

COMPARISON RESULTS ON THE PRECONDITIONED MIXED-TYPE SPLITTING ITERATIVE METHOD FOR M-MATRIX LINEAR SYSTEMS

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Communicated by Heydar Radjavi

ABSTRACT. Consider the linear system $Ax = b$ where the coefficient matrix A is an M-matrix. Here, it is proved that the rate of convergence of the Gauss-Seidel method is faster than the mixed-type splitting and AOR (SOR) iterative methods for solving M-matrix linear systems. Furthermore, we improve the rate of convergence of the mixed-type splitting iterative method by applying a preconditioned matrix. Comparison theorems show that the rate of convergence of the preconditioned Gauss-Seidel method is faster than the preconditioned mixed-type splitting and AOR (SOR) iterative methods. Finally, some numerical examples are presented to illustrate the reality of our results.

1. Introduction

Consider the iterative solution of the linear system

$$(1.1) \quad Ax = b,$$

where $x, b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is a nonsingular matrix with nonzero diagonal elements.

MSC(2010): Primary: 65F10.

Keywords: Linear system, mixed-type splitting iterative method, preconditioned matrix, M-matrix.

Received: 3 September 2010, Accepted: 4 December 2010.

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A basic iterative method for solving linear system (1.1) is defined by means of the splitting $A = M - N$, where M is a nonsingular matrix. The approximate solution $x^{(k+1)}$ is generated as follows:

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \quad k = 0, 1, 2, \dots .$$

Or equivalently,

$$(1.2) \quad x^{(k+1)} = Vx^{(k)} + M^{-1}b, \quad k = 0, 1, 2, \dots,$$

where the starting vector $x^{(0)}$ is given and $V = M^{-1}N$ is called the iteration matrix. The iterative method (1.2) is convergent to the unique solution $x = A^{-1}b$ for each $x^{(0)}$ if and only if $\rho(V) < 1$. The convergence analysis of the iterative method (1.2) is based on the spectral radius of the iteration matrix V , i.e., $\rho(V)$. For large values of k , at each step, the corresponding error decreases in magnitude approximately by a factor of $\rho(V)$. That is, the smaller $\rho(V)$ is the quicker the convergence is.

Definition 1.1. (Varga [13]). *Let $A = M - N$ be an arbitrary splitting for the matrix A and $\rho_1 = \rho(M^{-1}N)$ be the spectral radius of the iteration matrix based on the above splitting. The asymptotic rate of the convergence is defined by $R_\infty = -\ln \rho_1$.*

Here, we shall consider the following decomposition for a given matrix A ,

$$A = D - L - U,$$

where D is a nonsingular diagonal matrix, L and U are strictly lower and upper triangular matrices, respectively.

The classical iterative methods are defined as follows.

The **Jacobi** method (J):

$$M_J = D, \quad N_J = L + U \quad \text{and} \quad T_J = D^{-1}(L + U).$$

The **Gauss-Seidel** method (GS):

$$M_G = D - L, \quad N_G = U \quad \text{and} \quad T_G = (D - L)^{-1}U.$$

Definition 1.2. (Berman and Plemmons [2]). *$A \in \mathbb{R}^{n \times n}$ is called a Z-matrix if $a_{ij} \leq 0$, for $i, j = 1, 2, 3, \dots, n$ ($i \neq j$).*

Definition 1.3. (Berman and Plemmons [2]). *Let A be a Z-matrix with positive diagonal elements. Then, the matrix A is called an M-matrix if A is nonsingular and $A^{-1} \geq 0$.*

Definition 1.4. (Varga [13]). *A matrix is said to be reducible if there is a permutation matrix P such that PAP^T is a block upper triangular matrix. Otherwise, it is irreducible.*

Theorem 1.5. *Let $G \geq 0$ be an $n \times n$ irreducible matrix. Then, we have the followings:*

- (1) *G has a positive real eigenvalue which is equal to its spectral radius.*
- (2) *There exists an eigenvector $x > 0$ corresponding to the spectral radius of G .*
- (3) *$\rho(G)$ is a simple eigenvalue of G .*

Proof. See [13]. □

Theorem 1.6. *Let G be a nonnegative matrix. Then, we have the followings:*

- (1) *If $\alpha x \leq Gx$, for some nonnegative vector $x \neq 0$, then $\alpha \leq \rho(G)$.*
- (2) *If $Gx \leq \beta x$, for some nonnegative vector $x \neq 0$, then $\rho(G) \leq \beta$. Moreover, if G is irreducible and if $0 \neq \alpha x \leq Gx \leq \beta x$, $\alpha x \neq Gx$ and $Gx \neq \beta x$, for some nonnegative vector x , then $\alpha < \rho(G) < \beta$ and x is a positive vector.*

Proof. See [2]. □

Definition 1.7. (Woznicki [14]). *The splitting $A = M - N$ is called*

- (1) *a regular splitting of A if $M^{-1} \geq 0$ and $N \geq 0$,*
- (2) *a nonnegative splitting of A if $M^{-1} \geq 0$, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$,*
- (3) *a weak nonnegative splitting of A if $M^{-1} \geq 0$ and either $M^{-1}N \geq 0$ or $NM^{-1} \geq 0$,*
- (4) *a convergent splitting of A if $\rho(M^{-1}N) < 1$.*

Theorem 1.8. *Let $A = M - N$ be a regular splitting of A . Then, $\rho(M^{-1}N) < 1$ if and only if A is nonsingular and A^{-1} is nonnegative.*

Proof. See [13]. □

Theorem 1.9. *Let A be a Z -matrix with positive diagonal elements. Then, A is an M -matrix if and only if $\rho(T_J) < 1$.*

Proof. See [2]. □

The Mixed-type splitting iterative method is given as follows; for more details, see [3, 9, 10].

Mixed-type splitting iterative method:

$(D + D_1 + L_1 - L)x^{(k+1)} = (D_1 + L_1 + U)x^{(k)} + b, \quad k = 0, 1, 2, \dots$.
Therefore, the iteration matrix of the mixed-type splitting iterative method is defined by

$$T = (D + D_1 + L_1 - L)^{-1}(D_1 + L_1 + U),$$

where D_1 is an auxiliary nonnegative diagonal matrix, L_1 is an auxiliary strictly lower triangular matrix and $0 \leq L_1 \leq L$.

Classical SOR and AOR methods are special cases of the mixed-type splitting iterative method and are defined by the following choices of D_1 and L_1 .

1. The SOR method:

$$D_1 = \frac{1}{\omega}(1 - \omega)D, \quad L_1 = 0, \\ T_\omega = (D - \omega L)^{-1}[(1 - \omega)D + \omega U].$$

2. The AOR method:

$$D_1 = \frac{1}{\omega}(1 - \omega)D, \quad L_1 = \frac{1}{\omega}(\omega - r)L, \\ T_{r,\omega} = (D - rL)^{-1}[(1 - \omega)D + (\omega - r)L + \omega U],$$

where ω and r are real parameters with $0 \leq r \leq \omega < 1$ and $\omega \neq 0$.

Theorem 1.10. *Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of A , where $A^{-1} \geq 0$. If $N_2 \geq N_1$, then $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$.*

Proof. See [15]. □

Theorem 1.11. *Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of A , where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$.*

Proof. See [15]. □

Theorem 1.12. *Let B be a nonnegative matrix. Then, $\rho(B) < 1$ if and only if $I - B$ is nonsingular and $(I - B)^{-1}$ is nonnegative.*

Proof. See [2]. □

Proposition 1.13. *If $A^{-1} \geq 0$ and $A = M - N$ is a weak nonnegative splitting for the matrix A , then $\rho(M^{-1}N) = \rho(NM^{-1}) < 1$.*

Proof. Assume that $\sigma(B)$ shows the set of eigenvalues of an arbitrary matrix B . Clearly, $\sigma(M^{-1}N) = \sigma(NM^{-1})$. Without loss of generality, assume that $M^{-1}N \geq 0$. Now, the result can be concluded by Theorem 1.12 and the following relation,

$$A = M(I - M^{-1}N). \quad \square$$

The remainder of our work is organized as follows. In Section 2, some comparison results are established. Also, it is proved that the Gauss-Seidel method converges faster than the mixed-type splitting (AOR, SOR) iterative methods. Moreover, we improve the rate of convergence of the mixed-type splitting iterative method by applying a preconditioned matrix, in Section 3. In Section 4, some numerical examples are presented to illustrate the results established here. Finally, we conclude in Section 5 .

2. Comparison theorems

In this section, some comparison theorems are proved when the coefficient matrix A in (1.1) is a nonsingular M-matrix. First of all, it is shown that the mixed-type splitting is a regular convergent splitting for the matrix A . Then, we compare the spectral radii of the Gauss-Seidel splitting with the mixed-type, AOR and SOR splittings, respectively.

The AOR iterative method is given by

$$(D - rL)x^{(k+1)} = [(1 - \omega)D + (\omega - r)L + \omega U]x^{(k)} + \omega b, \quad k = 0, 1, 2, \dots,$$

with the iteration matrix

$$T_{r,\omega} = (D - rL)^{-1}[(1 - \omega)D + (\omega - r)L + \omega U],$$

where ω and r are real parameters with $0 \leq r \leq \omega < 1$ and $\omega \neq 0$.

Note that the classical SOR method is a special case of the AOR method with $r = \omega$.

Theorem 2.1. *Suppose that A and B are two square matrices which satisfy the inequalities $0 \leq A \leq B$. Then, $\rho(A) \leq \rho(B)$.*

Proof. See [13]. □

Theorem 2.2. *If A is an M-matrix, $D_1 \geq 0$ and $0 \leq L_1 \leq L$, then the mixed-type splitting is a convergent regular splitting.*

Proof. It is known that the mixed-type splitting for the matrix A is defined by

$$M = D + D_1 + L_1 - L, \quad N = D_1 + L_1 + U.$$

The matrix A is an M-matrix, and so Theorem 1.9 implies that the Jacobi splitting for the matrix A is convergent. It is easy to show that

$$0 \leq (D + D_1)^{-1}(L - L_1) \leq D^{-1}(L + U).$$

From Theorem 2.1, we can conclude that the Jacobi splitting of the matrix M is convergent, and hence Theorem 1.9 implies that $M^{-1} \geq 0$. Therefore, the result follows immediately from Theorem 1.8. \square

Corollary 2.3. *Let A be an M-matrix and $0 \leq r \leq \omega < 1, \omega \neq 0$. Then, the SOR and AOR splittings are convergent.*

Proof. For the SOR method, the result follows from Theorem 2.2 with $D_1 = \frac{1}{\omega}(1 - \omega)D$ and $L_1 = 0$. By considering $D_1 = \frac{1}{\omega}(1 - \omega)D$ and $L_1 = \frac{1}{\omega}(\omega - r)L$, Theorem 2.2 implies that the AOR method is also convergent. \square

In the following two theorems, the authors have assumed that the matrix A is a Z-matrix with positive diagonal elements. It has been established that in the case that the AOR (SOR) is a convergent method, the speed of convergence of the mixed-type splitting iterative method is faster than the speed of convergence of the AOR (SOR) method. We only state the theorems without proof; for more details, see [3].

Theorem 2.4. *Suppose that A is the coefficient matrix of the linear system (1.1). Let A be a Z-matrix with positive diagonal elements and*

$$0 \leq D_1 \leq \left(\frac{1}{\omega} - 1\right)D, \quad 0 \leq L_1 \leq \left(1 - \frac{r}{\omega}\right)L.$$

Moreover, assume that the matrices T and $T_{r,\omega}$ are the mixed-type splitting and AOR iteration matrices, respectively, where $0 < r < \omega < 1$. If T and $T_{r,\omega}$ are irreducible, then

- (1) $\rho(T) < \rho(T_{r,\omega})$, if $\rho(T_{r,\omega}) < 1$,
- (2) $\rho(T) = \rho(T_{r,\omega})$, if $\rho(T_{r,\omega}) = 1$,
- (3) $\rho(T) > \rho(T_{r,\omega})$, if $\rho(T_{r,\omega}) > 1$.

Theorem 2.5. *Suppose that A is the coefficient matrix of the linear system (1.1). Let A be a Z-matrix with positive diagonal elements and*

$$0 \leq D_1 \leq \left(\frac{1}{\omega} - 1\right)D, \quad L_1 = 0.$$

Moreover, assume that the matrices T and T_ω are the mixed-type splitting and SOR iteration matrices, respectively, where $0 < \omega < 1$. If T and T_ω are irreducible, then

- (1) $\rho(T) < \rho(T_\omega)$, if $\rho(T_\omega) < 1$,
- (2) $\rho(T) = \rho(T_\omega)$, if $\rho(T_\omega) = 1$,
- (3) $\rho(T) > \rho(T_\omega)$, if $\rho(T_\omega) > 1$.

Proposition 2.6. *Let A be a Z-matrix with positive diagonal elements. Assume that $0 \leq r \leq \omega < 1$, $\omega \neq 0$. Then, the AOR (SOR, $0 < \omega \leq 1$, or mixed-type) splitting is a convergent splitting for the matrix A if and only if A is an M-matrix.*

Proof. It is easy to see that the AOR (SOR or mixed-type) splitting is a regular splitting for the matrix A . Suppose that the AOR (SOR or mixed-type) splitting is a convergent splitting. Hence, Theorem 1.8 implies that A is an M-matrix.

Conversely, let A be an M-matrix. Theorem 1.9 implies that the Jacobi splitting is a convergent splitting for the matrix A , i.e., $\rho(T_J) < 1$. Evidently, the spectral radius of the iteration matrix of the AOR (SOR or mixed-type) method is smaller than the one corresponding to the Jacobi method. Hence, the result follows immediately from Theorem 1.10. □

Note that the Gauss-Seidel splitting is a special case of the SOR splitting with $\omega = 1$. Hence, Proposition 2.6 implies that the Gauss-Seidel splitting is a convergent splitting for an M-matrix.

Our aim is to show that if the AOR (SOR, mixed-type) method is a convergent method, then the Gauss-Seidel method converges faster than the AOR (SOR, mixed-type) method. On the other hand, it has been proved in Theorem 2.6 that if the AOR (SOR, mixed-type) is a convergent splitting for a given Z-matrix A , with positive diagonal elements, then the matrix A is an M-matrix. Hence, in the following we only consider M-matrix linear systems.

Theorem 2.7. *Let A be an M-matrix. Suppose that T_G and $T_{r,\omega}$ are the Gauss-Seidel and AOR iteration matrices, respectively, where $0 \leq r \leq \omega < 1$, $\omega \neq 0$. If T_G and $T_{r,\omega}$ are irreducible, then*

$$\rho(T_G) < \rho(T_{r,\omega}).$$

Proof. It is clear that $T_G = (D - L)^{-1}U \geq 0$. Hence, by Theorem 1.5, there exists a positive vector x such that

$$(2.1) \quad T_G x = (D - L)^{-1}U x = \rho(T_G)x.$$

Straightforward computations show that

$$\begin{aligned} T_{r,\omega}x - T_Gx &= (D - rL)^{-1}((1 - \omega)D + (\omega - r)L + \omega U)x - \rho(T_G)x \\ &= (D - rL)^{-1}[(1 - \omega)D + (\omega - r)L \\ &\quad + \omega U - \rho(T_G)(D - rL)]x. \end{aligned}$$

By computing Ux from (2.1) and substituting in the above equation, we have

$$(2.2) \quad T_{r,\omega}x - \rho(T_G)x = (1 - \rho(T_G))(D - rL)^{-1}[(1 - \omega)D + (\omega - r)L].$$

Evidently, Theorem 1.8 implies that $\rho(T_G) < 1$. It is easy to see that

$$(D - rL)^{-1}[(1 - \omega)D + (\omega - r)L] \geq 0,$$

but it is not equal to zero, and therefore the relation (2.2) implies that $T_{r,\omega}x \geq \rho(T_G)x$. Now, the result follows from Theorem 1.6 immediately. \square

As said earlier, the SOR method is a special case of the AOR method, and hence we can conclude the following corollary immediately.

Corollary 2.8. *Let A be an M -matrix. Suppose that T_G and T_ω are the Gauss-Seidel and SOR iteration matrices, respectively, where $0 < \omega < 1$. If T_G and T_ω are irreducible, then $\rho(T_G) < \rho(T_\omega)$.*

Theorem 2.9. *Let A be an M -matrix, $D_1 \geq 0$ and $0 \leq L_1 \leq L$. Suppose that T_G and T are the mixed-type splitting and Gauss-Seidel iteration matrices, respectively. If T_G and T are irreducible, then $\rho(T_G) < \rho(T)$.*

Proof. By Theorem 1.5, there exists a positive vector x such that $T_Gx = \rho(T_G)x$. From Theorem 2.2, we conclude that T is a positive matrix:

$$\begin{aligned} Tx - T_Gx &= (D + D_1 + L_1 - L)^{-1}(D_1 + L_1 + U)x - \rho(T_G)x \\ &= (1 - \rho(T_G))(D + D_1 + L_1 - L)^{-1}(D_1 + L_1)x. \end{aligned}$$

Since A is an M -matrix, by Proposition 1.13, it can be easily found that $\rho(T_G) < 1$. Evidently,

$$(D + D_1 + L_1 - L)^{-1}(D_1 + L_1) \geq 0.$$

Therefore, the result follows from Theorem 1.6. \square

Remark 2.10. If T, T_G, T_ω and $T_{r,\omega}$ are reducible matrices, then theorems 2.4, 2.5, 2.7, Corollary 2.8 and Theorem 2.9 still hold when “ $<$ ” changes to “ \leq ”.

3. Improving the mixed-type splitting method

Here, we shall consider the iterative methods for solving the linear system of equations $Ax = b$, where A is an M-matrix (M-matrix linear systems).

In order to improve the rate of convergence of the iterative methods for solving the linear system (1.1), several researchers applied one step or even more than one step of elimination methods. For more details in the case of one elimination, see [5, 6, 7, 8, 11, 12] and in the case of more than one elimination step, see [1].

In the present section, we consider the preconditioned matrix, introduced by the relation (3.1), to eliminate the first column below the diagonal of the matrix A . Then, the iterative methods, discussed in this paper, are employed. It is established that our approach leads to an improvement of the rate of convergence of the mixed-type, AOR, SOR and Gauss-Seidel iterative methods.

Consider the following *preconditioned* linear system,

$$\bar{A}x = \bar{b},$$

where $\bar{A} = (I + S)A$ and $\bar{b} = (I + S)b$ with

$$S = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -\frac{a_{21}}{a_{11}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{n1}}{a_{11}} & 0 & \cdots & 0 \end{pmatrix}_{n \times n}.$$

The matrix $P = (I + S)$ is called the *preconditioned* matrix.

The iterative methods defined by considering the classical iterative methods on \bar{A} is called preconditioned iterative methods. Furthermore, it is proved that the rate of convergence of the preconditioned Gauss-Seidel method is faster than the preconditioned AOR, preconditioned SOR and preconditioned mixed-type splitting iterative methods. At the end of this section, it is shown that more than one elimination step can be applied to get methods with even smaller spectral radii.

Lemma 3.1. *Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, $n \geq 2$ be an M-matrix, and suppose that*

$$(3.1) \quad P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ -\frac{a_{n1}}{a_{11}} & 0 & \cdots & 1 \end{pmatrix}_{n \times n}.$$

Then, the matrices $\bar{A} = PA$ and \bar{A}_1 , obtained from \bar{A} by deleting its first row and column, are M matrices.

Proof. See [4]. □

Assume that

$$E_1 = \bar{D} + \bar{D}_1 + \bar{L}_1 - \bar{L}, \quad F_1 = \bar{D}_1 + \bar{L}_1 + \bar{U},$$

where \bar{D} is a nonsingular diagonal matrix, \bar{L} and \bar{U} are strictly lower and upper triangular matrices, respectively, such that $\bar{A} = \bar{D} - \bar{L} - \bar{U}$.

We shall consider the special cases of the mixed-type and preconditioned mixed-type splittings which

$$(3.2) \quad D_1 = \alpha D, \quad \bar{D}_1 = \alpha \bar{D}, \quad L_1 = \beta L \quad \text{and} \quad \bar{L}_1 = \beta \bar{L},$$

where $0 \leq \alpha, \beta \leq 1$.

Since A is an M-matrix, from Lemma 3.1, we deduce that \bar{A} is an M-matrix. It can be easily shown that $\bar{D}_1 \leq D_1$. Also, it is clear that $\bar{D}, \bar{D}_1, \bar{L}, \bar{L}_1$ and \bar{U} are positive matrices.

Now, by the following theorem, it is proved that the rate of convergence of the preconditioned mixed-type splitting iterative method is faster than the mixed-type splitting iterative method for $D_1, \bar{D}_1, L_1, \bar{L}_1$, which satisfy in (3.2).

Theorem 3.2. *Suppose that A is an M-matrix. Furthermore, assume that T and $\bar{T} = E_1^{-1}F_1$ are the iteration matrices of the mixed-type and preconditioned mixed-type splittings, respectively. Then, $\rho(\bar{T}) \leq \rho(T)$.*

Proof. Consider the mixed-type splitting for the matrix A , i.e., $A = M - N$, where,

$$M = D + D_1 + L_1 - L, \quad N = D_1 + L_1 + U.$$

By Theorem 2.2, it is known that $M - N$ define a regular splitting for the matrix A . Also, it is easy to show that $E_1 - F_1$ is a regular splitting

for the matrix \bar{A} . Now, consider the following splitting for the matrix A ,

$$M_1 = (I + S)^{-1}E_1, \quad N_1 = (I + S)^{-1}F_1.$$

We can easily see that $M_1 - N_1$ defines a weak nonnegative splitting for the matrix A . Obviously $\bar{L} \geq L \geq 0$ and $E_1 \leq M$. By some easy computation, we get

$$E_1^{-1} - M^{-1} = E_1^{-1}(M - E_1)M^{-1}.$$

Clearly, $E_1^{-1} \geq M^{-1}$ and $M^{-1} \leq E_1^{-1} \leq E_1^{-1}(I + S) = M_1^{-1}$. Now, the result follows immediately from Theorem 1.11 . \square

It is obvious that the (preconditioned) Gauss-Seidel, SOR and AOR iterative methods are special cases of the (preconditioned) mixed-type splitting iterative method. Hence, we can conclude the following Corollaries from Theorem 3.2 by setting different values of the parameters α and β in the relation (3.2).

Corollary 3.3. *Suppose that A is an M -matrix and $0 \leq r \leq \omega < 1$, $\omega \neq 0$. Furthermore, assume that $T_{r,\omega}$ and $\bar{T}_{r,\omega}$ are the iteration matrices of the AOR and preconditioned AOR splittings, respectively. Then, $\rho(\bar{T}_{r,\omega}) \leq \rho(T_{r,\omega})$.*

Proof. We can conclude the result from Theorem 3.2, by setting $\alpha = \frac{1}{\omega}(1 - \omega)$ and $\beta = \frac{1}{\omega}(\omega - r)$ \square

It is well known that (preconditioned) SOR method is a special case of the (preconditioned) AOR method when $r = \omega$. Therefore, we can deduce the following Corollary from Corollary 3.3.

Corollary 3.4. *Suppose that A is an M -matrix and $0 < \omega < 1$. Furthermore, assume that T_ω and \bar{T}_ω are the iteration matrices of the SOR and preconditioned SOR splittings, respectively. Then, $\rho(\bar{T}_\omega) \leq \rho(T_\omega)$.*

Corollary 3.5. *Suppose that A is an M -matrix. Furthermore, assume that T_G and \bar{T}_G are the iteration matrices of the Gauss-Seidel and preconditioned Gauss-Seidel splittings, respectively. Then, $\rho(\bar{T}_G) \leq \rho(T_G)$.*

Proof. We can conclude the result from Theorem 3.2, by setting $\alpha = \beta = 0$ in (3.2). \square

Remark 3.6. Lemma 3.1, shows that \bar{A} is an M-matrix. Hence, Theorem 2.8 implies that the preconditioned Gauss-Seidel method converges faster than the preconditioned mixed-type splitting, preconditioned AOR and preconditioned SOR iterative methods.

Theorem 3.7. Suppose that A is a Z-matrix with positive diagonal elements. Moreover, assume that the Jacobi splitting is a convergent splitting for the matrix A . Then, the Jacobi splitting for each principal submatrix of the matrix A is convergent.

Proof. Assume that $A = [a_{ij}]_{n \times n}$ is a Z-matrix, where $a_{ii} > 0$ for $i = 1, 2, \dots, n$. Furthermore, suppose that the Jacobi splitting for A is convergent, i.e., $\rho(M_J^{-1}N_J) < 1$ where $A = M_J - N_J$. Let \hat{A} be an arbitrary principal submatrix of the matrix A . Without loss of generality, we may assume that

$$\hat{A} = \begin{pmatrix} a_{ii} & \cdots & a_{ij} \\ \vdots & & \vdots \\ a_{ji} & \cdots & a_{jj} \end{pmatrix}.$$

Consider the block diagonal matrix $P = \text{diag}\{\bar{D}, \hat{A}, \underline{D}\}$, where $\bar{D} = \text{diag}\{a_{11}, a_{22}, \dots, a_{i-1, i-1}\}$ and $\underline{D} = \text{diag}\{a_{j+1, j+1}, \dots, a_{nn}\}$. Suppose that $\hat{A} = \hat{M}_J - \hat{N}_J$ and $P = M_1 - N_1$ are the Jacobi splittings for \hat{A} and P , respectively.

It is clear that, $0 \leq M_1^{-1}N_1 \leq M_J^{-1}N_J$. Hence, Theorem 2.1 implies that $\rho(M_1^{-1}N_1) < 1$. Now, we can easily see that $\rho(\hat{M}_J^{-1}\hat{N}_J) < 1$. \square

The following Corollary is a result of Theorem 3.7. Also, it has been established by Varga[13], separately.

Corollary 3.8. Suppose that A is an M-matrix. Then, each principal submatrix of the matrix A is an M-matrix.

Theorem 3.9. Let $A = [a_{ij}]_{n \times n}$ be an M-matrix. Then, for each $1 \leq k \leq n$ the matrix $A^{(k)}$ obtained from A after performing k step(s) of Gauss elimination is an M-matrix.

Proof. For $k = 1$ the result follows from Lemma 3.1. For $k > 1$, assume that $A^{(k)}$ is an M-matrix obtained from A after applying k steps of Gauss elimination. The matrix $A^{(k)}$ has the following form

$$A^{(k)} = \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{pmatrix},$$

where $A_{11}^{(k)}$ is a $k \times k$ upper triangular matrix and $A_{22}^{(k)}$ is a $(n-k) \times (n-k)$ matrix.

By Corollary 3.8, $A_{11}^{(k)}$ and $A_{22}^{(k)}$ are M-matrices. Evidently, applying the $(k+1)$ th step of the Gauss elimination on the matrix $A^{(k)}$ is equivalent to applying the first step of the Gauss elimination on the matrix $A_{22}^{(k)}$. The matrix $A^{(k+1)}$ has the following form

$$A^{(k+1)} = \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k+1)} \end{pmatrix}.$$

By Lemma 3.1, $A_{22}^{(k+1)}$ is an M-matrix. On the other hand, $A^{(k+1)}$ is a Z-matrix with positive diagonal elements. Hence, we can conclude the result by computing the inverse of $A^{(k+1)}$. \square

Remark 3.10. *By Lemma 3.1, it can be easily shown that applying the preconditioned matrix P is equivalent to applying the first step of the Gauss elimination. Moreover, it has been proved that the preconditioned mixed-type splitting (AOR, SOR and Gauss-Seidel) iterative method converges faster than the mixed-type splitting (AOR, SOR and Gauss-Seidel) iterative method for solving an M-matrix linear system. Assume that $j (> 1)$ steps of Gauss elimination, on a given M-matrix A , have been performed. Denote the result matrix by $A^{(j)}$. By Theorem 3.9, $A^{(j)}$ itself is an M-matrix. Similar to the proof of Theorem 3.2, it can be established that applying j steps of Gauss elimination can improve the rate of convergence of the mixed-type splitting iterative method.*

4. Numerical results

In this section we present some numerical examples to illustrate the results established in the previous sections.

We define $E_{\infty} = \frac{R_{\infty,GS}}{R_{\infty}}$ as the *asymptotic coefficient of efficiency*. The value of E_{∞} shows that the Gauss-Seidel method is E_{∞} times faster asymptotically than a given method. In the following examples $E_{\infty, M}(E_{\infty, SOR}, E_{\infty, AOR})$ shows the *asymptotic coefficient of efficiency* for the mixed-type splitting (SOR, AOR) iterative method.

The matrix A in the following Example was used in [11].

Example 4.1. Suppose that the coefficient matrix, A^1 in the linear system (1.1), is given by

$$A = \begin{pmatrix} 1 & -\frac{1}{2 \times 10 + 1} & -\frac{1}{3 \times 10 + 1} & \cdots & -\frac{1}{n \times 10 + 1} \\ -\frac{1}{2 \times 10 + 2} & 1 & -\frac{1}{3 \times 10 + 2} & \cdots & -\frac{1}{n \times 10 + 2} \\ -\frac{1}{3 \times 10 + 3} & -\frac{1}{2 \times 10 + 3} & 1 & \cdots & -\frac{1}{n \times 10 + 3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n \times 10 + n} & -\frac{1}{(n-1) \times 10 + n} & -\frac{1}{(n-2) \times 10 + n} & \cdots & 1 \end{pmatrix}.$$

TABLE 1.

Spectral radii of mixed-type splitting and AOR when $D_1 = 0.5(1/\omega - 1)D$, $L_1 = 0.5(1 - r/\omega)L$.

n	ω	r	$\rho(T_G)$	$\rho(T)$	$\rho(T_{r,\omega})$	$E_{\infty,M}$	$E_{\infty,AOR}$
50	0.9	0.8	0.087219	0.172119	0.235915	1.38632	1.68895
100	0.95	0.8	0.104737	0.167120	0.215002	1.26493	1.47225
150	0.9	0.7	0.113111	0.216988	0.290315	1.42638	1.76213
200	0.8	0.65	0.119212	0.281770	0.385757	1.67910	2.23280

TABLE 2.

Spectral radii of mixed-type splitting and SOR when $D_1 = 0.7(1/\omega - 1)D$, $L_1 = 0$

n	ω	$\rho(T_G)$	$\rho(T)$	$\rho(T_\omega)$	$E_{\infty,M}$	$E_{\infty,SOR}$
50	0.6	0.087219	0.371838	0.515493	2.46571	3.68127
100	0.7	0.104737	0.314262	0.438784	1.95504	2.74721
150	0.9	0.113111	0.182754	0.239422	1.28228	1.52455
200	0.95	0.119212	0.154379	0.185694	1.13837	1.26324

The following example was used in [3, 7].

Example 4.2. Let

¹Note that the values of n are taken so that A is an M-matrix.

$$A = \begin{pmatrix} 1 & q & r & s & q & \cdots \\ s & 1 & q & r & \ddots & q \\ q & s & \ddots & \ddots & \ddots & s \\ r & \ddots & \ddots & \ddots & q & r \\ s & \ddots & q & s & 1 & q \\ \dots & s & r & q & s & 1 \end{pmatrix},$$

where $q = -\frac{p}{n}, r = -\frac{p}{n+1}$ and $s = -\frac{p}{n+2}$. For $n = 9$ and $p = 1$, the spectral radius of the Gauss-Seidel method is $\rho(T_G) = 0.66459$.

TABLE 3.

Spectral radii of mixed-type splitting and AOR when $D_1 = 0.4(1/\omega - 1)D, L_1 = 0.6(1 - r/\omega)L$.

ω	r	$\rho(T)$	$\rho(T_{r,\omega})$	$E_{\infty,M}$	$E_{\infty,AOR}$
0.5	0.3	0.85166	0.908023	2.5446	4.24340
0.6	0.4	0.83332	0.889844	2.24081	3.50080
0.7	0.6	0.81717	0.871485	2.02357	2.97029
0.8	0.65	0.80283	0.853125	1.86046	2.57216

TABLE 4.

Spectral radii of mixed-type splitting and SOR when $D_1 = 0.6(1/\omega - 1)D, L_1 = 0$.

ω	$\rho(T)$	$\rho(T_{r,\omega})$	$E_{\infty,M}$	$E_{\infty,AOR}$
0.4	0.871817	0.908762	2.97855	4.27066
0.6	0.805889	0.847752	1.89326	2.47377
0.7	0.772036	0.811464	1.57923	1.95574
0.8	0.737363	0.770021	1.34105	1.56344

The results, established in Section 3, are illustrated in the following tables. We have performed three steps of the Gauss elimination for the matrix A presented in Example 4.2. After applying three steps of Gauss elimination, the spectral radius of the preconditioned Gauss-Seidel method becomes $\rho(\overline{T}_G) = 0.541982$.

TABLE 5.

Spectral radii of mixed-type splitting and AOR when $D_1 = 0.4(1/\omega - 1)D$, $L_1 = 0.6(1 - r/\omega)L$.

ω	r	$\rho(\overline{T})$	$\rho(T_{r,\omega})$
0.5	0.3	0.789023	0.867772
0.6	0.4	0.763344	0.841326
0.7	0.6	0.740793	0.814881
0.8	0.65	0.720827	0.788435

TABLE 6.

Spectral radii of mixed-type splitting and SOR when $D_1 = 0.6(1/\omega - 1)D$, $L_1 = 0$.

ω	$\rho(\overline{T})$	$\rho(T_{r,\omega})$
0.4	0.820627	0.871482
0.6	0.731177	0.787804
0.7	0.685726	0.738688
0.8	0.639367	0.683028

The matrix A in the next example arises from discretization of the following differential equation, by using standard second-ordered differences; for more details see [14].

$$-u_{xx} - u_{yy} = f(x, y), \text{ in } \Omega = (0, 1) \times (0, 1), u = g(x, y) \text{ on } \partial\Omega.$$

Example 4.3. Suppose that A has the following form:

$$A = \begin{pmatrix} B & -I & & & & \\ -I & \ddots & \ddots & & & \\ & \ddots & \ddots & -I & & \\ & & -I & B & & \end{pmatrix}_{n \times n}, B = \begin{pmatrix} 4 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 4 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 4 & -1 & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & -1 & 4 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 4 \end{pmatrix}_{N \times N},$$

where $n = N^2$.

Consider the linear system $Ax = b$. Assume that the right hand side is generated as $b = Ae$, where $e^T = [1, 1, \dots, 1]$. Hence, the exact solution x is known in advance and all its components are equal to 1.

In the analysis of the reliability of iterative solution of $Ax = b$, it is convenient to consider the true error vector $e^{(k)} = x - x^{(k)}$, where x is the exact solution of the linear system $Ax = b$ and $x^{(k)}$ is the

k th approximate solution. The iteration process is terminated when $\|e^{(k)}\|_2 \leq 0.5 \times 10^{-8}$. In the following tables 7 and 8, m shows that the preconditioned system $\bar{A}x = \bar{b}$ is obtained after performing m steps of Gaussian elimination on the linear system $Ax = b$. In table 7, we apply the mixed-type splitting iterative method with $D_1 = 0.4(1/\omega - 1)D$, $L_1 = 0.6(1 - r/\omega)L$. For simplicity, in the following tables, we write PMixed-type (PSOR, PGS) instead of the preconditioned Mixed-type (SOR, GS) iterative method.

TABLE 7.

n	m	ω	Method	Iteration	$\ e^{(k)}\ _2$
400	140	0.85	SOR	669	4.9101×10^{-9}
			PSOR	543	4.8834×10^{-9}
			Mixed-type	581	4.8625×10^{-9}
			PMixed-type	471	4.9549×10^{-9}
900	450	0.9	SOR	1334	4.9234×10^{-9}
			PSOR	885	4.9121×10^{-9}
			Mixed-type	1212	4.9202×10^{-9}
			PMixed-type	775	4.89655×10^{-9}

For the Gauss-Seidel (GS) iterative method, the results are presented in the following Table 8.

TABLE 8.

Results for the Gauss-Seidel method					
n	m	Method	Iteration	$\ e^{(k)}\ _2$	
400	140	GS	492	4.9742×10^{-9}	
		PGS	399	4.9986×10^{-9}	
900	450	GS	1090	4.9073×10^{-9}	
		PGS	697	4.8632×10^{-9}	

5. Conclusion

We considered the mixed-type splitting iterative method for solving the M-matrix linear system $Ax = b$. It was established that the Gauss-Seidel iterative method converges faster than the mixed-type (SOR, AOR) iterative method. Furthermore, we improved the rate of convergence of the mixed-type (Gauss-Seidel, SOR, AOR) iterative method by applying a preconditioned matrix. It was shown that the preconditioned Gauss-Seidel method has smaller spectral radius than the preconditioned mixed-type splitting iterative (SOR, AOR) method. Finally, some numerical experiments were given to illustrate our results.

Acknowledgments

The authors thank the referee for the useful comments and helpful suggestions, which greatly improved our presentation.

This work has been partially supported by the Mahani Mathematical Research Center and by the Center of Excellence of Linear Algebra and Optimization of Shahid Bahonar University of Kerman.

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