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ON GENERALIZED TOPOLOGIES ARISING FROM MAPPINGS

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Abstract. Given a mapping $f : X \to X$, we naturally associate to it a monotonic map $\gamma_f : \exp X \to \exp X$ from the power set of X into itself, and thus inducing a generalized topology on X. Here, we investigate some properties of generalized topologies as defined by such a procedure.

1. Introduction

Various weakened forms of open sets and continuity have been considered in literature and vast research has been devoted to these concepts. In [1], Császár gave a common framework to all of these by introducing the notion of *generalized topologies*. Since then, the investigation of generalized topologies and generalized continuity has seen a rapid development over the past decade (see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11]) and our work here is continuation of these efforts.

Let us briefly describe the plan of the paper. Following [1], we use a method of generating generalized topologies on X via a specific $\gamma_f : \exp X \to \exp X$ (where $\exp X$ stands for the power set of X) that most naturally arises from a given mapping $f : X \to X$, where $\gamma_f A$ is defined to be the image of $A \subseteq X$ under f. We then examine the basic structure

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of the generalized topology (GT for short) thus obtained, and show that it is very unlikely for this GT to actually be a topology, indicating that it is reasonable to consider it only in the context of GTs, and finally investigate an aspect of products (as defined in [3]) of such GTs. By and large, we are concerned with how the properties of the mapping f reflect those of the induced GT.

2. **Preliminaries**

We begin with some notational explanations. $\mathbb{Z}$ is the set of all integers and $\mathbb{N}$ is the set of all positive integers. If $f : X \to Y$ is a mapping, then we shall use the notations $f^{\to}A$ and $f^{\leftarrow}B$ to denote, respectively, the image of $A \subseteq X$ and the inverse image of $B \subseteq Y$ under the mapping f. For an equivalence relation $\sim$ on a set X and $x \in X$, we denote by $x/_{\sim}$ the class of equivalence of x with respect to $\sim$, and $X/_{\sim}$ for the corresponding quotient set of the classes of equivalence. For a sequence $x = (x_n : \ n \in \mathbb{N})$, we put $\mathrm{Set}(x) = \{x_n : \ n \in \mathbb{N}\}$. For all unexplained topological notions, the reader is referred to [4].

Let us recall the basics of generalized topologies. A family σ of subsets of a given set X is said to be a *generalized topology* on X (see [2]) if $\emptyset \in \sigma$ and if $\bigcup \mathcal{A} \in \sigma$, whenever $\mathcal{A} \subseteq \sigma$. It is customary to call σ *strong* (see [8]) if $X \in \sigma$ (i.e., $\bigcup \sigma = X$).

Following [1], we call $\gamma : \exp X \to \exp X$ *monotonic* if $A \subseteq B \subseteq X$ implies $\gamma A \subseteq \gamma B$ (where γA stands for $\gamma(A)$) and denote by $\Gamma(X)$, the family of all such mappings γ. A set $A \subseteq X$ is said to be γ-open if $A \subseteq \gamma A$, and it is shown in [1] that the family g_γ of all γ-open subsets of X constitutes a GT on X. Actually, all GTs on a given set X can be obtained in this way (see Lemma 1.1 of [2]) and many papers use this approach to study GTs (see, e.g., [9, 10, 11]).

One of the most natural ways to produce elements of $\Gamma(X)$ is given by the following. Suppose $M \subseteq X$ and $f : X \to X$. Consider the mapping $\gamma_{f,M} : \exp X \to \exp X$, defined by $\gamma_{f,M}A = f^{\to}(A \setminus M) \setminus M$. Then, $\gamma_{f,M} \in \Gamma(X)$. Putting $\gamma_f = \gamma_{f,\emptyset}$ and denoting by λ_f the family of all γ_f-open sets (i.e., $\lambda_f = g_\gamma$), it shall be our goal to give some insights into the structure and basic properties of the generalized topology λ_f. Observe that for $A \subseteq X$, the set $\gamma_f A$ is exactly the image $f^{\to}A$.

3. **General considerations**

The proposition below gives a simple criterion for γ_f-openness of subsets, following directly from the definition of γ_f.

Proposition 3.1. *$A \subseteq X$ is γ_f-open if and only if for every $a \in A$ we have that $f^{\leftarrow}\{a\} \cap A \neq \emptyset$.*

Proof. Suppose that A is γ_f-open and let $a \in A$. Then, $a \in A \subseteq \gamma_f A = f^{\rightarrow}A$, and so there must be some $a_0 \in A$ with $f(a_0) = a$. Hence, $a_0 \in f^{\leftarrow}\{a\} \cap A$.

Now, suppose that $f^{\leftarrow}\{b\} \cap A \neq \emptyset$, for all $b \in A$ and let $a \in A$. There is an $x \in f^{\leftarrow}\{a\} \cap A$. Then, $x \in A$ and $a = f(x)$, and so $a \in f^{\rightarrow}A = \gamma_f A$. Thus, $A \subseteq \gamma_f A$, i.e., A is γ_f-open. □

Remark 3.2. By the preceding proposition λ_f is strong if and only if f is an onto mapping.

It may seem that for a generalized topology g_φ, $\varphi \in \Gamma(X)$, to be of the form λ_f, for some $f : X \to X$, the mapping φ would have to satisfy some very restrictive conditions. For example, it might seem obvious that the sets $\varphi\{x\}$ must be singletons, for all $x \in X$, (since this would yield a natural candidate for f defined by $\varphi\{x\} = \{f(x)\}$). The next example shows that this is actually not the case.

Example 3.3. *Let $1, 2 \in X$ and let $f : X \to X$ be given by $f(1) = 1$ and $f(i) = 2$, for $i \in X \setminus \{1\}$. Define $\varphi : \exp X \to \exp X$ by $\varphi\emptyset = \emptyset$, $\varphi\{1\} = \{1\}$, and $\varphi A = \{1, 2\}$ otherwise. We have $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\} = g_\varphi = \lambda_f$. However, the mapping φ obviously does not satisfy the condition of sending singletons to singletons.*

As introduced in [2], a mapping $\psi : X \to \exp\exp X$ is said to be a generalized neighborhood system if for each $x \in X$ and each $V \in \psi(x)$ we have $x \in V$. Following [2], we denote by $\mathbf{g}_\psi$ the family of all such $A \subseteq X$ with the property that for each $x \in A$ there exists $V \in \psi(x)$ such that $V \subseteq A$. Then, $\mathbf{g}_\psi$ is a GT on X (see Lemma 1.2 of [2]). All GTs can be obtained in this way (see Lemma 1.3 of [2]).

Given $f : X \to X$, let us call a sequence $x = (x_n : n \in \mathbb{N})$ of elements of X an f-sequence if for each $n \in \mathbb{N}$ we have $f(x_{n+1}) = x_n$. For such a sequence, we shall agree to say that it *starts at* x_1. Let us denote by $Seq_f(a)$ the set of all f-sequences starting at $a \in X$. Also, put $Str(f) = \{a \in X :$ there is an f-sequence starting at $a\}$ or equivalently, $Str(f) = \{a \in X : Seq_f(a) \neq \emptyset\}$. We define $\psi_f : X \to \exp\exp X$ by $\psi_f(x) = \{\mathrm{Set}(s) : s \in Seq_f(x)\}$, for $x \in X$. Then, ψ_f is a generalized neighborhood system and we have that the following proposition.

Proposition 3.4. *For any $f : X \to X$, the following statements hold:*

(1) *For each $x \in X$, we have $\psi_f(x) \subseteq \lambda_f$.*

(2) $\mathbf{g}_{\psi_f} = \lambda_f$.
(3) *The family* $\{\mathrm{Set}(x) : x \textit{ is an } f\textit{-sequence}\}$ *is a base for* λ_f.

Proof. (1) We show that $\mathrm{Set}(x) \in \lambda_f$, for any f-sequence x. Let $a \in \mathrm{Set}(x)$. Then, $a = x_n$, for some $n \in \mathbb{N}$. As x is an f-sequence, we have $f(x_{n+1}) = x_n = a$, and so $x_{n+1} \in f^{\leftarrow}\{a\} \cap \mathrm{Set}(x)$. As $a \in \mathrm{Set}(x)$ was arbitrary, $\mathrm{Set}(x)$ is in λ_f, by Proposition 3.1.

(2) Fix $A \in \lambda_f$ and $a \in A$. We shall construct recursively an f-sequence x of elements of A starting at a. This yields $a \in \mathrm{Set}(x) \subseteq A$, which is exactly what we need to show.

Put $x_1 = a$. If $x_i \in A$ have been constructed for $1 \leq i \leq n$ so that $f(x_{i+1}) = x_i$ holds, then proceed as follows: since $x_n \in A$ and A is γ_f-open, by Proposition 3.1, there is a $b \in f^{\leftarrow}\{x_n\} \cap A$. Define $x_{n+1} = b$. Then, obviously $x_{n+1} \in A$ and also $f(x_{n+1}) = x_n$.

The sequence $(x_n : n \in \mathbb{N})$ is as required.

(3) This is a direct consequence of (1) and (2). □

The following proposition gives a suitable description of the largest γ_f-open set.

Proposition 3.5. $\bigcup \lambda_f = Str(f)$.

Proof. We first show that $Str(f)$ is γ_f-open. Let $a \in Str(f)$. There is an f-sequence $x = (x_i : i \in \mathbb{N})$ with $x_1 = a$. Fix any $n \in \mathbb{N}$. $x' = (x_{n-1+i} : i \in \mathbb{N})$ is an f-sequence starting at x_n, and so $x_n \in Str(f)$. Thus, $a = x_1 \in \mathrm{Set}(x) \subseteq Str(f)$. By Proposition 3.4, the set $Str(f)$ is γ_f-open, and so clearly $Str(f) \subseteq \bigcup \lambda_f$.

Now, fix $a \in \bigcup \lambda_f$. $\bigcup \lambda_f$ is γ_f-open, and so there is an f-sequence x starting at a (such that $a \in \mathrm{Set}(x) \subseteq \bigcup \lambda_f$, but this is not needed here). Then, by the very definition of $Str(f)$, we have that $a \in Str(f)$. We have thus verified that $\bigcup \lambda_f \subseteq Str(f)$. □

The following few auxiliary notions to be introduced in the next paragraph will play crucial roles in depicting the structure of λ_f.

For $f : X \to X$ and $x \in X$, let us say that a is a *weak fixed point* of f if there is an $n \in \mathbb{N}$ such that $f^n(a) = a$. Let $Cycle(f)$ denote the set, possibly empty, of all weak fixed points of f. Clearly, $Cycle(f) \subseteq Str(f)$ and a mapping $k_f : Cycle(f) \to \mathbb{N}$ is defined by setting $k_f(x) = \min\{m \in \mathbb{N} : f^m(x) = x\}$, for $x \in Cycle(f)$. $k_f(x)$ shall be referred to as the *f-order* of the point $x \in Cycle(f)$.

The next two lemmas will be used in the sequel often without any explicit mention and are easy exercises, but we shall give proofs for the sake of completeness.

Lemma 3.6. *For any $x \in Cycle(f)$ the following statements hold:*

(1) *if m is a nonnegative integer, then $f^{mk_f(x)}(x) = x$;*
(2) *if m is a nonnegative integer and $f^m(x) = x$, then we have $m \equiv 0 \pmod{k_f(x)}$;*
(3) *if m and n are nonnegative integers, then $f^n(x) = f^m(x)$ if and only if $n \equiv m \pmod{k_f(x)}$.*

Proof. (1) Use induction on $m \geq 0$.

To prove (2), let $m = nk_f(x) + i$, where $0 \leq i < k_f(x)$. Then, $x = f^{nk_f(x)+i}(x) = f^i(x)$, where we have used (1). But, $i < k_f(x)$, and so $i = 0$.

We now prove (3). Suppose first that $n \equiv m \pmod{k_f(x)}$ and let $n \leq m$. Then $f^m(x) = f^n(f^{m-n}(x)) = f^n(x)$, by (1). Suppose now that $f^m(x) = f^n(x)$, $n \leq m$. Choose $l \in \mathbb{N}$ with $lk_f(x) \geq n$. Then, $f^{m+(lk_f(x)-n)}(x) = f^{n+(lk_f(x)-n)}(x) = x$, and so $m - n \equiv 0 \pmod{k_f(x)}$, by (2). $\square$

On the set $Cycle(f)$, define the relation $\sim_f$ by $a \sim_f b \overset{def}{\Longleftrightarrow} \exists n \in \mathbb{N}\ (f^n(a) = b)$.

Lemma 3.7. *With $f : X \to X$, we have that the following statements hold:*

(1) *if $a \in Cycle(f)$, $b \in X$ and n is a nonnegative integer such that $f^n(a) = b$, then $b \in Cycle(f)$ and $b \sim_f a$;*
(2) *$\sim_f$ is an equivalence relation;*
(3) *if $a \in Cycle(f)$, then $a/_{\sim_f} = \{f^i(a) :\ 0 \leq i < k_f(a)\}$;*
(4) *if a is an f-sequence with $Set(a) \subseteq Cycle(f)$, then $a_i = a_j$ if and only if $i \equiv j\ (mod\ k_f(a_1))$.*

Proof. (1) Write $n = mk_f(a) + i$, where $0 \leq i < k_f(a)$. Then, $b = f^n(a) = f^i(a)$, and so $f^{k_f(a)-i}(b) = f^{k_f(a)-i}(f^i(a)) = f^{k_f(a)}(a) = a$. Thus, we would have $b \sim_f a$ if we could show that $b \in Cycle(f)$. But, for $r = k_f(a) - i \in \mathbb{N}$, it follows from $f^r(b) = a$ that $f^{r+n}(b) = f^n(a) = b$, and hence $b \in Cycle(f)$.

(2) Reflexivity and transitivity of the relation $\sim_f$ are immediate from the definition, and by part (1) it follows that it is symmetric.

(3) $\{f^i(a):\ 0 \le i < k_f(a)\} \subseteq a/_{\sim_f}$ follows directly from the definition of $\sim_f$. On the other hand, if $a \sim_f b$, then we can find an integer i with $0 \le i < k_f(a)$ and $f^i(a) = b$ just as we did in the proof of (1).

(4) Let $a = (a_m : m \in \mathbb{N})$ be an f-sequence with $\{a_m : m \in \mathbb{N}\} \subseteq Cycle(f)$. Fix $i, j \in \mathbb{N}$ and denote $l = k_f(a_1)$. Take any integer $n > \max\{i, j\} + l$. We have $f^{n-1}(a_n) = a_1$ and $a_n, a_1 \in Cycle(f)$, and so $a_n \sim_f a_1$. Thus, $a_1 \sim_f a_n$, and therefore, as in the proof of (1), there is an integer r such that $0 \le r < l$ and $f^r(a_n) = a_1$. Now, $a_1 = f^r(a_n) = a_{n-r}$ and $n - r > n - l > \max\{i, j\}$, and so both $n - r - i$ and $n - r - j$ are positive integers, and also $f^{n-r-i}(a_1) = a_i$ and $f^{n-r-j}(a_1) = a_j$.

We now have $a_i = a_j \iff f^{n-r-i}(a_1) = f^{n-r-j}(a_1) \iff i \equiv j \pmod{k_f(a_1)}$, by Lemma 3.6. □

Theorem 3.8. *If $f : X \to X$ and X is a finite set then*

(1) *the set $\{a/_{\sim_f} :\ a \in Cycle(f)\}$ is a base for λ_f;*

(2) *$card(\lambda_f) = 2^{card(Cycle(f)/\sim_f)}$.*

Proof. (1) Let $\mathcal{B} = \{a/_{\sim_f} :\ a \in Cycle(f)\}$ and $\mathcal{L} = \{\mathrm{Set}(x) : x$ is an f-sequence$\}$. By Proposition, 3.4 it suffices to prove $\mathcal{B} = \mathcal{L}$.

Take an $a \in Cycle(f)$ and put $l = k_f(a)$. Define, for $n \in \mathbb{N}$, $x_n = f^{l+1-r}(a)$, where $0 \le i < l$ and $n \equiv r \pmod{l}$. Then, $x \in Seq_f(a)$ and $\mathrm{Set}(x) = \{f^i(a):\ 0 \le i < k_f(a)\} = a/_{\sim_f}$. Thus, $\mathcal{B} \subseteq \mathcal{L}$.

Let $x = (x_i : i \in \mathbb{N})$ be an f-sequence. As X is finite, there is some $n \in \mathbb{N}$ such that $\{x_i : i \in \mathbb{N}\} = \{x_i : 1 \le i \le n\}$.

Fix $i > n$. There is an integer j with $1 \le j \le n$ and $x_i = x_j = f^{i-j}(x_i)$, and so $x_i \in Cycle(f)$.

Fix any integer i such that $1 \le i \le n$. Now, $x_{n+1} \in Cycle(f)$ and $f^{n+1-i}(x_{n+1}) = x_i$, and so by (1) of Lemma 3.7, we get that $x_i \in Cycle(f)$. Hence $\mathrm{Set}(x) \subseteq Cycle(f)$, and by (4) of Lemma 3.7, $\mathrm{Set}(x) = \{x_i : 1 \le i \le k_f(x_1)\} = \{f^i(x_1) : 0 \le i < k_f(x_1)\} = x_1/_{\sim_f}$. This shows that $\mathcal{L} \subseteq \mathcal{B}$.

(2) Fix a transversal $T \subseteq Cycle(f)$ of the relation $\sim_f$ and put $\mathcal{B} = \{a/_{\sim_f} : a \in T\}$. Then, $\mathcal{B} = \{a/_{\sim_f} :\ a \in Cycle(f)\}$ and $card(\mathcal{B}) = card(T) = card(Cycle(f)/\sim_f)$. Now, $\mathcal{B}$ is a base for λ_f and elements of $\mathcal{B}$ are nonempty pairwise disjoint. Thus, for any $A \in \lambda_f$, there is a unique $\mathcal{A} \subseteq \mathcal{B}$ with $A = \bigcup \mathcal{A}$. The assertion of (2) now directly follows. □

Theorem 3.9. *If $f : X \to X$ is a bijection, then λ_f is a topology and there are cardinals κ and μ and $n_\alpha \in \mathbb{N}$, for $\alpha < \kappa$, such that the space (X, λ_f) is homeomorphic to the topological sum*

$$[\oplus_{0\leq\alpha<\kappa}(Z_{n_\alpha}, \{\emptyset, Z_{n_\alpha}\})] \oplus [\oplus_{0\leq\alpha<\mu}(\mathbb{Z}, \mathcal{R})],$$

where we have denoted $Z_m = \{k \in \mathbb{N} : 1 \leq k \leq m\}$, for $m \in \mathbb{N}$, and $\mathcal{R} = \{\{n \in \mathbb{Z} : n \geq m\} : m \in \mathbb{Z}\}$ is the right order topology on $\mathbb{Z}$ as called, e.g., in [12].

Proof. Fix $x \in X$. Since f is onto, λ_f is strong, i.e., $X = \bigcup \lambda_f \in \lambda_f$, and by Proposition 3.5, $X = Str(f)$. Thus, $Seq_f(x) \neq \emptyset$. Now, let $a, b \in Seq_f(x)$. Since f is injective, an easy induction on $n \in \mathbb{N}$ establishes $a_n = b_n$, i.e., $a = b$. Thus, there must be exactly one f-sequence starting at x. Denote it by $s(x) = (s_i(x) : i \in \mathbb{N})$.

To prove that λ_f is closed under taking intersections of finite subfamilies, it suffices to show, by virtue of Proposition 3.4, that if $a = (a_n : n \in \mathbb{N})$ and $b = (b_n : n \in \mathbb{N})$ are f-sequences and $x \in \text{Set}(a) \cap \text{Set}(b)$, then there is an f-sequence $c = (c_n : n \in \mathbb{N})$ with $x \in \text{Set}(c) \subseteq \text{Set}(a) \cap \text{Set}(b)$. Given such x, a and b, we can find positive integers n_1 and n_2 such that $x = a_{n_1} = b_{n_2}$. Then, $a' = (a_{n_1-1+m} : m \in \mathbb{N})$ and $b' = (b_{n_2-1+m} : m \in \mathbb{N})$ are two f-sequences both starting at x, and therefore they must coincide. But then, $x \in \text{Set}(a') = \text{Set}(b') \subseteq \text{Set}(a) \cap \text{Set}(b)$, as required.

We now turn to describing the structure of the topology λ_f.

From (1) of Lemma 3.7, it follows that $f^{\to}Cycle(f) = Cycle(f)$. But, f is a bijection, and so $f^{\to}[X \setminus Cycle(f)] = X \setminus Cycle(f)$.

Next, let $\kappa = card(Cycle(f)/\sim_f)$, fix a transversal $\{a_\alpha : \alpha \in \kappa\}$ of the relation $\sim_f$, put $n_\alpha = k_f(a_\alpha)$ and $X_\alpha = a_\alpha/_{\sim_f}$.

Let $\alpha \in \kappa$, $b \in X_\alpha$ and $t \in Seq_f(b)$. For $m \in \mathbb{N}$, define $z_m = f^{k_f(b)+1-r}(b)$, where $0 \leq r < k_f(b)$ and $m \equiv r \pmod{k_f(b)}$. Then, $z \in Seq_f(b)$. Therefore, as previously noted, it must be that $z = t$. Thus, $\text{Set}(t) = \text{Set}(z) = \{f^i(b) : 0 \leq i < k_f(b)\} = b/_{\sim_f} = a/_{\sim_f} = X_\alpha$, since $a \sim_f b$. This means that $X_\alpha \in \lambda_f$ and that the relative topology σ_f on X_α inherited from λ_f is the trivial topology $\sigma_\alpha = \{\emptyset, X_\alpha\}$. Thus, the space $(X_\alpha, \sigma_\alpha)$ is homeomorphic to the space $(Z_{n_\alpha}, \{\emptyset, Z_{n_\alpha}\})$.

Put $\mu = card(X \setminus Cycle(f))$ and let $x \in X \setminus Cycle(f)$. Define $l(x) = (l_i(x) : i \in \mathbb{Z})$ as follows: $l_i(x) = s_i(x)$, for $i \in \mathbb{N}$, and $l_i(x) = f^{1-i}(x)$, for integers $i \leq 0$. Also, put $L(x) = \{l_i(x) : i \in \mathbb{Z}\}$, for $x \in \setminus Cycle(f)$. It is easy to show that for $x, y \in X \setminus Cycle(f)$, the following two assertions hold:

- if $z \in L(x) \cap L(y)$, then $L(x) = L(y) = L(z)$ and
- if $i, j \in \mathbb{Z}$ and $i \neq j$, then $l_i(x) \neq l_j(x)$.

Thus, $\{L(x) : x \in X \setminus Cycle(f)\}$ is a partition of the set $X \setminus Cycle(f)$. Fix any transversal $\{b_\beta : \beta < \mu\}$ of this partition. Set $y_{\alpha,i} = l_i(b_\alpha)$, for $i \in \mathbb{Z}$ and $\alpha < \mu$. Denote $Y_\alpha = L(b_\alpha)$. It can easily be verified that $Y_\alpha \in \lambda_f$, for $\alpha < \mu$, and that the relative topology τ_α on Y_α inherited from λ_f is $\tau_\alpha = \{\{y_{\alpha,i} : i \in \mathbb{Z},\ k \leq i\} : k \in \mathbb{Z}\} \cup \{\emptyset, Y_\alpha\}$. Since $y_{\alpha,i} \neq y_{\alpha,j}$ for $i \neq j$, we can conclude that the topological space (Y_α, τ_α) is homeomorphic to the space $(\mathbb{Z}, \mathcal{R})$.

The discussion above together with $X = [\bigcup_{\alpha \in \kappa} X_\alpha] \cup [\bigcup_{\beta \in \mu} Y_\beta]$, where $X_{\alpha_1} \cap X_{\alpha_2} = \emptyset$, $X_{\alpha_2} \cap Y_{\beta_2} = \emptyset$ and $Y_{\beta_1} \cap Y_{\beta_2} = \emptyset$, if $\alpha_1 \in \alpha_2 \in \kappa$ and $\beta_1 \in \beta_2 \in \mu$, imply that the topological space (X, λ_f) is homeomorphic to the topological sum

$$[\oplus_{0 \leq \alpha < \kappa} (Z_{n_\alpha}, \{\emptyset, Z_{n_\alpha}\})] \oplus [\oplus_{0 \leq \alpha < \mu} (\mathbb{Z}, \mathcal{R})].$$

□

A partial converse of Theorem 3.9 is given below.

Theorem 3.10. *Suppose $f : X \to X$ has no fixed points and λ_f is a topology on X. Then, f must be a bijection.*

Proof. Since λ_f is a topology, it is strong generalized topology, and so, by Remark 3.2, f is an onto, mapping. To prove injectivity, suppose there are $b \neq c$ with $f(b) = f(c) = a$. As f is onto, we have $\bigcup \lambda_f = X$, and so, by Proposition 3.5, for each $x \in X$ we have $Seq_f(x) \neq \emptyset$. Thus, we can find some $y = (y_i : i \in \mathbb{N}) \in Seq_f(b)$ and $z = (z_i : i \in \mathbb{N}) \in Seq_f(c)$. Define $y' = (y'_i : i \in \mathbb{N})$ and $z' = (z'_i : i \in \mathbb{N})$ so that $y'_1 = z'_1 = a$ and $y'_i = y_{i-1}$, $z'_i = z_{i-1}$, for $i > 1$. As λ_f is a topology, the set $\mathrm{Set}(y') \cap \mathrm{Set}(z')$ is in λ_f and it contains the point a, and hence by Proposition 3.4, there is some $s \in Seq_f(a)$ such that $a \in \mathrm{Set}(s) \subseteq \mathrm{Set}(y') \cap \mathrm{Set}(z')$. From $a \neq f(a)$, it follows that s is not a constant sequence, and so there is some $x_0 \in \mathrm{Set}(s) \setminus \{a\}$. But then, $x_0 \in \mathrm{Set}(y) \cap \mathrm{Set}(z)$. Thus, there are $n_1, n_2 \in \mathbb{N}$ such that $x_0 = y_{n_1} = z_{n_2}$.

If $n_1 = n_2$, then $b = y_1 = f^{n_1 - 1}(y_{n_1}) = f^{n_1 - 1}(z_{n_2}) = f^{n_2 - 1}(z_{n_2}) = z_1 = c$, which contradicts our assumption. Therefore, $n_1 \neq n_2$ and we may suppose, without loss of generality, that $n_1 > n_2$. Now, $f^{n_2 - 1}(x_0) = f^{n_2 - 1}(z_{n_2}) = z_1 = c$ and $f^{n_2 - 1}(x_0) = f^{n_2 - 1}(y_{n_1}) = y_{n_1 - n_2 + 1}$ so that $y_{n_1 - n_2 + 1} = c$. Also, $a = f(c) = f(y_{n_1 - n_2 + 1}) = y_{n_1 - n_2}$. Thus, there is a least positive integer m_0 with $y_{m_0} = a$. Clearly, $2 \leq m_0 \leq n_1 - n_2$.

Set $y_0 = a$ and define $t_{nm_0+i} = y_{i-1}$, for any nonnegative integer n and $0 < i < m_0$ and $t_{nm_0} = y_{m_0-1}$, for $n \in \mathbb{N}$. The sequence $t = (t_i : i \in \mathbb{N})$ is an f-sequence starting at a, the set $\mathrm{Set}(t) \cap \mathrm{Set}(z')$ is in λ_f, and it contains the point a and $f(a) \neq a$, and so, with the same reasoning as before, we can find an $x_1 \in \mathrm{Set}(t) \cap \mathrm{Set}(z')$ with $x_1 \neq a$. Then, $x_1 = z_{n_3}$, for some $n_3 \geq 1$. $x_1 \in \mathrm{Set}(t) \setminus \{a\} = \{y_1, \ldots, y_{m_0-1}\}$, and so $x_1 = y_{n_4}$, for some $1 \leq n_4 \leq m_0 - 1$. Now, $c = f^{n_3-1}(z_{n_3}) = f^{n_3-1}(x_1) = f^{n_3-1}(y_{n_4}) \in \mathrm{Set}(t)$. Obviously, $c \neq a$, for otherwise, $f(a) = f(c) = a$. Thus, $c = y_{n_5}$, for some $1 \leq n_5 \leq m_0 - 1$. We cannot have that $n_5 = 1$, because then, $c = y_{n_5} = y_1 = b$. Thus, $n_5 - 1 \in \mathbb{N}$. Hence, $y_{n_5-1} = f(y_{n_5}) = f(c) = a$, but this is not possible by the choice of m_0, since $n_5 - 1 < m_0$, and $n_5 - 1 \in \mathbb{N}$. □

The last few results show that essentially λ_f is worth investigating exactly in the case when λ_f is not a topology.

4. On products

In [3], the notion of (Tychonoff) product of a family of topologies was generalized. Given sets X_s, $s \in S$, and generalized topologies σ_s on X_s, for $s \in S$, we define the product of the GTs, σ_s, as proposed by (see Császár in [3], and also Shen [8]), and denote it by $\mathbf{P}_{s\in S}\, \sigma_s$. In the case of only two GTs, σ_1 and σ_2, we shall write $\sigma_1 \otimes \sigma_2$ for their product.

Recall that the product of mappings $f_s : X_s \to Y_s$, $s \in S$, is the mapping $\otimes_{s\in S} f_s : \prod_{s\in S} X_s \to \prod_{s\in S} Y_s$, defined by $(\otimes_{s\in S} f_s)((x_s : s \in S)) = (f_s(x_s) : s \in S) \in \prod_{s\in S} Y_s$. In the case of only two mappings f_1 and f_2, we shall write $f_1 \otimes f_2$ for their product.

If $f_s : X_s \to X_s$, $s \in S$, then we can consider two GTs on the same set $\prod_{s\in S} X_s$. The first is the one induced by the product mapping $\otimes_{s\in S} f_s$, and the second is the product of the generalized topologies λ_{f_s}, $s \in S$. In the sequel, we shall look more closely into the relationship between these two.

For $s_0 \in S$, let us write $p_{s_0} : \prod_{s\in S} X_s \to X_{s_0}$ for the projection $p_{s_0}((x_s : s \in S)) = x_{s_0}$.

Theorem 4.1. *Let $f_s : X_s \to X_s$, for $s \in S$, and denote $f = \otimes_{s\in S} f_s$. Then,*

$$\mathbf{P}_{s\in S} \lambda_{f_s} \subseteq \lambda_f.$$

Proof. Let $A \in \mathbf{P}_{s\in S}\lambda_{f_s}$ be an element of the standard base for $\mathbf{P}_{s\in S}$. Then, there is a finite $T \subseteq S$ and $B_s \in \lambda_{f_s}$, for each $s \in T$, such that

$$A = [\bigcap_{s\in S\setminus T} p_s^{\leftarrow}(\bigcup \lambda_{f_s})] \cap [\bigcap_{s\in T} p_s^{\leftarrow} B_s].$$

We shall verify that $A \in \lambda_f$ by checking the condition in Proposition 3.1. Let $a \in A$. Fix $s \in S \setminus T$. Then, $p_s(a) \in \bigcup \lambda_{f_s} \in \lambda_{f_s}$, and so, by Proposition 3.1, there is some $x_s \in \bigcup \lambda_{f_s}$ with $f_s(x_s) = p_s(a)$. Now, fix $s \in T$. Then, $p_s(a) \in B_s \in \lambda_{f_s}$, and so there must be some $x_s \in B_s \cap f_s^{\leftarrow}\{p_s(a)\}$, i.e., $x_s \in B_s$ and $f_s(x_s) = p_s(a)$. Now, consider $x = (x_s : s \in S) \in \prod_{s\in S} X_s$. By the choice of the points x_s, we have that $x \in A$. Furthermore, $f(x) = a$. Indeed, denote $y = f(x)$. If $s_0 \in S$, then $p_{s_0}(y) = p_{s_0}((\otimes_{s\in S} f_s)(x)) = f_{s_0}(x) = p_{s_0}(a)$. Thus, $x \in A \cap f^{\leftarrow}\{a\}$. $\square$

The converse of Theorem 4.1 does not hold, in general. Instead of giving a specific counterexample, we shall describe how to produce (Theorem 4.3 below) all the counterexamples in the case of products of two mappings. But, first we prove an auxiliary lemma. As usual, we let $f \restriction A$ let restriction of function f to the subdomain A. Also, we let id_A to be the identity function of the set A.

Lemma 4.2. *$f \restriction Cycle(f)$ is injective.*

Proof. Suppose $x, y \in Cycle(f)$ and $f(x) = f(y) = z$. Then, $z \in Cycle(f)$, by (1) of Lemma 3.7, and thus we have $x \sim_f z$ and $y \sim_f z$. It follows now that $z \sim_f x$ and $z \sim_f y$, i.e., there are nonnegative integers n_1 and n_2 such that $f^{n_1}(z) = x$ and $f^{n_2}(z) = y$. Thus, $f^{n_1+1}(z) = f(x) = z = f(y) = f^{n_2+1}(z)$, and hence $n_1 \equiv n_2 \pmod{k_f(z)}$, and consequently $y = f^{n_2}(z) = f^{n_1}(z) = x$. $\square$

Theorem 4.3. *$\lambda_{f_1\otimes f_2} = \lambda_{f_1} \otimes \lambda_{f_2}$ if and only if one of the following three conditions holds:*

1. *$f_1 \restriction Str(f_1) = id_{Str(f_1)}$;*
2. *$f_2 \restriction Str(f_2) = id_{Str(f_2)}$;*
3. *$Cycle(f_1) = Str(f_1)$ and $Cycle(f_2) = Str(f_2)$ and for all $(u, v) \in Cycle(f_1) \times Cycle(f_2)$, we have that $gcd(k_{f_1}(u), k_{f_2}(v)) = 1$, where $gcd(n, m)$ stands for the greatest common divisor of n and m.*

Proof. Note first that $\lambda_{f_1\otimes f_2} = \lambda_{f_1} \otimes \lambda_{f_2}$ is actually equivalent, by Theorem 4.1, to $\lambda_{f_1\otimes f_2} \subseteq \lambda_{f_1} \otimes \lambda_{f_2}$.

For necessity, suppose $\lambda_{f_1\otimes f_2} \subseteq \lambda_{f_1} \otimes \lambda_{f_2}$ holds.

Case 1. There is some $u \in \mathit{Str}(f_1) \setminus \mathit{Cycle}(f_1)$.

Let $a \in \mathit{Seq}_{f_1}(u)$ be arbitrary. Take any $v \in \mathit{Str}(f_2)$ and any $b \in \mathit{Seq}_{f_2}(v)$. Define the sequence s by $s_i = \langle a_i, b_i \rangle$. Then, obviously, $s \in \mathit{Seq}_{f_1 \otimes f_2}((u,v)), \mathit{and}$ so $(u,v) \in \mathrm{Set}(s) \in \lambda_{f_1 \otimes f_2} \subseteq \lambda_{f_1} \otimes \lambda_{f_2}$. Therefore, there are $c \in \mathit{Seq}_{f_1}(u)$ and $d \in \mathit{Seq}_{f_2}(v)$ such that $(u,v) \in \mathrm{Set}(c) \times \mathrm{Set}(d) \subseteq \mathrm{Set}(s)$, i.e.,

$$(u,v) \in \{(c_i, d_j) : (i,j) \in \mathbb{N}^2\} \subseteq \{(a_i, b_i) : i \in \mathbb{N}\}.$$

In particular, $(a_1, d_2) = (c_1, d_2) = (a_i, b_i)$, for some $i \in \mathbb{N}$, and hence $a_1 = a_i$. But, u is not a weak fixed point of f_1, and so it must be $i = 1$. It now follows that $b_1 = d_2$, and consequently $f_2(v) = f_2(b_1) = f_2(d_2) = d_1 = v$. As $v \in \mathit{Str}(f_2)$ was arbitrary, we conclude that (2) holds.

Case 2. There is some $v \in \mathit{Str}(f_2) \setminus \mathit{Cycle}(f_2)$. Reasoning, in total analogy to the previous case, we get that (1) must be satisfied.

Case 3. Suppose now $\mathit{Cycle}(f_1) = \mathit{Str}(f_1)$ and $\mathit{Cycle}(f_2) = \mathit{Str}(f_2)$. Let $(u,v) \in \mathit{Cycle}(f_1) \times \mathit{Cycle}(f_2)$. Put $l_1 = k_{f_1}(u)$ and $l_2 = k_{f_2}(v)$ and suppose that $1 < \mathit{gcd}(l_1, l_2) = l'$.

Define the sequence $a \in \mathit{Seq}_{f_1}(u)$ by periodically repeating the finite sequence $(f_1^{l_1}(u), f_1^{l_1-1}(u), \ldots, f_1(u))$, or more formally let $a_n = f_1^{l_1+1-i}(u)$, where $0 \le i < l_1$ and $n \equiv i \pmod{l_1}$. Similarly, let $b \in \mathit{Seq}_{f_2}(v)$ be defined by $b_n = f_2^{l_1+1-i}(v)$, where $0 \le i < l_2$ and $n \equiv i \pmod{l_2}$. Define $s \in \mathit{Seq}_{f_1 \otimes f_2}((u,v))$ by $s_i = \langle a_i, b_i \rangle$. As before, we obtain sequences $c \in \mathit{Seq}_{f_1}(u)$ and $d \in \mathit{Seq}_{f_2}(v)$ such that $(u,v) \in \{(c_i, d_j) : (i,j) \in \mathbb{N}^2\} \subseteq \{(a_i, b_i) : i \in \mathbb{N}\}$. Using Lemma 4.2 and the fact that $\mathit{Cycle}(f_1) = \mathit{Str}(f_1)$ and $\mathit{Cycle}(f_2) = \mathit{Str}(f_2)$, we can conclude that $a = c$ and $b = d$. Hence, $\{(a_i, b_j) : (i,j) \in \mathbb{N}^2\} \subseteq \{(a_i, b_i) : i \in \mathbb{N}\}$.

Let $l_1 = n_1 l'$ and $l_2 = n_2 l'$. We must have that $(a_2, b_1) = (a_i, b_i)$ for some $i \in \mathbb{N}$. But then, $i \equiv 2 \pmod{l_1}$ and $i \equiv 1 \pmod{l_2}$, and so, for some nonnegative integers m_1 and m_2, $i = 2 + m_1 l_1 = 1 + m_1 l_2$, i.e., $1 = l'(m_2 n_2 - m_1 n_1)$, which is a contradiction, since $l' > 1$.

For sufficiency, suppose that (1) holds. Let $(u,v) \in \mathit{Str}(f_1 \otimes f_2)$ and $s \in \mathit{Seq}_{f_1 \otimes f_2}((u,v))$ be arbitrary. Then, $s = ((a_i, b_i) : i \in \mathbb{N})$, for some $a \in \mathit{Seq}_{f_1}(u)$ and $b \in \mathit{Seq}_{f_2}(v)$. As obviously $\{a_i : i \in \mathbb{N}\} \subseteq \mathit{Str}(f_1)$, we have that $a_i = u$, for all $i \in \mathbb{N}$. But then, $\mathrm{Set}(s) = \{u\} \times \mathrm{Set}(b) = \mathrm{Set}(a) \times \mathrm{Set}(b) \in \lambda_{f_1} \otimes \lambda_{f_2}$.

If (2) holds, then $\lambda_{f_1 \otimes f_2} = \lambda_{f_1} \otimes \lambda_{f_2}$ is proved exactly as in the case (1).

Suppose now that (3) holds. Let $(u,v) \in \mathit{Str}(f_1 \otimes f_2)$ and $s \in \mathit{Seq}_{f_1 \otimes f_2}$ $((u,v))$ be arbitrary. Then, $s = ((a_i, b_i) : i \in \mathbb{N})$, for some $a \in \mathit{Seq}_{f_1}(u)$

and $b \in Seq_{f_2}(v)$. As $Cycle(f_j) = Str(f_j)$, for $j \in \{1,2\}$, we have that $\{a_i : i \in \mathbb{N}\} \subseteq Cycle(f_1)$ and $\{b_i : i \in \mathbb{N}\} \subseteq Cycle(f_2)$. Thus, by Lemma 3.7, $a_i = a_j$ if and only if $i \equiv j \pmod{l_1}$ and $b_i = b_j$ if and only if $i \equiv j \pmod{l_2}$, where $l_1 = k_{f_1}(u)$ and $l_2 = k_{f_2}(v)$. Let us show that $\mathrm{Set}(s) = \mathrm{Set}(a) \times \mathrm{Set}(b)$ ($\in \lambda_{f_1} \otimes \lambda_{f_2}$). Take any $(a_i, b_j) \in \mathrm{Set}(a) \times \mathrm{Set}(b)$. By (3), we have that $gcd(l_1, l_2) = 1$, and so by the Chinese Remainder Theorem, there is some $n \in \mathbb{N}$ such that $n \equiv i \pmod{l_1}$ and $n \equiv j \pmod{l_2}$. But then, $(a_n, b_n) = (a_i, b_j) \in \mathrm{Set}(s)$. □

Corollary 4.4. *Let $f_1 : X_1 \to X_1$ and $f_2 : X_2 \to X_2$ be both onto. Then, $\lambda_{f_1 \otimes f_2} = \lambda_{f_1} \otimes \lambda_{f_2}$ if and only if*
– one of the mappings is the identity mapping, or
– all points of X_i are weak fixed points of f_i, $i \in \{1,2\}$, and any f_1-order of a point of X_1 is coprime with any f_2-order of a point of X_2. □

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