Bulletin of the Iranian Mathematical Society Vol. 38 *No. 2* (*2012*), *pp 507-515.*

SOME HOMOLOGICAL PROPERTIES OF AMALGAMATED DUPLICATION OF A RING ALONG AN IDEAL

M. CHHITI AND N. MAHDOU*

Communicated by Siamak Yassemi

Abstract. We investigate the transfer of some homological properties from a ring R to its amalgamated duplication along some ideal I of R $R \bowtie I$, and then generate new and original families of rings with these properties.

1. **Introduction**

Let R be a commutative ring with unit element 1 and let I be a proper ideal of R. The amalgamated duplication of a ring R along an ideal I is a ring that is defined as the following subring with unit element $(1, 1)$ of $R \times R$:

$$R \bowtie I := \{(r, r + i)/r \in R, i \in I\}.$$

This construction has been studied, in the general case, and from different points of view of pullbacks, by D'Anna and Fontana [7]. Also, D'Anna and Fontana, [5] have considered the case of the amalgamated duplication of a ring, not necessarily in a Noetherian setting, along a multiplicative-canonical ideal in the sense of Heinzer-Huckaba-Papick

MSC(2010): Primary: 13F05, 13B05; Secondary: 13D02, 13D05.
Keywords: Amalgamated duplication of a ring along an ideal, von Neumann regular ring, perfect ring, (n, d)-ring and weak (n, d)-ring.
Received: 17 October 2010, Accepted: 22 February 2011.
*Corresponding author

[10]. In [6], D'Anna has studied some properties of $R \bowtie I$, in order to construct reduced Gorenstein rings assosiated to Cohen-Macaulay rings and has applied this construction to curve singularities. On the other hand, Maimani and Yassemi [15] have studied the diameter and girth of the zero-divisor graph of the ring $R \bowtie I$. For instance, see [5, 6, 7, 15].

Let M be an R-module, the idealization $R \propto M$ (also called the trivial extention), introduced by Nagata in 1956 (cf. [16]) is defined as the R-module $R \oplus M$ with multiplication defined by $(r, m)(s, n) := (rs, rn + sm)$. For instance, see [8, 9, 11, 12].

When $I^2 = 0$, the new construction $R \bowtie I$ coincides with the idealization $R \propto I$. One main difference of amalgamated-duplication, with respect to idealization, is that the ring $R \bowtie I$ can be a reduced ring (and in fact, it is always reduced if R is a domain).

For two rings $A \subset B$, we say that A is a module retract (or a subring retract) of B if there exists an A-module homomorphism $\varphi : B \to A$ such that $\varphi\mid_A = id\mid_A$. In this case, φ is called a module retraction map. If such a map φ exists, then B contains A as an A- module direct summand. We can easily show that R is a module retract of $R \bowtie I$, where the module retraction map φ is defined by $\varphi(r, r + i) := r$.

Here, we study the transfer of some homological properties from a ring R to the ring $R \bowtie I$. Specially, we prove that $R \bowtie I$ is a von Neumann regular ring (resp., a perfect ring) if and only if so is R. Also, we prove that $\mathrm{gldim(R \bowtie I)} = \infty$ if R is a domain and I is a principal ideal of R.

Recall that if R is a ring and M is an R-module, as usual we use $\mathrm{pd_R(M)}$ and $\mathrm{fd_R(M)}$ to denote the usual projective and flat dimensions of M, respectively. The classical global and weak dimensions of R are respectively denoted by $\mathrm{gldim(R)}$ and $\mathrm{wdim(R)}$. Also, the Krull dimension of R is denoted by $\mathrm{dim(R)}$.

2. **Main results**

Let R be a commutative ring with identity element 1 and let I be an ideal of R. Recall that $R \bowtie I := \{(r, s)/r, s \in R, s - r \in I\}$. It is easy to check that $R \bowtie I$ is a subring with unit element $(1,1)$ of $R \times R$ (with the usual componentwise operations) and that $R \bowtie I = \{(r, r+i)/r \in R, i \in I\}$.

It is easy to see that if $\pi_i (i = 1,2)$ are the projections of $R \times R$ on R, then $\pi_i(R \bowtie I) = R$, and hence if $O_i := ker(\pi_i \backslash R \bowtie I)$, then $R \bowtie I/O_i \cong R$. Moreover, $O_1 = \{(0,i), i \in I\}$, $O_2 = \{(i,0), i \in I\}$ and $O_1 \cap O_2 = (0)$.

We begin by studying the transfer of von Neumann regular property.

Theorem 2.1. *Let R be a commutative ring and let I be a proper ideal of R. Then, R is a von Neumann regular ring if and only if $R \bowtie I$ is a von Neumann regular ring.*

The proof will use the following Lemma.

Lemma 2.2. [7, Theorem 3.5]

(1) *Let R be a commutative ring and let I be an ideal of R. Let P be a prime ideal of R and set*

$$\begin{aligned} P_0 &= \{(p, p+i)/p \in P, i \in I \cap P\}, \\ P_1 &= \{(p, p+i)/p \in P, i \in I\}, \end{aligned}$$

and $$P_2 = \{(p+i, p)/p \in P, i \in I\}$$

- *If $I \subseteq P$, then $P_0 = P_1 = P_2$ is a prime ideal of $R \bowtie I$ and it is the unique prime ideal of $R \bowtie I$ lying over P.*
- *If $I \nsubseteq P$, then $P_1 \neq P_2$, $P_1 \cap P_2 = P_0$ and P_1 and P_2 are the only prime ideals of $R \bowtie I$ lying over P.*

(2) *Let Q be a prime ideal of $R \bowtie I$ and let $O_1 = \{(0,i)/i \in I\}$. Two cases are possible: either $Q \nsupseteq O_1$ or $Q \supseteq O_1$.*

a: *If $Q \nsupseteq O_1$, then there exists a unique prime ideal P of R ($I \nsubseteq P$) such that*

$$Q = P_2 = \{(p+i, p)/p \in P, i \in I\}.$$

b: *If $Q \supseteq O_1$, then there exists a unique prime ideal P of R such that*

$$Q = P_1 = \{(p, p+i)/p \in P, i \in I\}.$$

Proof of Theorem 2.1. Assume that R is a von Neumann regular ring. Then, R is reduced and so $R \bowtie I$ is reduced by [7, Theorem 3.5 (a)(vi)].

It remains to show that $dim(R \bowtie I) = 0$ by [9, Remark, p. 5]. Let Q be a prime ideal of $R \bowtie I$. If $P = Q \cap R$, then $Q \in \{P_1, P_2\}$ (by Lemma 2.2(2)). But, P is a maximal ideal of R, since R is a von Neumann regular ring. Then, P_1 and P_2 are maximal ideals of $R \bowtie I$ (by [7, Theorem 3.5 (a)(vi)]). Hence, Q is a maximal ideal of $R \bowtie I$, as desired.

Conversely, assume that $R \bowtie I$ is a von Neumann regular ring. By [7, Theorem 3.5 (a)(vi)], R is reduced. Let P be a prime ideal of R. By Lemma 2.2(1), $P \bowtie I = \{(p, p + i)/p \in P, i \in I\}$ is a prime ideal of $R \bowtie I$. From [9, p. 7], we get $P \bowtie I$ to be a maximal ideal of $R \bowtie I$, and hence P is a maximal ideal of R. Therefore, $dim(R) = 0$, and so R is a von Neumann regular ring. □

A ring R is called semisimple if every R-module is projective, that is, gldim(R) = 0 (see [8, P. 26]). Recall that a ring is semisimple if and only if it is Noetherian von Neumann regular by [8, Theorems (1.4.2, 1.4.6, and 1.3.10(2)].

Corollary 2.3. *Let R be a commutative ring and let I be a proper ideal of R. Then, R is a semisimple ring if and only if $R \bowtie I$ is a semisimple ring.*

Proof. Assume that R be a semisimple ring. Then, R is a Noetherian von Neumann regular ring. By Theorem 2.1, $R \bowtie I$ is a von Neumann regular ring and by [7, Corollary 3.3], $R \bowtie I$ is Noetherian. Therefore, $R \bowtie I$ is semisimple.

Conversely, assume that $R \bowtie I$ is semisimple. Then, $R \bowtie I$ is a Noetherian von Neumann regular ring, and so R is a von Neumann regular ring (by Theorem 2.1) and Noetherian (by [7, Corollary 3.3]). Hence, R is semisimple. □

A ring R is called a stably coherent ring if for every positive integer n, the polynomial ring in n variables over R is a coherent ring. Recall that a ring R is is called a coherent ring if every finitely generated ideal of R is finitely presented.

Corollary 2.4. *Let R be a commutative ring and let I be a proper ideal of R. If R is a von Neumann regular ring, then $R \bowtie I$ is a stably coherent ring.*

Proof. Use Theorem 2.1 and [8, Theorem 7.3.1]. □

Now, we are able to construct a new class of non-Noetherian von Neumann regular rings.

Example 2.5. *Let R be a non-Noetherian von Neumann regular ring and I be a proper ideal of R. Then, $R \bowtie I$ is a non-Noetherian von Neumann regular ring by* [7, Corollary 3.3] *and Theorem 2.1.*

We recall that a ring R is called a perfect ring if every flat R-module is a projective R-module (see [1]). Secondly, we study the transfer of perfect property.

Theorem 2.6. *Let R be a commutative ring and let I be a proper ideal of R. Then, R is a perfect ring if and only if $R \bowtie I$ is a perfect ring.*

To prove Theorem 2.6, we need the following lemmas.

Lemma 2.7. ([13, Lemma 2.5.(2)]) *Let $(R_i)_{i=1,2}$ be a family of rings and E_i be an R_i-module, for $i = 1, 2$.*
Then, $pd_{R_1 \times R_2}(E_1 \times E_2)$=$\sup\{pd_{R_1}(E_1), pd_{R_2}(E_2)\}$.

Lemma 2.8. *Let $(R_i)_{i=1,2}$ be a family of rings and E_i be an R_i-module for $i = 1, 2$. Then, $fd_{R_1 \times R_2}(E_1 \times E_2)$=$\sup\{fd_{R_1}(E_1), fd_{R_2}(E_2)\}$.*

Proof. The proof is analogous to the proof of Lemma 2.7. □

Lemma 2.9. *Let $(R_i)_{i=1,...,m}$ be a family of rings. Then, $\prod_{i=1}^{m} R_i$ is a perfect ring, if and only if R_i is a perfect ring, for each $i = 1, ..., m$.*

Proof. The proof is done by induction on m and it suffices to check it for $m = 2$. Let R_1 and R_2 be two rings such that $R_1 \times R_2$ is perfect. Let E_1 be a flat R_1-module and let E_2 be a flat R_2-module. By Lemma 2.8, $E_1 \times E_2$ is a flat $(R_1 \times R_2)$-module, and so it is a projective $(R_1 \times R_2)$-module, since $R_1 \times R_2$ is a perfect ring. Hence, E_1 is a projective R_1-module, and E_2 is a projective R_2-module by Lemma 2.7; this means that R_1 and R_2 are perfect rings.

Conversely, assume that R_1 and R_2 are two perfect rings. Let $E_1 \times E_2$ be a flat $(R_1 \times R_2)$-module, where E_i is an R_i-module, for $i = 1, 2$. By Lemma 2.8, E_1 is a flat R_1-module and let E_2 be a flat R_2-module; so, E_1 is a projective R_1-module and E_2 is a projective R_2-module. Therefore, $E_1 \times E_2$ be a projective $(R_1 \times R_2)$-module by Lemma 2.7; this means that $R_1 \times R_2$ is a perfect rings. □

Lemma 2.10. *Let R be a commutative ring and let I be a proper ideal of R. Then,*

(1) *an* $(R \bowtie I)$*-module* M *is projective if and only if* $M \otimes_{R\bowtie I} (R \times R)$ *is a projective* $(R \times R)$*-module and* M/O_1M *is a projective* R*-module,*
(2) *an* $(R \bowtie I)$*-module* M *is flat if and only if* $M \otimes_{R\bowtie I} (R \times R)$ *is a flat* $(R \times R)$*-module and* M/O_1M *is a flat* R*-module.*

Proof. Note that $R \bowtie I$ is a subring of $R \times R$ and O_1 is a common ideal of $R \bowtie I$ and $R \times R$ by [7, Proposition 3.1]. The result follows from [8, Theorem 5.1.1]. □

Proof of Theorem 2.6. Assume that R is a perfect ring and let M be a flat $(R \bowtie I)$-module. By Lemma 2.10(2), $M \otimes_{R\bowtie I} (R \times R)$ is a flat $(R\times R)$-module and M/O_1M is a flat R-module. Then, $M \otimes_{R\bowtie I} (R\times R)$ is a projective $(R \times R)$-module (since $R \times R$ is perfect, by Lemma 2.9), and M/O_1M is a projective R-module, since R is perfect. By Lemma 2.10(1), M is a projective $(R \bowtie I)$-module, and so $R \bowtie I$ is a perfect ring.

Conversely, assume that $R \bowtie I$ is a perfect ring and let E be a flat R- module. Then, $E \otimes_R (R \bowtie I)$ is a flat $(R \bowtie I)$-module, and so it is a projective $(R \bowtie I)$-module, since $R \bowtie I$ is a perfect ring. In addition, for any R- module M and any $n \geq 1$, we have

$$Ext^n_R(E, M \otimes_R (R \bowtie I)) \cong Ext^n_R(E \otimes_R (R \bowtie I), M \otimes_R (R \bowtie I))$$

(see [3, P. 118]), and then $Ext^n_R(E, M \otimes_R (R \bowtie I)) = 0$. Since M is a direct summand of $M \otimes_R (R \bowtie I)$ because R is a module retract of $R \bowtie I$, $Ext^n_R(E, M) = 0$, for all $n \geq 1$ and all R-modules M. This means that E is a projective R-module, and so R is a perfect ring. □

We say that a ring R is Steinitz if any linearly independent subset of a free R-module F can be extended to a basis of F by adjoining elements of a given basis. In [4, Proposition 5.4], Cox and Pendleton showed that Steinitz rings are precisely the perfect local rings.

By Theorem 2.6 and since $R \bowtie I$ is local if and only if R is local, we obtain the following result.

Corollary 2.11. *Let* R *be a commutative ring and* I *be a proper ideal of* R*. Then,* R *is a Steinitz ring if and only if* $R \bowtie I$ *is a Steinitz ring.*

Example 2.12. *Let* $R = K[X]/(X^2)$*, where* K *is a field and* X *is indeterminate. Then,* $(K[X]/(X^2)) \bowtie I$ *is a Steinitz ring, where* $I := X(K[X]/(X^2))$*.*

For a nonnegative integer n, an R-module E is n-presented if there is an exact sequence $F_n \to F_{n-1} \to ... \to F_0 \to E \to 0$, in which each F_i is a finitely generated free R-module. In particular, "0-presented" means finitely generated and "1-presented" means finitely presented.
Given nonnegative integers n and d, a ring R is called an (n,d)-ring if every n-presented R-module has projective dimension $\leq d$; and R is called a weak (n,d)-ring if every n-presented cyclic R-module has projective dimension $\leq d$ (equivalently, if every $(n-1)$-presented ideal of R has projective dimension $\leq d-1$). For instance, the $(0,1)$-domains are the Dedekind domains, the $(1,1)$-domains are the Prüfer domains, and the $(1,0)$-rings are the von Neumann regular rings; see, for instance, [2, 11, 12, 13, 14].

Now, we give a wide class of rings which are not weak (n,d)-rings (and so not (n,d)-rings) for positive integers n and d.

Theorem 2.13. *Let R be an integral domain and let $I(\neq 0)$ be a principal ideal of R. Then, $R \bowtie I$ is not a weak (n,d)-rings (and so is not (n,d)-rings) for each positive integers n and d. In particular,* $\mathrm{wdim}(\mathrm{R} \bowtie \mathrm{I}) = \mathrm{gldim}(\mathrm{R} \bowtie \mathrm{I}) = \infty$.

To prove theorem 2.13, we need the following lemma.

Lemma 2.14. *Let R be a commutative ring and let $I(\neq 0)$ be a principal ideal of R. Then, $O_1 = \{(0,i), i \in I\}$ and $O_2 = \{(i,0), i \in I\}$ are principal ideals of $R \bowtie I$.*

Proof. Let $(0,i)$ be an element of O_1. Since I is a principal ideal of R, there exists $a \in I$ such that $I = Ra$, and so $(0,i) = (0,ra) = (r+j,r)(0,a)$, for some $r \in R$ and for all $j \in I$. Hence, O_1 is a principal ideal of $R \bowtie I$, generated by $(0,a)$. Also, O_2 is a principal ideal, generated by $(a,0)$, by the same argument, as desired. □

Proof of Theorem 2.13. Let $a \in I$ such that $I = Ra$. By Lemma 2.14, O_1 and O_2 are principal ideals of $R \bowtie I$. Consider the short exact sequence of $R \bowtie I$-modules:

$$(1) \qquad 0 \to ker(u) \to R \bowtie I \xrightarrow{u} O_1 \to 0,$$

where $u(r, r+i) = (r, r+i)(0,a) = (0,(r+i)a)$. Then, $ker(u) = \{(r,0) \in R \bowtie I / r \in I\} = O_2$. Consider the short exact sequence of $R \bowtie I$-modules:

$$(2) \qquad 0 \to ker(v) \to R \bowtie I \xrightarrow{v} O_2 \to 0,$$

where $v(r, r+i) = (r, r+i)(a, 0) = (ra, 0)$. Then, $ker(v) = \{(0, i) \in R \bowtie I / i \in I\} = O_1$. Therefore, O_1 (resp., O_2) is m-presented for each positive integer m by the above two exact sequences. It remains to show that $pd_{R \bowtie I}(O_1) = \infty$ (or $pd_{R \bowtie I}(O_2) = \infty$).

We claim that O_1 and O_2 are not projective. Deny. Then, O_1 is projective and so the short exact sequence (1) splits. Then, O_2 is generated by an idempotent element $(x, 0)$ such that $x(\neq 0) \in I$. Hence, $(x, 0)^2 = (x, 0)(x, 0) = (x^2, 0) = (x, 0)$. Then, $x^2 = x$, and so $x = 1$ or $x = 0$, a contradiction (since $x \in I$ and $x \neq 0$). Therefore, O_1 is not projective. Similar arguments show that O_2 is not projective. A combination of (1) and (2) yields $pd_{R \bowtie I}(O_1) = pd_{R \bowtie I}(O_2) + 1$ and $pd_{R \bowtie I}(O_2) = pd_{R \bowtie I}(O_1) + 1$. Then, $pd_{R \bowtie I}(O_1) = pd_{R \bowtie I}(O_2) + 1 + 1 = pd_{R \bowtie I}(O_1) + 2$. Consequently, the projective dimension of O_1 (resp., O_1) has to be infinite, as desired. □

If R is a principal domain, then we obtain the following result.

Corollary 2.15. *Let R be a principal domain and let I be a proper ideal of R. Then, $R \bowtie I$ is not a weak (n, d)-ring (and so is not an (n, d)-ring) for each positive integers n and d. In particular,* $\mathrm{wdim}(\mathrm{R} \bowtie \mathrm{I}) = \mathrm{gldim}(\mathrm{R} \bowtie \mathrm{I}) = \infty$.

Acknowledgments

The authors thank the referee for a careful reading of the manuscript.

References

[1] H. Bass, Finitistic dimension and a homological generalization of semi-primary rings, *Trans. Amer. Math.* Soc. **95** (1960) 466–488.
[2] D. L. Costa, Parameterising families of non-Noetherian rings, *Comm. Algebra* **22** (1994), no. 10, 3997–4011
[3] H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, N. J., 1956.
[4] S. H. Cox and R.L. Pendleton, Rings for which certain flat modules are projective, *Trans. Amer. Math. Soc.* **150** (1970) 139–156.
[5] M. D'Anna and M. Fontana, The amalgamated duplication of a ring along a multiplicative-canonical ideal, *Ark. Mat.* **45** (2007), no. 2, 241–252.
[6] M. D'Anna, A construction of Gorenstein rings, *J. Algebra* **306** (2006), no. 2, 507–519.

[7] M. D'Anna and M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties, *J. Algebra Appl.* **6** (2007), no. 3, 443–459.
[8] S. Glaz, Commutative Coherent Rings, *Lecture Notes in Mathematics*, Springer-Verlag, Springer-Verlag, Berlin, 1989.
[9] J. A. Huckaba, Commutative Rings with Zero Divizors, Marcel Dekker, Inc., New York, 1988.
[10] W. Heinzer, J. Huckaba and I. Papick, m-canonical ideals in integral domains, *Comm. Algebra* **26** (1998), no. 9, 3021–3043.
[11] S. Kabbaj and N. Mahdou, Trivial extensions defined by coherent-like condition, *Comm. Algebra* **32** (2004), no. 10, 3937–3953
[12] S. Kabbaj and N. Mahdou, Trivial extensions of local rings and a conjecture of Costa, *Lecture Notes in Pure and Appl. Math.* Marcel Dekker, New York, **231** (2003) 301–311.
[13] N. Mahdou, On Costa's conjecture, *Comm. Algebra* **29** (2001), no. 7, 2775–2785.
[14] N. Mahdou, On 2-von Neumann regular rings, *Comm. Algebra* **33** (2005), no. 10, 3489–3496.
[15] H. R. Maimani and S. Yassemi, Zero-divisor graphs of amalgamated duplication of a ring along an ideal, *J. Pure Appl. Algebra* **212** (2008), no. 1, 168-174.
[16] M. Nagata, Local Rings, Interscience Tracts in Pure and Applied Mathematics, Interscience Publishers a division of John Wiley & Sons, New york-London, 1962.
[17] B. Nashier and W. Nichols, On Steinitz properties, *Arch. Math. (Basel)* **57** (1991),
no. 3, 247–253.
[18] J. Rotman, An Introduction to Homological Algebra, Academic Press, Inc., New York-London, (1979).

Mohamed Chhiti
Department of Mathematics, Faculty of Science and Technology of Fez, University S.M. Ben Abdellah, P.O. Box 2202, Fez, Morocco
Email: chhiti.med@hotmail.com

Najib Mahdou
Department of Mathematics, Faculty of Science and Technology of Fez, University S.M. Ben Abdellah, P.O. Box 2202, Fez, Morocco
Email: mahdou@hotmail.com