SOME HOMOLOGICAL PROPERTIES OF AMALGAMATED DUPLICATION OF A RING ALONG AN IDEAL

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Abstract. We investigate the transfer of some homological properties from a ring $R$ to its amalgamated duplication along some ideal $I$ of $R \rhd \lhd I$, and then generate new and original families of rings with these properties.

1. Introduction

Let $R$ be a commutative ring with unit element $1$ and let $I$ be a proper ideal of $R$. The amalgamated duplication of a ring $R$ along an ideal $I$ is a ring that is defined as the following subring with unit element $(1, 1)$ of $R \times R$:

$$R \rhd \lhd I := \{(r, r + i)/r \in R, i \in I\}.$$ 

This construction has been studied, in the general case, and from different points of view of pullbacks, by D’Anna and Fontana [7]. Also, D’Anna and Fontana, [5] have considered the case of the amalgamated duplication of a ring, not necessarily in a Noetherian setting, along a multiplicative-canonical ideal in the sense of Heinzer-Huckaba-Papick...
In [6], D’Anna has studied some properties of $R \triangledown I$, in order to construct reduced Gorenstein rings associated to Cohen-Macaulay rings and has applied this construction to curve singularities. On the other hand, Maimani and Yassemi [15] have studied the diameter and girth of the zero-divisor graph of the ring $R \triangledown I$. For instance, see [5, 6, 7, 15].

Let $M$ be an $R$-module, the idealization $R \rtimes M$ (also called the trivial extension), introduced by Nagata in 1956 (cf. [16]) is defined as the $R$-module $R \oplus M$ with multiplication defined by $(r, m)(s, n) := (rs, rn + sm)$. For instance, see [8, 9, 11, 12].

When $I^2 = 0$, the new construction $R \bowtie I$ coincides with the idealization $R \rtimes I$. One main difference of amalgamated-duplication, with respect to idealization, is that the ring $R \bowtie I$ can be a reduced ring (and in fact, it is always reduced if $R$ is a domain).

For two rings $A \subset B$, we say that $A$ is a module retract (or a subring retract) of $B$ if there exists an $A$-module homomorphism $\varphi : B \to A$ such that $\varphi \mid_A = \text{id} \mid_A$. In this case, $\varphi$ is called a module retraction map. If such a map $\varphi$ exists, then $B$ contains $A$ as an $A$-module direct summand. We can easily show that $R$ is a module retract of $R \bowtie I$, where the module retraction map $\varphi$ is defined by $\varphi(r, r+i) := r$.

Here, we study the transfer of some homological properties from a ring $R$ to the ring $R \bowtie I$. Specially, we prove that $R \bowtie I$ is a von Neumann regular ring (resp., a perfect ring) if and only if so is $R$. Also, we prove that $\text{gldim}(R \bowtie I) = \infty$ if $R$ is a domain and $I$ is a principal ideal of $R$.

Recall that if $R$ is a ring and $M$ is an $R$-module, as usual we use $\text{pd}_R(M)$ and $\text{fd}_R(M)$ to denote the usual projective and flat dimensions of $M$, respectively. The classical global and weak dimensions of $R$ are respectively denoted by $\text{gldim}(R)$ and $\text{wdim}(R)$. Also, the Krull dimension of $R$ is denoted by $\text{dim}(R)$.

2. Main results

Let $R$ be a commutative ring with identity element 1 and let $I$ be an ideal of $R$. Recall that $R \bowtie I := \{(r, s)/r, s \in R, r - s \in I\}$. It is easy
to check that $R \Join I$ is a subring with unit element $(1,1)$ of $R \times R$ (with the usual componentwise operations) and that $R \Join I = \{(r, r + i)/r \in R, i \in I\}$.

It is easy to see that if $\pi_i (i = 1, 2)$ are the projections of $R \times R$ on $R$, then $\pi_i (R \Join I) = R$, and hence if $O_1 := \ker (\pi_1 \backslash R \Join I)$, then $R \Join I/O_1 \cong R$. Moreover, $O_1 = \{(0, i), i \in I\}$, $O_2 = \{(i, 0), i \in I\}$ and $O_1 \cap O_2 = (0)$.

We begin by studying the transfer of von Neumann regular property.

**Theorem 2.1.** Let $R$ be a commutative ring and let $I$ be a proper ideal of $R$. Then, $R$ is a von Neumann regular ring if and only if $R \Join I$ is a von Neumann regular ring.

The proof will use the following Lemma.

**Lemma 2.2.** [7, Theorem 3.5]

1. Let $R$ be a commutative ring and let $I$ be an ideal of $R$. Let $P$ be a prime ideal of $R$ and set
   \[ P_0 = \{(p, p + i)/p \in P, i \in I \cap P\}, \]
   \[ P_1 = \{(p, p + i)/p \in P, i \in I\}, \]
   \[ P_2 = \{(p + i, p)/p \in P, i \in I\} \]
   - If $I \subseteq P$, then $P_0 = P_1 = P_2$ is a prime ideal of $R \Join I$ and it is the unique prime ideal of $R \Join I$ lying over $P$.
   - If $I \not\subseteq P$, then $P_1 \neq P_2$, $P_1 \cap P_2 = P_0$ and $P_1$ and $P_2$ are the only prime ideals of $R \Join I$ lying over $P$.

2. Let $Q$ be a prime ideal of $R \Join I$ and let $O_1 = \{(0, i)/i \in I\}$. Two cases are possible: either $Q \not\subseteq O_1$ or $Q \supseteq O_1$.
   - **a:** If $Q \not\subseteq O_1$, then there exists a unique prime ideal $P$ of $R$ ($I \not\subseteq P$) such that $Q = P_2 = \{(p + i, p)/p \in P, i \in I\}$.
   - **b:** If $Q \supseteq O_1$, then there exists a unique prime ideal $P$ of $R$ such that $Q = P_1 = \{(p, p + i)/p \in P, i \in I\}$.

**Proof of Theorem 2.1.** Assume that $R$ is a von Neumann regular ring. Then, $R$ is reduced and so $R \Join I$ is reduced by [7, Theorem 3.5 (a)(vi)].
It remains to show that $\dim(R \bowtie I) = 0$ by [9, Remark, p. 5]. Let $Q$ be a prime ideal of $R \bowtie I$. If $P = Q \cap R$, then $Q \in \{P_1, P_2\}$ (by Lemma 2.2(2)). But, $P$ is a maximal ideal of $R$, since $R$ is a von Neumann regular ring. Then, $P_1$ and $P_2$ are maximal ideals of $R \bowtie I$ (by [7, Theorem 3.5 (a)(vi)]). Hence, $Q$ is a maximal ideal of $R \bowtie I$, as desired.

Conversely, assume that $R \bowtie I$ is a von Neumann regular ring. By [7, Theorem 3.5 (a)(vi)], $R$ is reduced. Let $P$ be a prime ideal of $R$. By Lemma 2.2(1), $P \bowtie I = \{(p, p + i)/p \in P, i \in I\}$ is a prime ideal of $R \bowtie I$. From [9, p. 7], we get $P \bowtie I$ to be a maximal ideal of $R \bowtie I$, and hence $P$ is a maximal ideal of $R$. Therefore, $\dim(R) = 0$, and so $R$ is a von Neumann regular ring.

A ring $R$ is called semisimple if every $R$-module is projective, that is, $\text{gldim}(R) = 0$ (see [8, P. 26]). Recall that a ring is semisimple if and only if it is Noetherian von Neumann regular by [8, Theorems (1.4.2, 1.4.6, and 1.3.10(2)].

**Corollary 2.3.** Let $R$ be a commutative ring and let $I$ be a proper ideal of $R$. Then, $R$ is a semisimple ring if and only if $R \bowtie I$ is a semisimple ring.

**Proof.** Assume that $R$ be a semisimple ring. Then, $R$ is a Noetherian von Neumann regular ring. By Theorem 2.1, $R \bowtie I$ is a von Neumann regular ring and by [7, Corollary 3.3], $R \bowtie I$ is Noetherian. Therefore, $R \bowtie I$ is semisimple.

Conversely, assume that $R \bowtie I$ is semisimple. Then, $R \bowtie I$ is a Noetherian von Neumann regular ring, and so $R$ is a von Neumann regular ring (by Theorem 2.1) and Noetherian (by [7, Corollary 3.3]). Hence, $R$ is semisimple.

A ring $R$ is called a stably coherent ring if for every positive integer $n$, the polynomial ring in $n$ variables over $R$ is a coherent ring. Recall that a ring $R$ is is called a coherent ring if every finitely generated ideal of $R$ is finitely presented.

**Corollary 2.4.** Let $R$ be a commutative ring and let $I$ be a proper ideal of $R$. If $R$ is a von Neumann regular ring, then $R \bowtie I$ is a stably coherent ring.

**Proof.** Use Theorem 2.1 and [8, Theorem 7.3.1].

Now, we are able to construct a new class of non-Noetherian von Neumann regular rings.
Example 2.5. Let $R$ be a non-Noetherian von Neumann regular ring and $I$ be a proper ideal of $R$. Then, $R \bowtie I$ is a non-Noetherian von Neumann regular ring by [7, Corollary 3.3] and Theorem 2.1.

We recall that a ring $R$ is called a perfect ring if every flat $R$-module is a projective $R$-module (see [1]). Secondly, we study the transfer of perfect property.

Theorem 2.6. Let $R$ be a commutative ring and let $I$ be a proper ideal of $R$. Then, $R$ is a perfect ring if and only if $R \bowtie I$ is a perfect ring.

To prove Theorem 2.6, we need the following lemmas.

Lemma 2.7. ([13, Lemma 2.5.(2)]) Let $(R_i)_{i=1,2}$ be a family of rings and $E_i$ be an $R_i$-module, for $i = 1, 2$. Then, $\text{pd}_{R_1 \times R_2}(E_1 \times E_2) = \sup\{\text{pd}_{R_1}(E_1), \text{pd}_{R_2}(E_2)\}$.

Lemma 2.8. Let $(R_i)_{i=1,2}$ be a family of rings and $E_i$ be an $R_i$-module for $i = 1, 2$. Then, $\text{fd}_{R_1 \times R_2}(E_1 \times E_2) = \sup\{\text{fd}_{R_1}(E_1), \text{fd}_{R_2}(E_2)\}$.

Proof. The proof is analogous to the proof of Lemma 2.7.

Lemma 2.9. Let $(R_i)_{i=1,...,m}$ be a family of rings. Then, $\prod_{i=1}^{m} R_i$ is a perfect ring, if and only if $R_i$ is a perfect ring, for each $i = 1, ..., m$.

Proof. The proof is done by induction on $m$ and it suffices to check it for $m = 2$. Let $R_1$ and $R_2$ be two rings such that $R_1 \times R_2$ is perfect. Let $E_1$ be a flat $R_1$-module and let $E_2$ be a flat $R_2$-module. By Lemma 2.8, $E_1 \times E_2$ is a flat $(R_1 \times R_2)$-module, and so it is a projective $(R_1 \times R_2)$-module, since $R_1 \times R_2$ is a perfect ring. Hence, $E_1$ is a projective $R_1$-module, and $E_2$ is a projective $R_2$-module by Lemma 2.7; this means that $R_1$ and $R_2$ are perfect rings.

Conversely, assume that $R_1$ and $R_2$ are two perfect rings. Let $E_1 \times E_2$ be a flat $(R_1 \times R_2)$-module, where $E_i$ is an $R_i$-module, for $i = 1, 2$. By Lemma 2.8, $E_1$ is a flat $R_1$-module and let $E_2$ be a flat $R_2$-module; so, $E_1$ is a projective $R_1$-module and $E_2$ is a projective $R_2$-module. Therefore, $E_1 \times E_2$ be a projective $(R_1 \times R_2)$-module by Lemma 2.7; this means that $R_1 \times R_2$ is a perfect rings.

Lemma 2.10. Let $R$ be a commutative ring and let $I$ be a proper ideal of $R$. Then,
(1) an \((R \triangleleft I)\)-module \(M\) is projective if and only if \(M \otimes_{R \triangleleft I} (R \times R)\) is a projective \((R \times R)\)-module and \(M/O_1M\) is a projective \(R\)-module.

(2) an \((R \triangleright I)\)-module \(M\) is flat if and only if \(M \otimes_{R \triangleright I} (R \times R)\) is a flat \((R \times R)\)-module and \(M/O_1M\) is a flat \(R\)-module.

Proof. Note that \(R \triangleleft I\) is a subring of \(R \times R\) and \(O_1\) is a common ideal of \(R \triangleright I\) and \(R \times R\) by [7, Proposition 3.1]. The result follows from [8, Theorem 5.1.1]. □

Proof of Theorem 2.6. Assume that \(R\) is a perfect ring and let \(M\) be a flat \((R \triangleright I)\)-module. By Lemma 2.10(2), \(M \otimes_{R \triangleright I} (R \times R)\) is a flat \((R \times R)\)-module and \(M/O_1M\) is a flat \(R\)-module. Then, \(M \otimes_{R \triangleright I} (R \times R)\) is a projective \((R \times R)\)-module (since \(R \times R\) is perfect, by Lemma 2.9), and \(M/O_1M\) is a projective \(R\)-module, since \(R\) is perfect. By Lemma 2.10(1), \(M\) is a projective \((R \triangleleft I)\)-module, and so \(R \triangleleft I\) is a perfect ring.

Conversely, assume that \(R \triangleright I\) is a perfect ring and let \(E\) be a flat \(R\)-module. Then, \(E \otimes_R (R \triangleright I)\) is a flat \((R \triangleright I)\)-module, and so it is a projective \((R \triangleright I)\)-module, since \(R \triangleright I\) is a perfect ring. In addition, for any \(R\)-module \(M\) and any \(n \geq 1\), we have

\[
\text{Ext}_R^n(E, M \otimes_R (R \triangleright I)) \cong \text{Ext}_R^n(E \otimes_R (R \triangleright I), M \otimes_R (R \triangleright I))
\]

(see [3, P. 118]), and then \(\text{Ext}_R^n(E, M \otimes_R (R \triangleright I)) = 0\). Since \(M\) is a direct summand of \(M \otimes_R (R \triangleright I)\) because \(R\) is a module retract of \(R \triangleright I\), \(\text{Ext}_R^n(E, M) = 0\), for all \(n \geq 1\) and all \(R\)-modules \(M\). This means that \(E\) is a projective \(R\)-module, and so \(R\) is a perfect ring. □

We say that a ring \(R\) is Steinitz if any linearly independent subset of a free \(R\)-module \(F\) can be extended to a basis of \(F\) by adjoining elements of a given basis. In [4, Proposition 5.4], Cox and Pendleton showed that Steinitz rings are precisely the perfect local rings.

By Theorem 2.6 and since \(R \triangleright I\) is local if and only if \(R\) is local, we obtain the following result.

Corollary 2.11. Let \(R\) be a commutative ring and \(I\) be a proper ideal of \(R\). Then, \(R\) is a Steinitz ring if and only if \(R \triangleright I\) is a Steinitz ring.

Example 2.12. Let \(R = K[X]/(X^2)\), where \(K\) is a field and \(X\) is indeterminate. Then, \((K[X]/(X^2)) \triangleright I\) is a Steinitz ring, where \(I := X(K[X]/(X^2))\).
For a nonnegative integer $n$, an $R$-module $E$ is $n$-presented if there is an exact sequence $F_n \to F_{n-1} \to \ldots \to F_0 \to E \to 0$, in which each $F_i$ is a finitely generated free $R$-module. In particular, “0-presented” means finitely generated and “1-presented” means finitely presented.

Given nonnegative integers $n$ and $d$, a ring $R$ is called an $(n, d)$-ring if every $n$-presented $R$-module has projective dimension $\leq d$; and $R$ is called a weak $(n, d)$-ring if every $n$-presented cyclic $R$-module has projective dimension $\leq d$ (equivalently, if every $(n - 1)$-presented ideal of $R$ has projective dimension $\leq d - 1$). For instance, the $(0, 1)$-domains are the Dedekind domains, the $(1, 1)$-domains are the Prüfer domains, and the $(1, 0)$-rings are the von Neumann regular rings; see, for instance, [2, 11, 12, 13, 14].

Now, we give a wide class of rings which are not weak $(n, d)$-rings (and so not $(n, d)$-rings) for positive integers $n$ and $d$.

**Theorem 2.13.** Let $R$ be an integral domain and let $I(\neq 0)$ be a principal ideal of $R$. Then, $R \bowtie I$ is not a weak $(n, d)$-rings (and so is not $(n, d)$-rings) for each positive integers $n$ and $d$. In particular, $\text{wdim}(R \bowtie I) = \text{gldim}(R \bowtie I) = \infty$.

To prove theorem 2.13, we need the following lemma.

**Lemma 2.14.** Let $R$ be a commutative ring and let $I(\neq 0)$ be a principal ideal of $R$. Then, $O_1 = \{(0, i), i \in I\}$ and $O_2 = \{(i, 0), i \in I\}$ are principal ideals of $R \bowtie I$.

**Proof.** Let $(0, i)$ be an element of $O_1$. Since $I$ is a principal ideal of $R$, there exists $a \in I$ such that $I = Ra$, and so $(0, i) = (0, ra) = (r + j, r)(0, a)$, for some $r \in R$ and for all $j \in I$. Hence, $O_1$ is a principal ideal of $R \bowtie I$, generated by $(0, a)$. Also, $O_2$ is a principal ideal, generated by $(a, 0)$, by the same argument, as desired. \qed

**Proof of Theorem 2.13.** Let $a \in I$ such that $I = Ra$. By Lemma 2.14, $O_1$ and $O_2$ are principal ideals of $R \bowtie I$. Consider the short exact sequence of $R \bowtie I$-modules:

$$(1) \quad 0 \to \ker(u) \to R \bowtie I \overset{u}{\to} O_1 \to 0,$$

where $u(r, r + i) = (r, r + i)(0, a) = (0, (r + i)a)$. Then, $\ker(u) = \{(r, 0) \in R \bowtie I/r \in I\} = O_2$. Consider the short exact sequence of $R \bowtie I$-modules:

$$(2) \quad 0 \to \ker(v) \to R \bowtie I \overset{v}{\to} O_2 \to 0,$$
where $v(r, r+i) = (r, r+i)(a, 0) = (ra, 0)$. Then, $\ker(v) = \{(0, i) \in R \otimes I/i \in I\} = O_1$. Therefore, $O_1$ (resp., $O_2$) is $m$-presented for each positive integer $m$ by the above two exact sequences. It remains to show that $pd_{R \otimes I}(O_1) = \infty$ (or $pd_{R \otimes I}(O_2) = \infty$).

We claim that $O_1$ and $O_2$ are not projective. Deny. Then, $O_1$ is projective and so the short exact sequence (1) splits. Then, $O_2$ is generated by an idempotent element $(x, 0)$ such that $x(\neq 0) \in I$. Hence, $(x, 0)^2 = (x, 0)(x, 0) = (x^2, 0) = (x, 0)$. Then, $x^2 = x$, and so $x = 1$ or $x = 0$, a contradiction (since $x \in I$ and $x \neq 0$). Therefore, $O_1$ is not projective. Similar arguments show that $O_2$ is not projective. A combination of (1) and (2) yields $pd_{R \otimes I}(O_1) = pd_{R \otimes I}(O_2) + 1$ and $pd_{R \otimes I}(O_2) = pd_{R \otimes I}(O_1) + 1$. Then, $pd_{R \otimes I}(O_1) = pd_{R \otimes I}(O_2) + 1 + 1 = pd_{R \otimes I}(O_1) + 2$. Consequently, the projective dimension of $O_1$ (resp., $O_1$) has to be infinite, as desired. □

If $R$ is a principal domain, then we obtain the following result.

**Corollary 2.15.** Let $R$ be a principal domain and let $I$ be a proper ideal of $R$. Then, $R \otimes I$ is not a weak $(n,d)$-ring (and so is not an $(n,d)$-ring) for each positive integers $n$ and $d$. In particular, $\text{wdim}(R \otimes I) = \text{gldim}(R \otimes I) = \infty$.

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