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RING STRUCTURES OF MOD p EQUIVARIANT COHOMOLOGY RINGS AND RING HOMOMORPHISMS BETWEEN THEM

Y. CHEN AND Y. WANG∗

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Abstract. We consider a class of connected oriented (with respect to $\mathbb{Z}/p$) closed G-manifolds with a non-empty finite fixed point set, each of which is G-equivariantly formal, where $G = \mathbb{Z}/p$ and p is an odd prime. Using localization theorem and equivariant index, we give an explicit description of the mod p equivariant cohomology ring of such a G-manifold in terms of algebra. This makes it possible to determine the number of equivariant cohomology rings (up to isomorphism) of such 2-dimensional G-manifolds. Moreover, we obtain a description of the ring homomorphism between equivariant cohomology rings of such two G-manifolds induced by a G-equivariant map, and show a characterization of the ring homomorphism.

1. Introduction

Assume that $G = \mathbb{Z}/p$ and p is an odd prime unless stated otherwise. Let X be a G-space and EG be the universal free G-space. Then, the Borel construction $X_G := EG \times_G X$, the orbit space of the diagonal action on the product $EG \times X$, is the total space of the bundle $X \to X_G \to BG$ associated to the universal principal bundle

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∗Corresponding author

$G \to EG \to BG$ ($BG := EG/G$, the classifying space of G). Applying cohomology with coefficients $\mathbb{Z}/p$ to X_G gives the equivariant cohomology ring $H^*_G(X;\mathbb{Z}/p) := H^*(X_G;\mathbb{Z}/p)$. It is well-known that the equivariant cohomology ring $H^*_G(X;\mathbb{Z}/p)$ is an $H^*(BG;\mathbb{Z}/p)$-module and $H^*_G(X^G;\mathbb{Z}/p)$ is a free $H^*(BG;\mathbb{Z}/p)$-module, where X^G denotes the fixed point set of the $G-$action.

Suppose that M is a connected oriented (with respect to $\mathbb{Z}/p$) closed manifold and admits a G-action with a non-empty finite fixed set M^G. For the fibration $M \to M_G \to BG$, if the restriction to a typical fiber $H^*_G(M;\mathbb{Z}/p) \longrightarrow H^*(M;\mathbb{Z}/p)$ is surjective, then M is called totally nonhomologous to zero in M_G (cf. [3]). If M satisfies this condition, then M is also called G-equivariantly formal (cf. [5]). In 1998, Goresky, Kottwitz and MacPherson showed that the equivariant cohomology rings of a class of T^n-manifolds (i.e., GKM manifolds) can be explicitly expressed in terms of their associated graphs (cf. [5, 6]). Correspondingly, there is a mod 2 GKM theory (cf. [2]). Note that any odd dimensional oriented closed G-manifold must not have a non-empty finite fixed point set (cf. [4]). Then, we shall give explicit descriptions of the mod p equivariant cohomology rings of G-equivariantly formal manifolds at any even dimension in terms of algebra.

Let Λ_{2n} denote the set of all $2n$-dimensional connected oriented (with respect to $\mathbb{Z}/p$) closed G-manifolds with a non-empty finite fixed point set, each of which is G-equivariantly formal. Given an M in Λ_{2n}, we know from [1, Theorem 3.10.4] and [3, pp. 371-374] that M has the following properties:

(1) The order $|M^G|$ of M^G equals $\sum_{i=0}^{2n} b_i$, where b_i is the ith mod p Betti number of M;

(2) $H^*_G(M;\mathbb{Z}/p)$ is a free $H^*(BG;\mathbb{Z}/p)$-module;

(3) The inclusion $i : M^G \hookrightarrow M$ induces a monomorphism $i^* : H^*_G(M;\mathbb{Z}/p) \longrightarrow H^*_G(M^G;\mathbb{Z}/p)$.

Note that $b_i = b_{2n-i}$, by the Poincaré duality, and $b_0 = b_{2n} = 1$, since M is connected. For each $M \in \Lambda_{2n}$, by the property (1), we have that $|M^G| \geq 2$. Let $r \geq 2$ be a positive integer, and write $\Lambda^r_{2n} = \{M \in \Lambda_{2n} || M^G| = r\}$. Then, $\Lambda_{2n} = \bigcup_{r\geq 2} \Lambda^r_{2n}$. From [1], we also have that $H^*(BG;\mathbb{Z}/p) = \Lambda(s) \bigotimes \mathbb{Z}/p[t] = \mathbb{Z}/p[s,t]/(s^2)$, with deg($s$)=1, deg($t$)=2, and $t = \beta(s)$, where β is the Bockstein homomorphism associated with the coefficient sequence $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$. If $|M^G| = r$, since $H^*_G(M^G;\mathbb{Z}/p) = \bigoplus_{a\in M^G} H^*_G(\{a\};\mathbb{Z}/p)$ and the equivariant cohomology ring of a point is isomorphic to $H^*(BG;\mathbb{Z}/p) = \mathbb{Z}/p[s,t]/(s^2)$ where s and t are as above, we have that $H^*_G(M^G;\mathbb{Z}/p) \cong (\mathbb{Z}/p)^r[s,t]/(s^2)$ is a polynomial ring (or algebra). Thus, we obtain a monomorphism from $H^*_G(M;\mathbb{Z}/p)$ into $(\mathbb{Z}/p)^r[s,t]/(s^2)$, also denoted by i^*, and so $H^*_G(M;\mathbb{Z}/p)$ may be identified with a subring (or subalgebra) of $(\mathbb{Z}/p)^r[s,t]/(s^2)$.

Using the localization theorem and equivariant index, we give an explicit description of $H^*_G(M;\mathbb{Z}/p)$ in $(\mathbb{Z}/p)^r[s,t]/(s^2)$ (see Theorem 3.3). By using this result, we find that there is only one equivariant cohomology ring (up to isomorphism) of G-manifolds in Λ_2^r if Λ_2^r is non-empty (see Theorem 4.1). Furthermore, we give a description for the homomorphism between equivariant cohomology rings of two G-manifolds in Λ_{2n} induced by a G-equivariant map (see Theorem 5.1), obtaining a characterization of the homomorphism.

The reminder of our work is organized as follows. In Section 2, we review the localization theorem and reformulate the equivariant index from [1]. In Section 3, we study the equivariant cohomology structure of a G-manifold in Λ_{2n} and obtain an explicit description in terms of algebra. In Section 4, we determine the number of equivariant cohomology rings (up to isomorphism) of G-manifolds in Λ_2. In Section 5, we give a description of the homomorphism between equivariant cohomology rings of two G-manifolds in Λ_{2n} induced by a G-equivariant map, obtaining a characterization of the homomorphism.

2. **Preliminaries**

Let M be a $2n$-dimensional G-manifold with a non-empty finite set M^G. Let R denote the polynomial part of $H^*(BG;\mathbb{Z}/p)$, i.e., $R = \mathbb{Z}/p[t]$ and $S = R - (0)$. Then, we have the following well-known localization theorem (cf. [1, 7]).

Theorem 2.1. *$S^{-1}i^* : S^{-1}H^*_G(M;\mathbb{Z}/p) \longrightarrow S^{-1}H^*_G(M^G;\mathbb{Z}/p)$ is an isomorphism of $S^{-1}H^*(BG;\mathbb{Z}/p)$-algebras, where i is the inclusion of M^G into M.* □

Take an isolated point $a \in M^G$. Let i_a be the inclusion of a into M. Then, we have the equivariant Gysin homomorphism

$$i_{a!} : H_G^*(\{a\}; \mathbb{Z}/p) \longrightarrow H_G^{*+2n}(M; \mathbb{Z}/p).$$

On the other hand, we also have a natural induced homomorphism

$$i_a^* : H_G^*(M; \mathbb{Z}/p) \longrightarrow H_G^*(\{a\}; \mathbb{Z}/p)$$

and we see that $i^* = \bigoplus_{a \in M^G} i_a^*$. Moreover, we have that the equivariant Euler class at a is

$$\chi_G(a) = i_a^* i_{a!}(1_a) \in H_G^{2n}(\{a\}; \mathbb{Z}/p) = H^{2n}(BG; \mathbb{Z}/p) = (\mathbb{Z}/p)t^n,$$

where $1_a \in H_G^*(\{a\}; \mathbb{Z}/p)$ is the identity and $(\mathbb{Z}/p)t^n = \{kt^n | k \in \mathbb{Z}/p\}$. So, we may write

$$\chi_G(a) = N_a t^n,$$

where $N_a \in \mathbb{Z}/p$. Let $\theta_a = i_{a!}(1_a)$. Then, $\theta_a \in H_G^{2n}(M; \mathbb{Z}/p)$ and $i_a^*(\theta_a) = \chi_G(a)$.

Lemma 2.2. *All elements $\theta_a, a \in M^G$ are linearly independent over $H^*(BG; \mathbb{Z}/p)$.*

Proof. Let $\sum_{a \in M^G} l_a \theta_a = 0$, where $l_a \in H^*(BG; \mathbb{Z}/p)$. By Lemma 5.3.14(2) of [1], we have that $i_b^*(\theta_a) = 0$, for $b \neq a$ in M^G, and so

$$i_b^*(\sum_{a \in M^G} l_a \theta_a) = \sum_{a \in M^G} l_a i_b^*(\theta_a) = l_b i_b^*(\theta_b) = l_b \chi_G(b) = 0.$$

Since $\chi_G(b)$ is a unit in $S^{-1}H_G^*(\{b\}; \mathbb{Z}/p) \cong S^{-1}H^*(BG; \mathbb{Z}/p)$, we have $l_b = 0$.

□

Lemma 2.3. *Let $\alpha \in S^{-1}H_G^*(M; \mathbb{Z}/p)$. Then,*

$$\alpha = \sum_{a \in M^G} \frac{f_a \theta_a}{\chi_G(a)},$$

where $f_a = S^{-1}i_a^*(\alpha) \in S^{-1}H^*(BG; \mathbb{Z}/p)$.

Proof. From Proposition 5.3.18(1) of [1], we know that

$$\alpha = \sum_{a \in M^G} S^{-1}i_{a!}(S^{-1}i_a^*(\alpha)/\chi_G(a)).$$

Since $f_a = S^{-1}i_a^*(\alpha) \in S^{-1}H^*(BG; \mathbb{Z}/p)$, we have that $\frac{f_a}{\chi_G(a)} \in S^{-1}H^*(BG; \mathbb{Z}/p)$. Since $S^{-1}i_{a!}$ is a $S^{-1}H^*(BG; \mathbb{Z}/p)$-algebra homomorphism, we have

$$S^{-1}i_{a!}(S^{-1}i_a^*(\alpha)/\chi_G(a)) = S^{-1}i_{a!}(\tfrac{f_a}{\chi_G(a)}) = \tfrac{f_a}{\chi_G(a)}S^{-1}i_{a!}(1_a)$$
$$= \tfrac{f_a}{\chi_G(a)}i_{a!}(1_a) = \tfrac{f_a\theta_a}{\chi_G(a)}.$$

Thus, $\alpha = \sum_{a\in M^G} \frac{f_a\theta_a}{\chi_G(a)}$. □

Remark 2.4. *From Lemma 2.2 and Lemma 2.3, we see that* $\{\frac{\theta_a}{\chi_G(a)} | a \in M^G\}$ *forms a basis of* $S^{-1}H_G^*(M;\mathbb{Z}/p)$ *as a* $S^{-1}H^*(BG;\mathbb{Z}/p)$*-algebra.*

The equivariant Gysin homomorphism of collapsing M to a point gives the G-index of M, i.e.,

$$\mathrm{Ind}_G : H_G^*(M;\mathbb{Z}/p) \longrightarrow H^{*-2n}(BG;\mathbb{Z}/p).$$

Theorem 2.5. *For any* $\alpha \in S^{-1}H_G^*(M;\mathbb{Z}/p)$,

$$S^{-1}\ \mathit{Ind}_G(\alpha) = \sum_{a\in M^G} \frac{f_a}{\chi_G(a)},$$

where $f_a = S^{-1}i_a^*(\alpha) \in S^{-1}H^*$
$(BG;Z/p)$. *In particular, if* $\alpha \in H_G^*(M;\mathbb{Z}/p)$, *then* $f_a = i_a^*(\alpha) \in H^*(BG;\mathbb{Z}/p)$ *and*

$$\mathit{Ind}_G(\alpha) = \sum_{a\in M^G} \frac{f_a}{\chi_G(a)} \in H^*(BG;\mathbb{Z}/p) = \mathbb{Z}/p[s,t]/(s^2).$$

Proof. From Lemma 5.3.19 of [1], we have that $\mathrm{Ind}_G(\theta_a) = 1_a$, and so by Lemma 2.3,

$$S^{-1}\mathrm{Ind}_G(\alpha) = \sum_{a\in M^G} \frac{f_aS^{-1}\mathrm{Ind}_G(\theta_a)}{\chi_G(a)} = \sum_{a\in M^G} \frac{f_a\cdot 1_a}{\chi_G(a)} = \sum_{a\in M^G} \frac{f_a}{\chi_G(a)}.$$

Since $H^*(BG;\mathbb{Z}/p) \longrightarrow S^{-1}H^*(BG;\mathbb{Z}/p)$ is injective, the last part of Theorem 2.5 follows immediately.

□

3. Equivariant cohomology structure

The purpose of this section is to study the structures of mod p equivariant cohomology rings of G-manifolds in Λ_{2n}.

Lemma 3.1. *Let* $M \in \Lambda_{2n}^r (r \geq 2)$. *Then,*

$$\dim_{\mathbb{Z}/p} H_G^i(M;\mathbb{Z}/p) = \begin{cases} \sum_{j=0}^{i} b_j, & if\ i \le 2n-1, \\ \\ r, & if\ i \ge 2n, \end{cases}$$

where b_j is the jth mod p Betti number of M.

Proof. Let

$$P_z(M_G) = \sum_{i=0}^{\infty} \dim_{\mathbb{Z}/p} H_G^i(M;\mathbb{Z}/p) z^i$$

be the equivariant Poincaré polynomial of $H_G^*(M;\mathbb{Z}/p)$. Since $H_G^*(M;\mathbb{Z}/p)$ is a free $H^*(BG;\mathbb{Z}/p)$-module, we have that $H_G^*(M;\mathbb{Z}/p) = H^*(M;\mathbb{Z}/p) \bigotimes_{\mathbb{Z}/p} H^*(BG;\mathbb{Z}/p)$, and so

$$\begin{aligned} P_z(M_G) &= \sum_{i=0}^{\infty} \dim_{\mathbb{Z}/p} H_G^i(M;\mathbb{Z}/p) z^i = \frac{1}{1-z} \sum_{i=0}^{2n} \dim_{\mathbb{Z}/p} H^i(M;\mathbb{Z}/p) z^i \\ &= b_0 + (b_0+b_1)z + \cdots + (b_0+b_1+\cdots+b_{2n-1})z^{2n-1} \\ &\quad + (b_0+b_1+\cdots+b_{2n})(z^{2n}+\cdots) \\ &= b_0 + (b_0+b_1)z + \cdots + (b_0+b_1+\cdots+b_{2n-1})z^{2n-1} \\ &\quad + r(z^{2n}+\cdots). \end{aligned}$$

So, the result follows. □

Let $x = (x_1, \cdots, x_r)^T$ and $y = (y_1, \cdots, y_r)^T$ be two vectors in $(\mathbb{Z}/p)^r$. Define $x \circ y$ by

$$x \circ y = (x_1 y_1, \cdots, x_r y_r)^T.$$

Then, $(\mathbb{Z}/p)^r$ forms a commutative ring with respect to two operations $+$ and $\circ$. Let $a_1, \cdots, a_r$ be all fixed points in M^G and

$$\mathcal{V}_r^{(M)} = \{x = (x_1, \cdots, x_r)^T \in (\mathbb{Z}/p)^r \mid |x| = \sum_{i=1}^{r} \frac{x_i}{N_{a_i}} = 0\},$$

where N_{a_i} is as above. Then, it is easy to see that $\mathcal{V}_r^{(M)}$ is an $(r-1)$-dimensional subspace of $(\mathbb{Z}/p)^r$. Generally speaking, the operation $\circ$ in $\mathcal{V}_r^{(M)}$ is not closed.

If $M \in \Lambda_{2n}^r$, then we have that the inclusion $i : M^G \hookrightarrow M$ induces a monomorphism

$$i^* : H_G^*(M;\mathbb{Z}/p) \longrightarrow (\mathbb{Z}/p)^r[s,t]/(s^2).$$

By Lemma 3.1, there are subspaces V_i^M with $\dim V_i^M = \sum_{j=0}^{i} b_j$ ($i = 0, \cdots, 2n-1$) of $(\mathbb{Z}/p)^r$ such that

$$H_G^i(M;\mathbb{Z}/p) \cong i^*(H_G^i(M;\mathbb{Z}/p)) = \begin{cases} V_i^M t^{\frac{i}{2}}, & if\ i \le 2n-2\ and\ i\ is\ even, \\ V_i^M st^{\frac{i-1}{2}}, & if\ i \le 2n-1\ and\ i\ is\ odd, \\ (\mathbb{Z}/p)^r t^{\frac{i}{2}}, & if\ i \ge 2n\ and\ i\ is\ even, \\ (\mathbb{Z}/p)^r st^{\frac{i-1}{2}}, & if\ i \ge 2n+1\ and\ i\ is\ odd, \end{cases}$$

where $V_i^M t^{\frac{i}{2}} = \{vt^{\frac{i}{2}} | v \in V_i^M\}$ and $V_i^M st^{\frac{i-1}{2}} = \{vst^{\frac{i-1}{2}} | v \in V_i^M\}$.

Lemma 3.2. *There are the following properties:*

(1) $V_i^M \subset V_{2n-1}^M = \mathcal{V}_r^{(M)}$, for $i < 2n-1$, $V_0^M \cong \mathbb{Z}/p$ is generated by $(1, \cdots, 1)^\top \in (\mathbb{Z}/p)^r$, $V_i^M \subset V_{i+1}^M$, for $i < 2n-1$ and i even, and $V_i^M \subset V_{i+2}^M$, for $i+1 < 2n-1$.

(2) For $d = \sum_{i \le 2n-2\ and\ i\ even} i \cdot d_i < 2n$, with each $d_i \ge 0$, $v_{\omega_{d_0}} \circ v_{\omega_{d_2}} \circ \cdots \circ v_{\omega_{d_{2n-2}}} \in V_d^M$, where $v_{\omega_{d_i}} = v_1^{(i)} \circ \cdots \circ v_{d_i}^{(i)}$, with each $v_j^{(i)} \in V_i^M$.

(3) For $d = \sum_{i \le 2n-2\ and\ i\ even} i \cdot d_i + k < 2n$, with k odd, $1 \le k \le 2n-1$ and each $d_i \ge 0$, $v_{\omega_{d_0}} \circ v_{\omega_{d_2}} \circ \cdots \circ v_{\omega_{d_{2n-2}}} \circ v_1^{(k)} \in V_d^M$, where $v_{\omega_{d_i}} = v_1^{(i)} \circ \cdots \circ v_{d_i}^{(i)}$, with each $v_j^{(i)} \in V_i^M$ and $v_1^{(k)} \in V_k^M$.

Proof. (1) For an element $\alpha \in H_G^*(M;\mathbb{Z}/p)$ of even degree d, we have that $i^*(\alpha) = vt^{\frac{d}{2}}$, where $v \in (\mathbb{Z}/p)^r$. Since $i^* = \bigoplus_{a \in M^G} i_a^*$, by Theorem 2.5 we have that

$$\begin{aligned} \mathrm{Ind}_G(\alpha) &= \sum_{a \in M^G} \frac{i_a^*(\alpha)}{\chi_G(a)} = \sum_{a \in M^G} \frac{i_a^*(\alpha)}{N_a t^n} = \frac{1}{t^n} \sum_{a \in M^G} \frac{i_a^*(\alpha)}{N_a} \\ &= |v| t^{\frac{d}{2}-n} \in \mathbb{Z}/p[s,t]/(s^2). \end{aligned}$$

Thus, if $d < 2n$ and d is even, then $|v|$ must be zero. For an element $\alpha' \in H_G^*(M;\mathbb{Z}/p)$ of odd degree d, we have that $i^*(\alpha') = v'st^{\frac{d-1}{2}}$, where $v' \in (\mathbb{Z}/p)^r$. Similarly, we have that if $d < 2n$ and d is odd, $|v'|$ also must be zero. Thus, if $i < 2n$, V_i^M is a subspace of $\mathcal{V}_r^{(M)}$, and $V_{2n-1}^M = \mathcal{V}_r^{(M)}$, by reason of dimension.

In particular, when $\alpha = 1$ (the identity of $H_G^*(M;\mathbb{Z}/p)$), $i^*(\alpha) = \bigoplus_{a \in M^G} i_a^*(1) = (1, \cdots, 1)^\top \in (\mathbb{Z}/p)^r$. So, $V_0^M \cong \mathbb{Z}/p$

is generated by $(1, \cdots, 1)^{\top} \in (\mathbb{Z}/p)^r$, since $\dim V_0^M = b_0 = 1$. Since $H_G^*(M;\mathbb{Z}/p) = H^*(M;\mathbb{Z}/p) \bigotimes_{\mathbb{Z}/p} H^*(BG;\mathbb{Z}/p)$, we have that $(1, \cdots, 1)^{\top} s \in i^*(H_G^1(M;$
$\mathbb{Z}/p))$ and $(1, \cdots, 1)^{\top} t \in i^*(H_G^2(M;\mathbb{Z}/p))$. Thus, for any $v \in V_i^M$ with $i < 2n-1$ and i even, $[(1, \cdots, 1)^{\top} s] \circ (vt^{\frac{i}{2}}) = vst^{\frac{i}{2}} \in V_{i+1}^M st^{\frac{i}{2}}$, and so we have $v \in V_{i+1}^M$. For any $v \in V_i^M$ with $i+1 < 2n-1$ and i odd, $[(1, \cdots, 1)^{\top} t] \circ (vst^{\frac{i-1}{2}}) = vst^{\frac{i+1}{2}} \in V_{i+2}^M st^{\frac{i+1}{2}}$, and so $v \in V_{i+2}^M$. Similarly, we have that $V_i^M \subset V_{i+2}^M$ for $i+1 < 2n-1$ and i even. This completes the proof of (1).

(2) If i is even and $v_j^{(i)} \in V_i^M$, since $i^* : H_G^*(M;\mathbb{Z}/p) \longrightarrow (\mathbb{Z}/p)^r[s,t]/(s^2)$ is injective, there is a class $\alpha_j^{(i)}$ of degree i in $H_G^*(M;\mathbb{Z}/p)$ such that $i^*(\alpha_j^{(i)}) = v_j^{(i)} t^{\frac{i}{2}}$. If k is odd and $v_1^{(k)} \in V_k^M$, there is also a class $\beta_1^{(k)}$ of degree k in $H_G^*(M;\mathbb{Z}/p)$ such that $i^*(\beta_1^{(k)}) = v_1^{(k)} st^{\frac{k-1}{2}}$.

For $d = \sum_{i \le 2n-2 \text{ and } i \text{ even}} i \cdot d_i < 2n$ with each $d_i \ge 0$, since $i^* = \bigoplus_{a \in M^G} i_a^*$ is a ring homomorphism, we have

$$\begin{aligned} i^*\Big(\prod_{i \le 2n-2 \text{ and } i \text{ even}} \prod_{j=1}^{d_i} \alpha_j^{(i)}\Big) &= \bigoplus_{a \in M^G} i_a^*\Big(\prod_{i \le 2n-2 \text{ and } i \text{ even}} \prod_{j=1}^{d_i} \alpha_j^{(i)}\Big) \\ &= \bigoplus_{a \in M^G} \prod_{i \le 2n-2 \text{ and } i \text{ even}} \prod_{j=1}^{d_i} i_a^*(\alpha_j^{(i)}) \\ &= v_{\omega_{d_0}} \circ v_{\omega_{d_2}} \circ \cdots \circ v_{\omega_{d_{2n-2}}} t^{\frac{d}{2}}, \end{aligned}$$

and so $v_{\omega_{d_0}} \circ v_{\omega_{d_2}} \circ \cdots \circ v_{\omega_{d_{2n-2}}} \in V_d^M$. The proof of (2) is now complete.

(3) Similarly, for $d = \sum_{i \le 2n-2 \text{ and } i \text{ even}} i \cdot d_i + k < 2n$ with k odd, $1 \le k \le 2n-1$ and each $d_i \ge 0$, we have that

$$\begin{aligned} & i^*\Big(\Big[\prod_{i \le 2n-2 \text{ and } i \text{ even}} \prod_{j=1}^{d_i} \alpha_j^{(i)}\Big] \cdot \beta_1^{(k)}\Big) \\ &= \bigoplus_{a \in M^G} i_a^*\Big(\Big[\prod_{i \le 2n-2 \text{ and } i \text{ even}} \prod_{j=1}^{d_i} \alpha_j^{(i)}\Big] \cdot \beta_1^{(k)}\Big) \\ &= \bigoplus_{a \in M^G} \Big(\Big[\prod_{i \le 2n-2 \text{ and } i \text{ even}} \prod_{j=1}^{d_i} i_a^*(\alpha_j^{(i)})\Big] \cdot i_a^*(\beta_1^{(k)})\Big) \end{aligned}$$

$$= v_{\omega_{d_0}} \circ v_{\omega_{d_2}} \circ \cdots \circ v_{\omega_{d_{2n-2}}} \circ v_1^{(k)} st^{\frac{d-1}{2}},$$

and so $v_{\omega_{d_0}} \circ v_{\omega_{d_2}} \circ \cdots \circ v_{\omega_{d_{2n-2}}} \circ v_1^{(k)} \in V_d^M$, which completes the proof.

□

Remark 3.3. *An observation shows that Lemma 3.2 gives a subring structure of*

$$\mathcal{R}_M = V_0^M + V_1^M s + \cdots + V_{2n-2}^M t^{n-1} + V_{2n-1}^M st^{n-1} + (\mathbb{Z}/p)^r (t^n + st^n + \cdots)$$

in $(\mathbb{Z}/p)^r[s,t]/(s^2)$.

Combining lemmas 3.1, 3.2, and Remark 3.3, we have the following.

Theorem 3.4. *Let* $M \in \Lambda_{2n}^r (r \geq 2)$*. Then, there are subspaces* V_i^M *with* $dim V_i^M = \sum_{j=0}^{i} b_j (i = 0, \cdots, 2n-1)$ *of* $\mathcal{V}_r^{(M)}$ *such that* $H_G^*(M; \mathbb{Z}/p)$ *is isomorphic to the graded ring*

$\mathcal{R}_M = V_0^M + V_1^M s + \cdots + V_{2n-2}^M t^{n-1} + V_{2n-1}^M st^{n-1} + (\mathbb{Z}/p)^r (t^n + st^n + \cdots)$*, where the ring structure of* $\mathcal{R}_M$ *is determined by*

(1) $V_i^M \subset V_{2n-1}^M = \mathcal{V}_r^{(M)}$*, for* $i < 2n-1$*,* $V_0^M \cong \mathbb{Z}/p$ *is generated by* $(1, \cdots, 1)^\top \in (\mathbb{Z}/p)^r$*,* $V_i^M \subset V_{i+1}^M$*, for* $i < 2n-1$ *and* i *even, and* $V_i^M \subset V_{i+2}^M$*, for* $i+1 < 2n-1$.

(2) *For* $d = \sum_{i \leq 2n-2 \text{ and } i \text{ even}} i \cdot d_i < 2n$ *with each* $d_i \geq 0$*,* $v_{\omega_{d_0}} \circ v_{\omega_{d_2}} \circ \cdots \circ v_{\omega_{d_{2n-2}}} \in V_d^M$*, where* $v_{\omega_{d_i}} = v_1^{(i)} \circ \cdots \circ v_{d_i}^{(i)}$ *with each* $v_j^{(i)} \in V_i^M$.

(3) *For* $d = \sum_{i \leq 2n-2 \text{ and } i \text{ even}} i \cdot d_i + k < 2n$ *with* k *odd,* $1 \leq k \leq 2n-1$ *and each* $d_i \geq 0$*,* $v_{\omega_{d_0}} \circ v_{\omega_{d_2}} \circ \cdots \circ v_{\omega_{d_{2n-2}}} \circ v_1^{(k)} \in V_d^M$*, where* $v_{\omega_{d_i}} = v_1^{(i)} \circ \cdots \circ v_{d_i}^{(i)}$ *with each* $v_j^{(i)} \in V_i^M$ *and* $v_1^{(k)} \in V_k^M$.

Remark 3.5. *Since* $H_G^*(M; \mathbb{Z}/p)$ *is a free* $H^*(BG; \mathbb{Z}/p)$*-module, we have that* $\mathcal{R}_M$ *is also a free* $\mathbb{Z}/p[s,t]/(s^2)$*-module.*

4. **2-dimensional case**

Let $M \in \Lambda_2^r (r \geq 2)$. From Theorem 3.4, we have that

$$H_G^*(M;\mathbb{Z}/p)\cong V_0^M+\mathcal{V}_r^{(M)}s+(\mathbb{Z}/p)^r(t+st+\cdots).$$

Theorem 4.1. *Let $M_1, M_2 \in \Lambda_2$. Then, $H_G^*(M_1;\mathbb{Z}/p)$ and $H_G^*(M_2;\mathbb{Z}/p)$ are isomorphic as graded rings if and only if $|M_1^G| = |M_2^G|$.*

Proof. If $|M_1^G| = |M_2^G| = r$, then let $a_1, \cdots, a_r$ be all fixed points in M_1^G and $b_1, \cdots, b_r$ be all fixed points in M_2^G. Thus,

$$\mathcal{V}_r^{(M_1)} = \{x=(x_1,\cdots,x_r)^\top \in (\mathbb{Z}/p)^r \mid |x| = \sum_{i=1}^r \tfrac{x_i}{N_{a_i}} = 0\}$$

and

$$\mathcal{V}_r^{(M_2)} = \{y=(y_1,\cdots,y_r)^T \in (\mathbb{Z}/p)^r \mid |y| = \sum_{i=1}^r \tfrac{y_i}{N_{b_i}} = 0\}.$$

By Theorem 3.4, we have that

$$H_G^*(M_1;\mathbb{Z}/p)\cong \mathcal{R}_{M_1} = V_0^{M_1}+\mathcal{V}_r^{(M_1)}s+(\mathbb{Z}/p)^r(t+st+\cdots)$$

and

$$H_G^*(M_2;\mathbb{Z}/p)\cong \mathcal{R}_{M_2} = V_0^{M_2}+\mathcal{V}_r^{(M_2)}s+(\mathbb{Z}/p)^r(t+st+\cdots).$$

Let $f_0 : V_0^{M_1} \longrightarrow V_0^{M_2}$ be the identity map,

$f_1 : \mathcal{V}_r^{(M_1)}s \longrightarrow \mathcal{V}_r^{(M_2)}s$ be $f_1((x_1,\cdots,x_r)^\top s) = (\frac{x_1}{N_{a_1}}N_{b_1},\cdots,\frac{x_r}{N_{a_r}}N_{b_r})^T s$,

$f_i : (\mathbb{Z}/p)^r t^{\frac{i}{2}} \longrightarrow (\mathbb{Z}/p)^r t^{\frac{i}{2}}$ be the identity map for $i \geq 2$ and i even,

$f_i : (\mathbb{Z}/p)^r st^{\frac{i-1}{2}} \longrightarrow (\mathbb{Z}/p)^r st^{\frac{i-1}{2}}$ be $f_i((x_1,\cdots,x_r)^\top st^{\frac{i-1}{2}}) = (\frac{x_1}{N_{a_1}}N_{b_1},$
$\cdots, \frac{x_r}{N_{a_r}}N_{b_r})^\top st^{\frac{i-1}{2}}$ for $i \geq 3$ and i odd.

Then, it is easy to check that $f = \sum_{i=0}^{\infty} f_i$ is an isomorphism between $\mathcal{R}_{M_1}$ and $\mathcal{R}_{M_2}$. Thus, $H_G^*(M_1;\mathbb{Z}/p)$ and $H_G^*(M_2;\mathbb{Z}/p)$ are isomorphic as graded rings.

If $|M_1^G| \neq |M_2^G|$, then obviously $H_G^*(M_1;\mathbb{Z}/p)$ and $H_G^*(M_2;\mathbb{Z}/p)$ are not isomorphic by their ring structures. □

An observation shows that S^2 admits a G-action such that $|(S^2)^G| = 2$. Thus, Λ_2 is non-empty.

As a consequence of Theorem 4.1, we have the following.

Corollary 4.2. *Let $r \geq 2$ be a positive integer. All G-manifolds in Λ_2^r determine a unique equivariant cohomology up to isomorphism if Λ_2^r is non-empty.*

Remark 4.3. *For each $M \in \Lambda_2^r (r \geq 2)$, its equivariant cohomology ring $H_G^*(M;\mathbb{Z}/p)$ can be expressed in a simpler way. We have that*

$$H_G^*(M;\mathbb{Z}/p) \cong$$

$$\{\alpha = (\alpha_1, \cdots, \alpha_r) \in (\mathbb{Z}/p)^r[s,t]/(s^2) | \begin{cases} \alpha_1 = \cdots = \alpha_r, \, if\, deg\alpha = 0 \\ \sum_{i=1}^r \frac{\alpha_i}{N_{a_i}} = 0, \quad if\, deg\alpha = 1 \end{cases} \},$$

where $a_1, \cdots, a_r$ are all fixed points in M^G and $\chi_G(a_i) = N_{a_i}t$.

5. Ring homomorphisms induced by G-equivariant maps

In this section, the task is to give a description for the homomorphism between equivariant cohomology rings of two G-manifolds in Λ_{2n} induced by a G-equivariant map and to show a characterization of the homomorphism.

Theorem 5.1. *Let $f : M_1 \to M_2$ be a G-equivariant map for $M_1 \in \Lambda_{2n}^{r_1}$, $M_2 \in \Lambda_{2n}^{r_2}(r_1, r_2 \geq 2)$ and f^* be the induced homomorphism between graded rings*

$$\begin{aligned} \mathcal{R}_{M_2} &= V_0^{M_2} + V_1^{M_2}s + \cdots + V_{2n-2}^{M_2}t^{n-1} + V_{2n-1}^{M_2}st^{n-1} + \\ &\quad (\mathbb{Z}/p)^{r_2}(t^n + st^n + \cdots) \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}_{M_1} &= V_0^{M_1} + V_1^{M_1}s + \cdots + V_{2n-2}^{M_1}t^{n-1} + V_{2n-1}^{M_1}st^{n-1} + \\ &\quad (\mathbb{Z}/p)^{r_1}(t^n + st^n + \cdots). \end{aligned}$$

Then, there is a linear map σ from $(\mathbb{Z}/p)^{r_2}$ to $(\mathbb{Z}/p)^{r_1}$ such that $f^* = \sum_{i\ even} \sigma t^{\frac{i}{2}} + \sum_{j\ odd} \sigma st^{\frac{j-1}{2}}$, where $f^*(\beta) = \sum_{i\ even} \sigma(v_i)t^{\frac{i}{2}} + \sum_{j\ odd} \sigma(v_j)st^{\frac{j-1}{2}}$, for $\beta = \sum_{i\ even} v_i t^{\frac{i}{2}} + \sum_{j\ odd} v_j st^{\frac{j-1}{2}} \in \mathcal{R}_{M_2}$. In particular, if $r_1 = r_2 = r$ and f^* is an isomorphism, then $\sigma \in \mathrm{GL}(r, \mathbb{Z}/p)$.

Proof. Since f^* is a ring homomorphism, $f^*((1\underbrace{,\cdots,}_{r_2}1)^T) = (1\underbrace{,\cdots,}_{r_1}1)^T$ and $f^*((k\underbrace{,\cdots,}_{r_2}k)^T) = (k\underbrace{,\cdots,}_{r_1}k)^T$, for $k \in \mathbb{Z}/p$. It is easy to check that f^* is linear. Since the restriction $f^*|_{(\mathbb{Z}/p)^{r_2}t^n} : (\mathbb{Z}/p)^{r_2}t^n \longrightarrow (\mathbb{Z}/p)^{r_1}t^n$ is linear, there exists a linear map σ from $(\mathbb{Z}/p)^{r_2}$ to $(\mathbb{Z}/p)^{r_1}$ such that $f^*|_{(\mathbb{Z}/p)^{r_2}t^n} = \sigma t^n$.

By Remark 3.5 we know that $\mathcal{R}_{M_i}, i = 1, 2$, are free $\mathbb{Z}/p[s,t]/(s^2)$-modules, and that f^* is a module homomorphism between $\mathcal{R}_{M_2}$ and $\mathcal{R}_{M_1}$. If $i > 2n$ and i is even, since $f^*|_{(\mathbb{Z}/p)^{r_2}t^n} = \sigma t^n$, we have that for $x \in (\mathbb{Z}/p)^{r_2}$,

$$f^*(xt^{\frac{i}{2}}) = f^*(xt^n)t^{\frac{i}{2}-n} = \sigma(x)t^{\frac{i}{2}}.$$

Thus, $f^*|_{(\mathbb{Z}/p)^{r_2}t^{\frac{i}{2}}} = \sigma t^{\frac{i}{2}}$, for $i > 2n$ and i even. Similarly, we have that $f^*|_{(\mathbb{Z}/p)^{r_2}st^{\frac{i-1}{2}}} = \sigma st^{\frac{i-1}{2}}$, for $i > 2n$ and i odd.

Finally we consider the case of the dimension being less than $2n$. For $0 \leq i < 2n$ and i even, let $v \in V_i^{M_2} \subset (\mathbb{Z}/p)^{r_2}$. Since $f^*|_{(\mathbb{Z}/p)^{r_2}t^n} = \sigma t^n$, we have that

$$f^*(vt^{\frac{i}{2}})t^{n-\frac{i}{2}} = f^*(vt^n) = \sigma(v)t^n,$$

and so $f^*(vt^{\frac{i}{2}}) = \sigma(v)t^{\frac{i}{2}}$. Thus, $f^*|_{V_i^{M_2}t^{\frac{i}{2}}} = \sigma t^{\frac{i}{2}}$, for $0 \leq i < 2n$ and i even.

In a similar way, using $f^*|_{(\mathbb{Z}/p)^{r_2}st^n} = \sigma st^n$, we have that $f^*|_{V_i^{M_2}st^{\frac{i-1}{2}}} = \sigma st^{\frac{i-1}{2}}$, for $0 \leq i < 2n$ and i odd.

If $r_1 = r_2 = r$ and f^* is an isomorphism, it is easy to see that $\sigma \in \mathrm{GL}(r, \mathbb{Z}/p)$. This completes the proof. □

Remark 5.2. *σ in Theorem 5.1 has the following property:*
$\sigma(x \circ y) = \sigma(x) \circ \sigma(y)$, for any $x, y \in (\mathbb{Z}/p)^{r_2}$.
In fact, since $f^*|_{(\mathbb{Z}/p)^{r_2}t^n} = \sigma t^n$ and $f^*|_{(\mathbb{Z}/p)^{r_2}t^{2n}} = \sigma t^{2n}$, for any $x, y \in (\mathbb{Z}/p)^{r_2}$, we have

$$\sigma(x \circ y)t^{2n} = f^*(x \circ yt^{2n}) = f^*(xt^n)f^*(yt^n) = \sigma(x) \circ \sigma(y)t^{2n},$$

and so $\sigma(x \circ y) = \sigma(x) \circ \sigma(y)$.

Combining Theorem 5.1 and Remark 5.2, we have the following.

Proposition 5.3. *With the notation of Theorem 5.1, let* $f : M_1 \to M_2$ *be a* G*-equivariant map for* $M_1 \in \Lambda_{2n}^{r_1}$ *and* $M_2 \in \Lambda_{2n}^{r_2}$*. Then, there is a linear map* σ *from* $(\mathbb{Z}/p)^{r_2}$ *to* $(\mathbb{Z}/p)^{r_1}$ *satisfying the property that* $\sigma(x \circ y) = \sigma(x) \circ \sigma(y)$*, for any* $x, y \in (\mathbb{Z}/p)^{r_2}$ *and that* σ *linearly maps* $V_i^{M_2}$ *to* $V_i^{M_1}$*, for* $i < 2n$*.*

Remark 5.4. *Let* $M_1, M_2 \in \Lambda_{2n}$*. We see that if* $|M_1^G| \neq |M_2^G|$*, then* $H_G^*(M_1; \mathbb{Z}/p)$ *and* $H_G^*(M_2; \mathbb{Z}/p)$ *must not be isomorphic as graded rings.*

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Yanchang Chen
College of Mathematics and Information Science, Hebei Normal University, Yuhua Road 113, Shijiazhuang 050016, People's Republic of China
and
College of Mathematics and Information Science, Henan Normal University, Jianshe Road 46, Xinxiang 453007, People's Republic of China
Email: cyc810707@163.com

Yanying Wang
College of Mathematics and Information Science, Hebei Normal University, Yuhua Road 113, Shijiazhuang 050016, People's Republic of China
Email: wyanying2003@yahoo.com.cn