|^2 = |x_0 - q_T(x_0)|^2$. It follows from (2.4) that

$$|c - q_T(c)|^2 \leq |x_0 - q_T(x_0)|^2 - |x_0 - c|^2 = |c - q_T(x_0)|^2 \leq |c - q_T(c)|^2.$$

Hence, $|c - q_T(c)| = |c - q_T(x_0)|$. Due to the fact that T is uniquely remotal, $q_T(c) = q_T(x_0)$.

If $q_T(c) \in F$ and $\langle c - q_T(c), c - q_T(q_T(c)) \rangle + \langle c - q_T(q_T(c)), c - q_T(c) \rangle = 0$, then by the first part of (iii) with $x_0 = q_T(c)$, we have $q_T(c) = q_T(q_T(c))$. Hence, $T = \{x_0\}$. Conversely, if T is a singleton set, then $q_T(c) = q_T(q_T(c))$ and $|c - q_T(c)| \leq |q_T(c) - q_T(q_T(c))| = 0$. So $c - q_T(c) = 0$, i.e., $T = \{c\}$. Therefore, $c = q_T(q_T(c))$ and we conclude that $\langle c - q_T(c), c - q_T(q_T(c)) \rangle = 0$.

Corollary 2.4. *Let T be a uniquely remotal subset M such that $d(T)$ exists, and let c be a Chebyshev center of T . Then, the following assertions are satisfied:*

- (i) $|x - q_T(x)|^2 \geq |x - c|^2 + r^2(T)$.
- (ii) If T is not a singleton, then $d(T) \geq \sqrt{2}r(T)$.

Proof. (i) This part follows immediately from assertion (i) of Theorem 2.4 with $F = M$.

(ii) We know that $d(T)^2 \geq |q_T(c) - q_T(q_T(c))|^2$. We infer therefore that $|q_T(c) - q_T(q_T(c))|^2 \geq 2r_F^2(T) \geq 2r^2(T)$, by part (i) of Theorem 2.4. \square

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A. Niknam and S. Shadkam

Department of Mathematics and Center of Excellence of Analysis on Algebraic Structure, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad, Iran.

Email: niknam@math.um.ac.ir

Email: shadkam.s@wali.um.ac.ir