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A NOTE ON THE SOCLE OF CERTAIN TYPES OF f-RINGS

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Abstract. For any reduced commutative f-ring with identity and bounded inversion, we show that a condition which is obviously necessary for the socle of the ring to coincide with the socle of its bounded part, is actually also sufficient. The condition is that every minimal ideal of the ring consists entirely of bounded elements. This is not too stringent, and is satisfied, for instance, by rings of continuous functions.

1. **Introduction**

Throughout, L denotes a completely regular frame, X denotes a Tychonoff space, $\mathfrak{R}L$ denotes the ring of real-valued continuous functions on L, and $C(X)$ has its usual meaning, as in [11]. By a "function ring", we mean any ring which is isomorphic to some $\mathfrak{R}L$. All rings $C(X)$ are function rings in this sense, but not every function ring is (isomorphic to) a $C(X)$, as observed by Banaschewski [5]. Recall that the socle of a ring A (throughout, understood to be commutative with identity) is the ideal generated by minimal ideals of A. In [13], the socle of $C(X)$ is characterized by the ideal consisting of functions which vanish everywhere except on a finite number of points. This is extended in [9] to

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arbitrary $\mathfrak{R}L$, where it is shown that a function is in $\mathrm{Soc}(\mathfrak{R}L)$ if and only if its cozero is a join of finitely many atoms.

The approaches in [13] and [9] are completely different. In the former case, the authors start by characterizing minimal ideals of $C(X)$. In [9], minimal ideals of $\mathfrak{R}L$ are not considered, and the description of the socle is arrived at via the fact that, for a ring A and $a \in A$, we have that $a \in \mathrm{Soc}\, A$ if and only if $\mathrm{Ann}(a)$ is an intersection of finitely many maximal ideals.

One byproduct of this note is a description of minimal ideals of $\mathfrak{R}L$. Interestingly, unlike the case of maximal ideals where the description (even for $C(X)$) requires the Stone-Čech compactification, minimal ideals are describable "internally" in terms of elements of the frame. Just as maximal ideals are in one-one correspondence with points of the Stone-Čech compactification, minimal ideals of $\mathfrak{R}L$ are in one-one correspondence with atoms of L.

Let A be an f-ring with bounded inversion, meaning that every element above the identity is invertible. Its bounded elements form a subring A^* is called its bounded part. It is obvious from the definition of the socle that if $\mathrm{Soc}\, A = \mathrm{Soc}\, A^*$, then every minimal ideal of A consists entirely of bounded elements. Conversely, we will show that if every minimal ideal of A consists entirely of bounded elements, then $\mathrm{Soc}\, A = \mathrm{Soc}\, A^*$. Having characterized minimal ideals of $\mathfrak{R}L$, we will see that function rings are f-rings of the kind mentioned here. It will then follow that the socle of a function ring coincides with the socle of its bounded part.

We are trying to treat the socle of function rings, more or less, in the same way as the socle of $C(X)$ is treated in [13]; see also [2] and [10].

2. A bit of background

Because this is intended to be a short note, we will keep the exposition brief and refer to [12] or [15] for details concerning frames, and to [3] or [4] for a background on the ring $\mathfrak{R}L$ and its bounded part $\mathfrak{R}^*L$. Our notation is standard. The *pseudocomplement* of an element a is denoted by a^*. An element a of a frame L is said to be

(1) *dense* if $a^* = 0$,
(2) an *atom* if $a > 0$ and, for any $s \in L$, $0 \leq s \leq a$ implies $s = 0$ or $s = a$,
(3) *complemented* if $a \vee a^* = 1$,

(4) a *point* if $a < 1$ and, for any $x, y \in L$, $x \wedge y \leq a$ implies $x \leq a$ or $y \leq a$.

Points are also called *prime* elements. Points of a regular frame are precisely the elements which are maximal strictly below the top. Atoms in a regular frame are complemented. A point, in any regular frame, is either dense or complemented.

A frame homomorphism $h\colon L \to M$ is *dense* if, for any $a \in L$, $h(a) = 0$ implies $a = 0$. If $h\colon L \to M$ is a dense onto frame homomorphism, then $h(a^*) = h(a)^*$, for every $a \in L$, so that $h(a)$ is dense if and only if a is dense.

We briefly outline the construction of the compact completely regular coreflection, βL, of a frame L, the frame analogue of the Stone-Čech compactification, βX, of a Tychonoff space X. Recall that an element a of L is said to be *rather below* an element b, written as $a \prec b$, if there is an element s such that $a \wedge s = 0$ and $s \vee b = 1$. On the other hand, a is said to be *completely below* b, written as $a \prec\!\prec b$, if there is a sequence $\{x_q \mid q \in \mathbb{Q} \cap [0,1]\}$ such that $a = x_0$, $b = x_1$ and $x_r \prec x_s$, whenever $r < s$. An ideal J of L is *completely regular* if for every $x \in J$, there exists $y \in J$ such that $x \prec\!\prec y$. The set of all completely regular ideals of L is a compact regular frame, denoted by βL, and the join map $\beta L \to L$ is the coreflection map from compact completely regular frames to L. We denote its right adjoint by r_L, and recall that

$$r_L(a) = \{x \in L \mid x \prec\!\prec a\}.$$

One checks easily that for any $I \in \beta L$, $I^* = r_L\big((\bigvee I)^*\big)$. If $I \prec J$ in βL, then $\bigvee I \in J$. To see this, take $H \in \beta L$ such that $I \wedge H = 0$ and $H \vee J = 1$. Next, take $x \in H$ and $y \in J$ such that $x \vee y = 1$. Since $\bigvee I \wedge \bigvee H = 0$, it follows that $\bigvee I \wedge x = 0$, and therefore $\bigvee I \leq y \in J$.

The *cozero map* (see [4] for details) is the map $\operatorname{coz}\colon \mathfrak{R}L \to L$, defined by

$$\operatorname{coz}\varphi = \bigvee \{\varphi(p,0) \vee \varphi(0,q) \mid p, q \in \mathbb{Q}\} = \varphi\big((-,0) \vee (0,-)\big),$$

where

$$(-,0) = \bigvee \{(p,0) \mid p \in \mathbb{Q}, p < 0\}$$

and

$$(0,-) = \bigvee \{(0,q) \mid q \in \mathbb{Q}, q > 0\}.$$

Concerning f-rings, recall that a *lattice-ordered ring* is a ring A which is also a lattice such that, for all $a, b, c \in A$,

$$a + (b \vee c) = (a + b) \vee (a + c),$$

and

$$ab \geq 0, \quad \text{whenever} \quad a \geq 0 \text{ and } b \geq 0.$$

An f-ring is a lattice-ordered ring A in which the identity

$$(a \wedge b)c = (ac) \wedge (bc)$$

holds for all $a, b \in A$ and $c \geq 0$ in A. A good reference is [6]. An element a of an f-ring is said to be *positive* if $a \geq 0$. An f-ring has *bounded inversion* if every element $a \geq 1$ is invertible. Squares are positive in any f-ring. In any f-ring, idempotents are bounded. Indeed, for any idempotent e, $0 \leq e^2 = e$, and since $1 - e$ is also an idempotent, $0 \leq 1 - e$, so that $e \leq 1$. Thus, $0 \leq e \leq 1$.

Let u be a positive invertible element in an f-ring. Then, the inequalities

$$u \geq 0 \quad \text{and} \quad (u^{-1})^2 \geq 0$$

yield

$$0 \leq u(u^{-1})^2 = u^{-1},$$

showing that the inverse of u is positive.

Considering the previous comment, we immediately have the following fact.

Lemma 2.1. *Let A be an f-ring with bounded inversion. Then, for any $a \in A$, $\frac{a^2}{1+a^2}$ is bounded.*

3. **Minimal ideals of $\mathfrak{R}L$**

We recall that a minimal ideal of a reduced ring is generated by an idempotent. Furthermore, if A is a reduced ring and e is an idempotent of A, then the principal ideal generated by e is a minimal ideal if and only if it is a field with multiplicative identity e, (see [14], p. 62).

We start with the f-ring result mentioned in the abstract. The condition mentioned there is that minimal ideals consist entirely of bounded elements. We shall see that function rings satisfy this condition. Alas, we do not have an example of a reduced f-ring which has a minimal ideal containing an unbounded element.

Lemma 3.1. *Let A be a reduced f-ring with bounded inversion. Suppose every minimal ideal of A consists entirely of bounded elements. Then, the set of minimal ideals of A coincides with the set of minimal ideals of A^*.*

Proof. For any $a \in A^*$, we write $\langle a \rangle$ for the ideal of A generated by a, and $\langle a \rangle_*$ for the ideal of A^* generated by a. Let M be a minimal ideal of A^*. Then, there is an idempotent $e \in A$ such that $M = \langle e \rangle_*$ is a field. We show that $\langle e \rangle$ is a field. Consider any $a \in A$ such that $ae \neq 0$. By Lemma 2.1, $\frac{a^2}{1+a^2} \in A^*$, and hence $\frac{a^2 e}{1+a^2} \in \langle e \rangle_*$. The latter element is nonzero, lest we have $a^2 e = 0$, implying $(ae)^2 = 0$, and hence $ae = 0$, since A is reduced. Therefore, there exists $b \in A^*$ such that

$$\frac{a^2 e}{1+a^2} \cdot be = e,$$

that is,

$$ae \cdot \frac{abe}{1+a^2} = e,$$

showing that $\langle e \rangle$ is a field. Hence, $\langle e \rangle$ is a minimal ideal of A. We aim to show that $M = \langle e \rangle$. Clearly, $M \subseteq \langle e \rangle$. Let $c \in \langle e \rangle$. Since $\langle e \rangle$ is a minimal ideal of A, c is bounded, by the hypothesis on minimal ideals of A. Now, $c = de$, for some $d \in A$, and hence $ce = de^2 = de = c$. This shows that c is a product of e with an element of A^*. Therefore, $c \in \langle e \rangle_*$, and hence $M = \langle e \rangle$. We have thus shown that every minimal ideal of A^* is a minimal ideal of A.

Now, we show that every minimal ideal of A is a minimal ideal of A^*. Let $\langle e \rangle$ be a minimal ideal of A. Then, by the hypothesis on A, $\langle e \rangle \subseteq A^*$. Arguing as above, we have that $\langle e \rangle = \langle e \rangle_*$. Since $\langle e \rangle$ is a minimal ideal of A, it is a field. But, now $\langle e \rangle_*$ is a principal ideal of A^*, generated by an idempotent, and is a field. So, $\langle e \rangle_*$ is a minimal ideal of A^*. Therefore, $\langle e \rangle$ is a minimal ideal of A^*. □

In light of the fact that the socle of any ring is the sum of its minimal ideals, the following proposition follows from the foregoing lemma.

Proposition 3.2. *Let A be a reduced f-ring with bounded inversion. Then,* $\operatorname{Soc} A = \operatorname{Soc} A^*$ *if and only if every minimal ideal of A consists entirely of bounded elements.*

We now apply this to the function ring $\mathfrak{R}L$. We first show that every minimal ideal of $\mathfrak{R}L$ consists entirely of bounded functions. We do this by actually describing minimal ideals of this ring. For any $a \in L$ with $a < 1$, the set

$$\mathfrak{R}(a) = \{\varphi \in \mathfrak{R}L \mid \operatorname{coz}\varphi \leq a\}$$

is clearly an ideal of $\mathfrak{R}L$. We recite from [7, Lemma 4.4] the following lemma.

Lemma 3.3. *If α and β are elements of $\mathfrak{R}L$ such that $\operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\beta$, then α is a multiple of β.*

Note that if η is an idempotent in $\mathfrak{R}L$, then $\operatorname{coz}\eta$ is complemented, because $\mathbf{1} = \eta + (\mathbf{1} - \eta)$ and $\eta(\mathbf{1} - \eta) = \mathbf{0}$ imply $\operatorname{coz}\eta \vee \operatorname{coz}(\mathbf{1} - \eta) = 1$, and $\operatorname{coz}\eta \wedge \operatorname{coz}(\mathbf{1} - \eta) = 0$.

Lemma 3.4. *An ideal of $\mathfrak{R}L$ is minimal if and only if it is of the form $\mathfrak{R}(a)$, for some atom a of L.*

Proof. Let a be an atom of L, and let Q be a nonzero ideal of $\mathfrak{R}L$ such that $Q \subseteq \mathfrak{R}(a)$. Take any $\mathbf{0} \neq \alpha \in Q$. Then, $0 \neq \operatorname{coz}\alpha \leq a$, and hence $\operatorname{coz}\alpha = a$, since a is an atom. Now, for any $\gamma \in \mathfrak{R}(a)$, $\operatorname{coz}\gamma \leq \operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\alpha$. Therefore, γ is a multiple of α, and hence $\gamma \in Q$, showing that $\mathfrak{R}(a) \subseteq Q$. Thus, $\mathfrak{R}(a)$ is a minimal ideal.

Now, let M be a minimal ideal of $\mathfrak{R}L$. Take an idempotent η in $\mathfrak{R}L$ such that $M = \langle \eta \rangle$. We show that $\operatorname{coz}\eta$ is an atom. Suppose s is an element of L such that $0 < s \leq \operatorname{coz}\eta$. By complete regularity, there is an element γ of $\mathfrak{R}L$ such that $0 < \operatorname{coz}\gamma \prec\!\!\prec s$. Then, $\operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\eta$, and so γ is a multiple of η and is therefore in M. Since any minimal ideal in any ring is generated by each of its nonzero elements, given any two nonzero elements in a minimal ideal, each is a multiple of the other. Thus, by the rules of the coz map, we deduce that $\operatorname{coz}\gamma = \operatorname{coz}\eta$, which shows that $s = \operatorname{coz}\eta$, and hence $\operatorname{coz}\eta$ is an atom. Now, by what we have shown in the first paragraph of the proof, $\mathfrak{R}(\operatorname{coz}\eta)$ is a minimal ideal with $M \subseteq \mathfrak{R}(\operatorname{coz}\eta)$. Therefore, $M = \mathfrak{R}(\operatorname{coz}\eta)$, and we are done. □

As an immediate corollary of Lemma 3.4 we have the following result which generalizes [13, Proposition 3.1].

Corollary 3.5. *An ideal Q of $\mathfrak{R}L$ is minimal if and only if $\operatorname{coz}[Q]$ consists of only two elements.*

Proof. The left-to-right implication is immediate. For the converse, suppose $\operatorname{coz}[Q] = \{0, a\}$, for some $0 \neq a \in \operatorname{Coz} L$. Then, reasoning as in the proof of the preceding lemma, we see that a is an atom such that $Q = \mathfrak{R}(a)$. □

Next, we show that the condition we imposed on minimal ideals of an f-ring in Lemma 3.1 automatically holds in function rings.

Lemma 3.6. *Every minimal ideal of $\mathfrak{R}L$ consists entirely of bounded functions.*

Proof. Let a be an atom of L, and consider the ideal $\mathfrak{R}(a)$. Let $\mathbf{0} \neq \varphi \in \mathfrak{R}(a)$. Then, $\operatorname{coz}\varphi = a$; that is, $\varphi\big((-,0) \vee (0,-)\big) = a$. Since $\big((-,0) \vee (0,-)\big) \vee (-1,1) = 1_{\mathfrak{L}(\mathbb{R})}$, it follows that

$$(\dagger) \qquad a \vee \varphi(-1,1) = 1_L.$$

Now, the equality

$$(-,0) \vee (0,-) = \bigvee_{n \in \mathbb{N}} \big((-n,0) \vee (0,n)\big)$$

implies

$$a = \bigvee_{n \in \mathbb{N}} \varphi\big((-n,0) \vee (0,n)\big).$$

Since $a \neq 0_L$, there is an $n \in \mathbb{N}$ such that

$$0_L \neq \varphi\big((-n,0) \vee (0,n)\big) \leq a.$$

Since a is an atom, this implies $\varphi\big((-n,0) \vee (0,n)\big) = a$, whence $\varphi(-n,n) = 1_L$, from (†). This shows that φ is bounded, and therefore $\mathfrak{R}(a) \subseteq \mathfrak{R}^*L$. □

We observe that minimal ideals (the socle) of $C(X)$ coincide with minimal ideals (the socle) of its bounded functions; see [13]. We conclude this section with the following result which gives the counterpart of the latter facts for function rings.

Proposition 3.7. *The set of minimal ideals of $\mathfrak{R}L$ coincides with the set of minimal ideals of $\mathfrak{R}^*L$. Hence,* $\operatorname{Soc}(\mathfrak{R}L) = \operatorname{Soc}(\mathfrak{R}^*L)$.

4. **Some properties of the socle of function rings**

In this section, we are to address two questions pertaining $\operatorname{Soc}(\mathfrak{R}L)$. Namely, one is its primeness and the other is its essentiality. Whereas in [9] the Stone-Čech compactification was invoked (albeit indirectly), in the proof of the description of the socle of $\mathfrak{R}L$ given below, Lemma 3.4 allows us to give a proof free of the Stone-Čech compactification.

Proposition 4.1. *The socle of $\mathfrak{R}L$ consists of those φ for which* $\operatorname{coz}\varphi$ *is a join of finitely many atoms.*

Proof. If $\mathrm{Soc}(\mathfrak{R}L) = 0$, then there is nothing to prove. So, suppose it is nonzero. If $\mathbf{0} \neq \varphi \in \mathrm{Soc}\,\mathfrak{R}L$, then $\varphi = \varphi_1 + \cdots + \varphi_m$, where each φ_i is a nonzero element of some minimal ideal. So, from Lemma 3.4, there are atoms $a_1, \ldots, a_m$ of L such that $\mathrm{coz}\,\varphi_i = a_i$, for each i. Thus, by the rules of the coz map, $\mathrm{coz}\,\varphi \leq a_1 \vee \cdots \vee a_m$. Consequently, $\mathrm{coz}\,\varphi = (\mathrm{coz}\,\varphi \wedge a_1) \vee \cdots \vee (\mathrm{coz}\,\varphi \wedge a_m)$. Since each a_i is an atom so that $\mathrm{coz}\,\varphi \wedge a_i = 0$ or a_i, it follows that $\mathrm{coz}\,\varphi$ is a join of finitely many atoms.

On the other hand, suppose $\mathrm{coz}\,\gamma = c_1 \vee \cdots \vee c_m$, where each c_i is an atom. For each i, take a positive $\gamma_i \in \mathfrak{R}L$ such that $\mathrm{coz}\,\gamma_i = c_i$. Then, γ_i is an element of the minimal ideal $\mathfrak{R}(c_i)$. Since $c_1 \vee \cdots \vee c_m$ is complemented, and since the γ_i are positive, we have

$$\mathrm{coz}\,\gamma = \mathrm{coz}(\gamma_1 + \cdots + \gamma_m) \prec\!\prec \mathrm{coz}(\gamma_1 + \cdots + \gamma_m).$$

Therefore, γ is a multiple of $\gamma_1 + \cdots + \gamma_m$, and hence a sum of multiples of the γ_i. This shows that $\gamma \in \sum_i \mathfrak{R}(c_i) \subseteq \mathrm{Soc}(\mathfrak{R}L)$. □

This description makes it clear that $\mathrm{Soc}(\mathfrak{R}L)$ is a radical ideal (i.e., it does not contain powers of non-members), and hence, by [11, Corollary 0.18], is an intersection of prime ideals. However, as observed in [10, Proposition 1.2], if X is an infinite Tychonoff space, then $\mathrm{Soc}(C(X))$ is never a prime ideal. If X is finite and has cardinality at least 2, then $\mathrm{Soc}(C(X)) = C(X)$, and is therefore not a prime ideal. If X is a singleton, then $C(X) \cong \mathbb{R}$, where $\mathbb{R}$ denotes the field of real numbers. Thus, $\mathrm{Soc}(C(X)) = 0$, and of course the zero ideal is a prime ideal in $\mathbb{R}$. So, if, as in [5], we call function rings of the form $C(X)$ "classical", we can state:

> *The socle of a classical function ring is a prime ideal if and only if the ring is isomorphic to the field of real numbers.*

Now, what about the socle of the function rings $\mathfrak{R}L$, in general? We argue that unless L is the two-element frame $\mathbf{2}$, so that $\mathfrak{R}L = \mathfrak{R}\mathbf{2} \cong \mathbb{R}$, the socle of $\mathfrak{R}L$ is never a prime ideal.

Let L be a completely regular frame such that $S = \mathrm{Soc}(\mathfrak{R}L)$ is a prime ideal of $\mathfrak{R}L$. Since, by Proposition 3.7, S is an ideal of the subring $\mathfrak{R}^*L$ of $\mathfrak{R}L$, it is clear that S is a prime ideal in the ring $\mathfrak{R}^*L$. This ring is known to be isomorphic to $\mathfrak{R}(\beta L)$. Since compact regular frames are (up to isomorphism) topologies of compact regular spaces, there is a compact regular space X such that $\mathfrak{R}(\beta L) \cong C(X)$. Thus, for this space X, $\mathrm{Soc}(C(X))$ is prime, and hence X is a singleton, whence $\beta L = \mathbf{2}$, thence $L = \mathbf{2}$. Consequently, we have the following result.

Proposition 4.2. *The socle of a function ring* $\mathfrak{R}L$ *is a prime ideal if and only if* $L = \mathbf{2}$.

Recall that an ideal of a ring is said to be *essential* if it intersects every nonzero ideal of the ring nontrivially. We give necessary and sufficient conditions for the socle of a function ring to be essential. We start by recalling some frame-theoretic resources needed for the discussion. These will include a description of maximal ideals of $\mathfrak{R}L$, which, in turn, requires the Stone-Čech compactification.

For each $I \in \beta L$ with $I < 1_{\beta L}$, the sets $\mathbf{O}^I$ and $\mathbf{M}^I$ defined by

$$\mathbf{O}^I = \{\varphi \in \mathfrak{R}L \mid \operatorname{coz}\varphi \in I\} \quad \text{and} \quad \mathbf{M}^I = \{\varphi \in \mathfrak{R}L \mid r_L(\operatorname{coz}\varphi) \subseteq I\}$$

are ideals of $\mathfrak{R}L$. It is shown in [7]:

(1) Maximal ideals of $\mathfrak{R}L$ are precisely the ideals $\mathbf{M}^I$, for I a point of βL.
(2) For every prime ideal P of $\mathfrak{R}L$, there is a unique point I of βL such that $\mathbf{O}^I \subseteq P \subseteq \mathbf{M}^I$.

A frame is *atomic* if below every nonzero element there is an atom. In [9], it was shown that $\operatorname{Soc}(\mathfrak{R}L)$ is essential if and only if L is atomic. Here, we include other conditions which are equivalent to essentiality. These should be compared with similar conditions for the classical case established in [13, Proposition 2.1]. We shall need a result about non-essential maximal ideals of $\mathfrak{R}L$. First, we recall the following lemmas from [7] and [8]. For an ideal Q of $\mathfrak{R}L$, by $\operatorname{coz}[Q]$, we mean the set

$$\operatorname{coz}[Q] = \{\operatorname{coz}\varphi \mid \varphi \in Q\}.$$

Lemma 4.3. *An ideal* Q *of* $\mathfrak{R}L$ *is essential if and only if* $\bigvee \operatorname{coz}[Q]$ *is dense.*

Lemma 4.4. *For any* $I \in \beta L$, $\bigvee \operatorname{coz}[\mathbf{O}^I] = \bigvee \operatorname{coz}[\mathbf{M}^I] = \bigvee I$.

We recall that each maximal ideal in a commutative ring (even a maximal right ideal in a non-commutative ring) is either essential or a direct summand (i.e., generated by an idempotent). In [1, Corollary 3.3]), Azarpanah shows that each pseudoprime ideal of $C(X)$ is either essential or simultaneously maximal and generated by an idempotent. In the latter case, it is also a minimal prime ideal. The preliminary result we want is a variant of this.

Lemma 4.5. *The non-essential prime ideals of* $\mathfrak{R}L$ *are precisely the ideals* $\mathbf{M}^I$, *for* I, *a complemented point of* βL. *Each is therefore principal, generated by an idempotent. Furthermore, each is minimal prime.*

Proof. Since, for any point I of βL, $\bigvee \operatorname{coz}[\mathbf{M}^I] = \bigvee I$, the ideal $\mathbf{M}^I$ is non-essential precisely when $\bigvee I$ is not a dense element of L. Since the join map $\beta L \to L$ is a dense onto homomorphism, the latter is true precisely when I is complemented in βL. Now, let P be a non-essential prime ideal of $\mathfrak{R}L$. Take a point I of βL such that $\mathbf{O}^I \subseteq P \subseteq \mathbf{M}^I$. Then, in light of Lemma 4.4, $\bigvee \operatorname{coz}[P] = \bigvee I$. Since P is not essential, $\bigvee I$ is not dense, and so $\mathbf{M}^I$ is not essential, because $\bigvee \operatorname{coz}[\mathbf{M}^I] = \bigvee I$. Since $\mathbf{M}^I$ is a maximal ideal, there exists an idempotent $\eta \in \mathfrak{R}L$ such that $\mathbf{M}^I = \langle \eta \rangle$. We claim that $P = \langle \eta \rangle$. Since η is an idempotent, $\eta(\mathbf{1} - \eta) = \mathbf{0} \in P$. By primeness, we have that $\eta \in P$ or $\mathbf{1} - \eta \in P$. The latter is not possible, lest $\mathbf{M}^I$ be improper. So $\eta \in P$, which implies that $P = \mathbf{M}^I$. Lastly, if $Q \subseteq P$ is a prime ideal, then $\eta \in Q$, and so $Q = P$, showing the claimed minimality. □

Note that if a is an atom in a frame L, then a^* is a point of L. For, if $s \in L$ is such that $a^* < s \leq 1$, then $s^* \leq a^{**} = a$. But, $s^* \neq a$, and so $s^* = 0$, implying that $s = 1$. Conversely, if p is a complemented point, then p^* is an atom. Indeed, suppose $0 \leq t \leq p^*$. Then, $p = p^{**} \leq t^* \leq 1$. Now, we cannot have $p = t^*$, for that would mean $t \vee t^* \leq p$, which would make p dense, which is not true. Thus, $t^* = 1$, implying $t = 0$, and hence showing that p^* is an atom. Consequently, if a is an atom, then $r_L(a^*)$ is a point of βL, because the right adjoint of any frame homomorphism preserves points. Note also the following. If $h\colon L \to M$ is a dense frame homomorphism and a is a complemented element with $h(a) = 1$, then $0 = h(a^*) \wedge h(a) = h(a^*)$, so that $a^* = 0$, by density of h, and hence $a = 1$. Thus, if I is a complemented point of βL, then $\bigvee I < 1$. Lastly, note that if I is a complemented element of βL, then $\mathbf{O}^I = \mathbf{M}^I$, because $\bigvee J \in K$ whenever $J \prec K$ in βL.

Noting that an ideal in reduced rings is essential if and only if its annihilator is zero, and also the annihilator of the socle in these rings is the intersection of the non-essential maximal ideals, we next present a result which gives the counterpart of [13, Proposition 2.1] for function rings.

Proposition 4.6. *If* $\operatorname{Soc}(\mathfrak{R}L)$ *is nonzero, then the followings are equivalent:*

(1) $\operatorname{Soc}(\mathfrak{R}L)$ *is essential.*
(2) *For every nonzero* $b \in L$*, there exists a non-essential maximal ideal* M *of* $\mathfrak{R}L$ *such that* $b \vee \bigvee \operatorname{coz}[M] = 1$.

(3) *The intersection of all non-essential maximal ideals of* $\mathfrak{R}L$ *is the zero ideal.*

Proof. (1) $\Rightarrow$ (2): Since $\mathrm{Soc}(\mathfrak{R}L)$ is essential, L is atomic, and therefore, there is an atom a such that $a \leq b$. As observed above, $r_L(a^*)$ is a complemented point of βL. Therefore, $M = \mathbf{M}^{r_L(a^*)}$ is a maximal ideal of $\mathfrak{R}L$ such that

$$b \vee \bigvee \mathrm{coz}[M] = b \vee \bigvee r_L(a^*) = b \vee a^* = 1.$$

Furthermore, by Lemma 4.3, M is non-essential, because $\bigvee \mathrm{coz}[M] = a^*$, which is not dense.

(2) $\Rightarrow$ (3): Suppose, by way of contradiction, that there is a nonzero φ in the intersection in question. By Lemma 4.5, the current hypothesis implies that there is a complemented point I of βL such that $\mathrm{coz}\,\varphi \vee \bigvee \mathrm{coz}[\mathbf{M}^I] = \mathrm{coz}\,\varphi \vee \bigvee I = 1$. Now, $\bigvee I \neq 1$, as observed above. Since φ is in every non-essential maximal ideal, we have that $\varphi \in \mathbf{M}^I = \mathbf{O}^I$. But, this implies $\mathrm{coz}\,\varphi \in I$, and hence $\bigvee I = 1$, yielding a contradiction.

(3) $\Rightarrow$ (1): We aim to demonstrate that $\bigvee\{\mathrm{coz}\,\gamma \mid \gamma \in \mathrm{Soc}(\mathfrak{R}L)\}$ is dense. Let $\alpha \in \mathfrak{R}L$ be such that

$$0 = \mathrm{coz}\,\alpha \wedge \bigvee \mathrm{coz}[\mathrm{Soc}(\mathfrak{R}L)] = \bigvee\{\mathrm{coz}\,\alpha \wedge \mathrm{coz}\,\gamma \mid \gamma \in \mathrm{Soc}\,\mathfrak{R}L\}.$$

We must show that $\alpha = \mathbf{0}$, which, by complete regularity, will complete the proof. For any atom $a \in L$, $a = \mathrm{coz}\,\tau$, for some $\tau \in \mathfrak{R}L$. Thus, $\tau \in \mathrm{Soc}(\mathfrak{R}L)$, and so $\mathrm{coz}\,\alpha \wedge a = 0$, whence $\mathrm{coz}\,\alpha \leq a^*$. Let I be a complemented point of βL. Then, I^* is an atom of βL. We show that this implies $(\bigvee I)^*$ to be an atom of L. Suppose s is an element of L such that $0 \neq s \leq (\bigvee I)^*$. Then, $0 \neq r_L(s) \subseteq r_L\big((\bigvee I)^*\big) = I^*$. Therefore, $r_L(s) = I^*$. Acting the join map and recalling that, as a dense onto homomorphism, it preserves pseudocomplements, we have that $s = \bigvee I^* = (\bigvee I)^*$. So, in view of the fact that $I \prec I$ as it is complemented, we have

$$\mathrm{coz}\,\alpha \leq \Big(\bigvee I\Big)^{**} = \bigvee I \in I,$$

implying that $\alpha \in \mathbf{O}^I = \mathbf{M}^I$. This shows that α is in the intersection of all non-essential maximal ideals of $\mathfrak{R}L$, and hence $\alpha = \mathbf{0}$, by the hypothesis. $\square$

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References

[1] F. Azarpanah, Essential ideals in $C(X)$, *Period. Math. Hungar.* **31** (1995), no. 2, 105–112.
[2] F. Azarpanah, O.A.S. Karamzadeh and S. Rahmati, $C(X)$ vs. $C(X)$ modulo its socle, *Colloq. Math.* **111** (2008), no. 2, 315–336.
[3] R. N. Ball and J. Walters-Wayland, C- and C^*-quotients in pointfree topology, *Dissertationes Math. (Rozprawy Mat.)* **412** (2002) 1–61.
[4] B. Banaschewski, The Real Numbers in Pointfree Topology, Textos de Matemática Série B, No. 12, Departamento de Matemática da Universidade de Coimbra, Coimbra, 1997.
[5] B. Banaschewski, On the function rings of pointfree topology, *Kyungpook Math. J.* **48** (2008), no. 2, 195–206.
[6] A. Bigard, K. Keimel and S. Wolfenstein, Groupes et Anneaux Réticulés, *Lect. Notes in Math.* **608**, Springer-Verlag, Berlin, 1977.
[7] T. Dube, Some ring-theoretic properties of almost P-frames, *Algebra Universalis* **60** (2009), no. 2, 145–162.
[8] T. Dube, Some algebraic characterizations of F-frames, *Algebra Universalis* **62** (2009), no. 2-3, 273–288.
[9] T. Dube, Contracting the socle in rings of continuous functions, *Rend. Semin. Mat. Univ. Padova* **123** (2010), 37–53.
[10] A. A. Estaji and O. A. S. Karamzadeh, On $C(X)$ modulo its socle, *Comm. Algebra* **31** (2003), no. 4, 1561–1571.
[11] L. Gillman and M. Jerison, Rings of Continuous Functions, Van Nostrand Co., Inc., Princeton, 1960.
[12] P. T. Johnstone, Stone Spaces, Cambridge Univ. Press, Cambridge, 1982.
[13] O. A. S. Karamzadeh and M. Rostami, On the intrinsic topology and some related ideals of $C(X)$, *Proc. Amer. Math. Soc.* **93** (1985), no. 1, 179–184.
[14] J. Lambek, Lecture Notes on Rings and Modules, Chelsea Publishing Co., New York, 1976.
[15] J. Picado, A. Pultr and A. Tozzi, Locales, In: Categorical foundations, 49–101, Encyclopedia Math. Appl., 97, Cambridge Univ. Press, Cambridge, 2004.

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