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SOLVING INTEGRAL EQUATIONS OF THE THIRD KIND IN THE REPRODUCING KERNEL SPACE

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Abstract. A reproducing kernel Hilbert space restricts the space of functions to smooth functions and has structure for function approximation and some aspects in learning theory. Here, the solution of an integral equation of the third kind is constructed analytically using a new method. The analytical solution is represented in the form of series in the reproducing kernel space. Some numerical examples are studied to demonstrate the accuracy of the given method. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement.

1. Introduction

Reproducing Kernel Hilbert Spaces (RKHS) are wonderful objects and can be used in a wide variety of areas such as curve fitting, function estimation and model description, differential equation, probability, statistics, and so on [1, 2]. Recently, using the RKHS method, we discussed singular linear two-point boundary value problems, singular nonlinear two-point periodic boundary value problems, nonlinear system of boundary value problems and nonlinear Burgers equations [3, 4, 5, 6]. Nowadays, kernel methods are among the fastest growing and most exciting methods in machine learning.

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Here, we consider the following integral equations of the third kind in the RKHS,

$$(1.1)\ Lu(x) = \alpha(x)u(x) + \int_a^b K(x,s)u(s)ds = f(x), \quad a \le x \le b,$$

where $\alpha(x)$ is continuous and vanishes at some but not all points in [a, b], K is a continuous function, $u(x) \in W_2^1[a,b]$, and $f(x) \in W_2^1[a,b]$, with $W_2^1[a,b]$ being defined in the following section.

Such integral equations contain a variable coefficient, multiplying the identity operator, and vanishing at a number of points in the domain of definition of the equation.

Integral equations of the third kind are widely investigated in theory and used in applications. A number of important problems in elasticity, neutron transport, particle scattering lead to such equations. The third kind integral equations of the form (1.1) arises in the theories of singular integral equations with degenerate symbol and boundary value problems for mixed type partial differential equations. Therefore, the investigations in this area are of great interest. Integral equations of the third kind were the object of specin ial investigations by Bateman, Picard, Fubini, and Platrier. Friedrichs [7] performed, in the Hilbert space, spectral analysis of the operator corresponding to (1.1) under the assumption that $\alpha(x) = x$. Bart and Warnock [8]investigated the solvability of the equation in the class of generalized functions. Shulaia [9] discussed the solvability of the equation in the class of Holder functions assuming that $\alpha(x)$ has a simple zero. However, as we know, there are a few valid methods for solving integral equations of the third kind. Gabbasor [10, 11, 12] studied the equations using a new direct method and a special collocation method. Shulaia [13, 14, 15] investigated the equations basing on the ideas of the theory of spectral expansions.

2. The reproducing kernel space $W_2^1[a,b]$

Definition 2.1. (*Reproducing kernel*) *Let E be a nonempty abstract set. A function $K : E \times E \to C$ is a reproducing kernel of the Hilbert space H if and only if*

$$(a)\forall\ t \in E, \quad K(\cdot,\ t) \in H,$$
$$(b)\forall\ t \in E, \forall\ \varphi \in H, \quad (\varphi,\ K(\cdot,\ t)) = \varphi(t).$$

The last condition is called "the reproducing property": the value of the function φ at the point t is reproduced by the inner product of φ with $K(\cdot,\ t)$.

A Hilbert space which possesses a reproducing kernel is called a reproducing kernel Hilbert space (RKHS).

The inner product space $W_2^1[a,b]$ is defined by $W_2^1[a,b] = \{u(x) \mid u$ is absolutely continuous real valued function, $u, u' \in L^2[a,b]\}$. The inner product and norm in $W_2^1[a,b]$ are given respectively by

$$(u(x), v(x))_{W_2^1} = \int_a^b (uv + u'v')dx,\ \parallel u \parallel_{W_2^1} = \sqrt{(u,u)_{W_2^1}},$$

where $u(x), v(x) \in W_2^1[a,b]$. In [15]; Li and Cui proved that $W_2^1[a,b]$ is an RKHS and its reproducing kernel is

$$K(x,y) = \frac{1}{2\sinh(b-a)}[\cosh(x+y-b-a) + \cosh(|x-y|-b+a)].$$

3. **The solution of eq.** (1.1)

In this section, the solution of Eq.(1.1) is given in the RKHS, $W_2^1[a,b]$. In Eq. (1.1), it is clear that $L : W_2^1[a,b] \to W_2^1[a,b]$ is a bounded linear operator. Put $\varphi_i(x) = K(x_i, x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where L^* is the adjoint operator of L . The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^1[a,b]$ can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), (\beta_{ii} > 0, i = 1, 2, ...).$$

Theorem 3.1. *For Eq.* (1.1)*, if* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[a,b]$*, then* $\{\psi_i(x)\}_{i=1}^{\infty}$ *is the complete system of* $W_2^1[a,b]$ *and* $\psi_i(x) = L_yK(x,y)|_{y=x_i}$*. The subscript* y *by the operator* L *indicates that the operator* L *applies to the function of* y*.*

Proof. We have

$$\begin{aligned}\psi_i(x) &= (L^*\varphi_i)(x) = ((L^*\varphi_i)(y), K(x,y)) \\ &= (\varphi_i(y), L_yK(x,y)) = L_yK(x,y)|_{y=x_i}.\end{aligned}$$

Clearly, $\psi_i(x) \in W_2^1[a,b]$.

For each fixed $u(x) \in W_2^1[a,b]$, let $(u(x), \psi_i(x)) = 0, (i = 1, 2, ...)$, which means

$$(u(x),(L^*\varphi_i)(x)) = (Lu(\cdot),\varphi_i(\cdot)) = (Lu)(x_i) = 0.$$

Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, and hence $(Lu)(x) = 0$. It follows that $u \equiv 0$, from the existence of L^{-1}. So, the proof is complete. □

Theorem 3.2. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$ and the solution of Eq.* (1.1) *is unique, then the solution of Eq.*(1.1) *is*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k)\overline{\psi}_i(x). \tag{3.1}$$

Proof. Applying Theorem 3.1, it is easy to see that $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W_2^1[a,b]$.
For each $v(x) \in W_2^1[a,b]$, $(v(x),\varphi_i(x)) = (v(x),K(x_i,x))$. By the reproducing property of $K(x,y)$, it follows that $(v(x),\varphi_i(x)) = v(x_i)$, and hence we have

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty}(u(x),\overline{\psi}_i(x))\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(u(x),L^*\varphi_k(x))\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(Lu(x),\varphi_k(x))\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(f(x),\varphi_k(x))\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k)\overline{\psi}_i(x),
\end{aligned}$$

and the proof is complete. □

Now, the approximate solution $u_n(x)$ can be obtained by taking finitely many terms in the series representation of the analytical solution $u(x)$ as follows:

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}f(x_k)\overline{\psi}_i(x). \tag{3.2}$$

Remark: Since $W_2^1[a,b]$ is a Hilbert space, it is clear that

$$\sum_{i=1}^{\infty}\Big(\sum_{k=1}^{i}\beta_{ik}f(x_k)\Big)^2 < \infty.$$

Therefore, the sequence u_n is convergent.

In the following theorem, we will give the error estimate.

Theorem 3.3. *Assume that $u(x)$ is the solution of Eq. (1.1) and $r_n(x)$ is the error between the approximate solution $u_n(x)$ and the exact solution $u(x)$. Then, the error $r_n(x)$ is monotonically decreasing in $\| \cdot \|_{W_2^1}$ and $\| r_n \| = O(h)$, where $h = \max \mid x_{j+1} - x_j \mid$.*

Proof. Note that

$$\| r_n \|_{W_2^1}^2 = \| \sum_{i=n+1}^{\infty} B_i \overline{\psi}_i(x) \|_{W_2^1} = \sum_{i=n+1}^{\infty} (B_i)^2, \tag{3.3}$$

where $B_i = \sum_{k=1}^{i} \beta_{ik} f(x_k)$. Eq. (3.3) shows that the error r_n is monotonically decreasing in the sense of $\| \cdot \|_{W_2^1}$.

Note here that

$$Lu(x) = \sum_{i=1}^{\infty} B_i L\overline{\psi}_i(x)$$

and

$$(Lu)(x_n) = \sum_{i=1}^{\infty} B_i (L\overline{\psi}_i, \varphi_n) = \sum_{i=1}^{\infty} B_i (\overline{\psi}_i, L^* \varphi_n) = \sum_{i=1}^{\infty} B_i (\overline{\psi}_i, \psi_n).$$

Therefore,

$$\sum_{j=1}^{n} \beta_{nj} (Lu)(x_j) = \sum_{i=1}^{\infty} B_i (\overline{\psi}_i, \sum_{j=1}^{n} \beta_{nj} \psi_j) = \sum_{i=1}^{\infty} B_i (\overline{\psi}_i, \overline{\psi}_n) = B_n.$$

If $n = 1$, then $(Lu)(x_1) = f(x_1)$.

If $n = 2$, then $\beta_{21}(Lu)(x_1) + \beta_{22}(Lu)(x_2) = \beta_{21} f(x_1) + \beta_{22} f(x_2)$.

It is clear that $(Lu)(x_2) = f(x_2)$.

Moreover, it is easy to see, by induction, that

$$(Lu)(x_j) = f(x_j), j = 1, 2, \cdots. \tag{3.4}$$

Likewise, one can show that

$$(Lu_n)(x_j) = f(x_j), j = 1, 2, \cdots, n. \tag{3.5}$$

By (3.4), (3.5), we have $Lu(x_j) = Lu_n(x_j), \ \ j = 1, 2, \cdots, n$.

Therefore, $Lu_n(x)$ is the interpolating function of $Lu(x)$, where $x_j (j = 1, 2, \cdots, n)$ are the interpolation nodes in $[a, b]$. Noting that

$$Lu(x) - Lu_n(x) = Lu(x) - Lu(x_j) + \sum_{i=1}^{n} B_i L\overline{\psi}_i(x_j) - \sum_{i=1}^{n} B_i L\overline{\psi}_i(x),$$

by mean value theorem for differentials, there exist $\xi, \eta \in (x, x_j)$ such that

$$
\begin{aligned}
Lu(x) - Lu_n(x) &= (Lu)'(\xi)(x - x_j) + \sum_{i=1}^{n} B_i(L\overline{\psi}_i)'(\eta)(x - x_j) \\
&= (x_{j+1} - x_j)((Lu)'(\xi)\tfrac{x-x_j}{x_{j+1}-x_j} + \omega(\eta)\tfrac{x-x_j}{x_{j+1}-x_j}) \\
&= (x_{j+1} - x_j)\alpha_0(x) \\
&= h\alpha_0(x),
\end{aligned} \tag{3.6}
$$

where $\omega(\eta) = \sum_{i=1}^{n} B_i(L\overline{\psi}_i)'(\eta)$ and $\alpha_0(x) = f'(\xi)\frac{x-x_j}{x_{j+1}-x_j} + \omega(\eta)\frac{x-x_j}{x_{j+1}-x_j} \in W_2^1[a, b]$. If L^{-1} exists, then

$$
u(x) = L^{-1}f(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x). \tag{3.7}
$$

Thus, L^{-1} is determined by (3.7). In view of $L^{-1}\alpha_0(x) \in W_2^3[0, 1]$, one obtains that $\| L^{-1}\alpha_0(x) \|$ is bounded. Form (3.6) and (3.7), it follows that

$$
\| r_n \| = \| u(x) - u_n(x) \| = \| L^{-1}(h\alpha(x)) \| = h \| L^{-1}\alpha(x) \| = O(h).
$$

The proof is complete. □

4. **Numerical examples**

In this section, some numerical examples are given to demonstrate the accuracy of the given method. The examples are computed using Mathematica 5.0. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement.

In the following examples, RE, relative error between $u(x)$ and $u_n(x)$, is $\frac{|u(x)-u_n(x)|}{|u(x)|}$.

Example 4.1. *Consider the integral equation of the third kind,*

$$
a(x)u(x) + \int_0^1 \sin(x + s)u(s)ds = f(x), \;\; 0 \le x \le 1,
$$

where $a(x) = 20(x - 0.001)(x - 0.01)(x - 0.61)(x - 0.99)^2$.

The exact solution $u(x)$ *is* $\sinh(x) + 5$. *Using our method, we choose three sets of 26 points, 51 points and 100 points on* $[0, 1]$ ($x_i = (i - 1)/(n - 1), i = 1, 2, \cdots, n,$ *with* n *being the number of points). The*

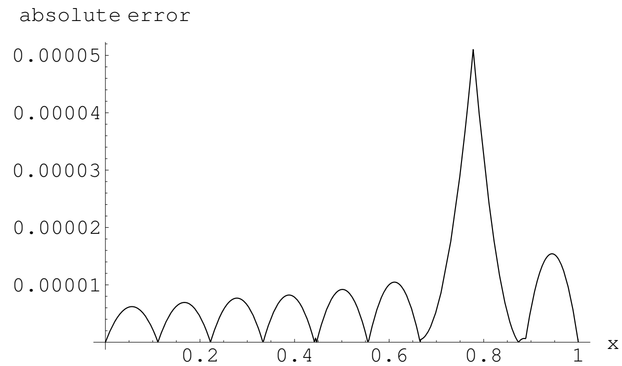

FIGURE 1. Absolute errors $|u(x) - u_{10}(x)|$ for Example 4.2

numerical results are given in Table 1. From the table, we can see that the relative error decreases as the number of points n increases.

TABLE 1. Numerical results for Example 4.1.

x	True solution	RE (u_{26})	RE (u_{51})	RE (u_{100})
0.001	5.00100	3.7E-03	2.5E-03	5.1E-05
0.01	5.01000	2.1E-03	1.1E-03	2.7E-04
0.16	5.16068	4.9E-05	3.1E-06	3.5E-12
0.32	5.32549	8.8E-06	2.1E-06	4.3E-12
0.48	5.49865	1.9E-05	4.2E-06	9.0E-12
0.61	5.64854	5.1E-04	8.6E-05	2.1E-04
0.80	5.88811	4.7E-05	8.6E-06	2.7E-11
0.96	6.11440	7.7E-04	1.3E-04	5.0E-10
0.99	6.15983	3.8E-03	7.6E-04	1.1E-03

Example 4.2. *Consider the integral equation of the third kind,*

$$(x-0.4)^2(x-0.8)^3u(x) + \int_0^1 e^{x+s}u(s)ds = f(x), \;\; 0 \le x \le 1.$$

The exact solution $u(x)$ is e^x. Using our method, we choose 10 points ($x_i = (i-1)/(10-1), i = 1, 2, \cdots, 10$. The numerical results are given in Figure 1.

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