Bulletin of the Iranian Mathematical Society Vol. 38 *No. 3* (*2012*), *pp 581-596.*

THE STARLIKENESS, CONVEXITY, COVERING THEOREM AND EXTREME POINTS OF p-HARMONIC MAPPINGS

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Communicated by Behzad Djafari-Rouhani

Abstract. The main aim of this paper is to introduce three classes $H^0_{p,q}$, $H^1_{p,q}$ and TH^*_p of p-harmonic mappings and discuss the properties of mappings in these classes. First, we discuss the starlikeness and convexity of mappings in $H^0_{p,q}$ and $H^1_{p,q}$. Then establish the covering theorem for mappings in $H^1_{p,q}$. Finally, we determine the extreme points of the class TH^*_p.

1. **Introduction**

A $2p$ times continuously differentiable complex-valued mapping $F = u + iv$ in a domain $D \subseteq \mathbb{C}$ is *p-harmonic* if F satisfies the p-harmonic equation $\underbrace{\Delta \cdots \Delta}_{p} F = 0$, where p (≥ 1) is an integer and Δ represents the complex Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

It is known that a mapping F is p-harmonic in a simply connected domain D if and only if F has the following representation:

MSC(2010): Primary: 65F05; Secondary: 46L05, 11Y50.
Keywords: p-harmonic mapping, univalence, starlikeness, convexity, extreme point.
Received: 5 January 2011, Accepted: 13 March 2011.
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$$F(z)=\sum_{k=1}^{p}|z|^{2(k-1)}G_{p-k+1}(z)$$

where $\Delta G_{p-k+1}(z)=0$, i.e., $G_{p-k+1}(z)$ is harmonic in D for each $k\in\{1,\cdots,p\}$ (see [4]).

Obviously, when $p=1$ (respectively 2), F is harmonic (respectively biharmonic). The properties of harmonic mappings have been investigated by many authors, see [1, 4, 6, 7, 9, 16, 17]. Biharmonic mappings arise in a lot of physical situations, particularly in fluid dynamics and elasticity problems, and have many important applications in engineering and biology. See [12, 14] for the details. Nowadays, the study of biharmonic mappings attracts much attention, see [1, 2, 3, 5, 8, 15].

Let $\mathbb{D}_r=\{z:|z|<r\}$ $(r>0)$. In particular, we use $\mathbb{D}$ to denote the unit disk $\mathbb{D}_1$. Throughout this paper, we consider p-harmonic mappings in $\mathbb{D}$.

In [6], Clunie and Sheil-Small introduced the class S_H^0 of univalent harmonic mappings in $\mathbb{D}$, consisting of all harmonic mappings F with the series expansion:

$$F(z)=z+\sum_{n=2}^{\infty}(a_nz^n+\overline{b}_n\overline{z}^n).$$

The main aim of Ganczar [9] was to discuss the starlikeness and convexity of mappings F in S_H^0 under the coefficient condition:

$$\sum_{n=2}^{\infty}n^q\big(|a_n|+|b_n|\big)\leq 1 \tag{1.1}$$

for $q>0$. For convenience, we denote by $H_{1,q}^0$ the subclass of S_H^0 with the coefficient condition (1.1).

Let H_1^* denote the set of all mappings in S_H^0 mapping $\mathbb{D}$ onto starlike domains, and let TH_1^* denote the subclass of H_1^* whose elements satisfy that $F=h+\overline{g}$, where

$$h(z)=z-\sum_{n=2}^{\infty}a_nz^n \text{ with } a_n\geq 0 \text{ and } g(z)=-\sum_{n=2}^{\infty}b_nz^n \text{ with } b_n\geq 0.$$

In [18], Silverman obtained many properties of mappings in [18]. For example, he proved that $F\in TH_1^*$ if and only if $\sum_{n=2}^{\infty}n\big(|a_n|+|b_n|\big)\leq 1$ (cf. [18, Theorems 2]). Also the extreme points of TH_1^* were determined (cf. [18, Theorem 4(a)]). See [10, 11, 13] for other discussions in this line.

We use $H^1_{p,1}$ to denote the set of all p-harmonic mappings F in $\mathbb{D}$ with the following series expansion:

$$F(z)=\sum_{k=1}^{p}|z|^{2(k-1)}\big(\sum_{j=1}^{\infty}a_{j,p-k+1}z^j+\sum_{j=1}^{\infty}\overline{b}_{j,p-k+1}\bar{z}^j\big),$$

where $a_{1,p}=1$ and $b_{1,p}=0$, and satisfying the following coefficient condition:

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}(2(k-1)+j)\big(|a_{j,p-k+1}|+|b_{j,p-k+1}|\big)\le 2.$$

In [16], Qiao and Wang proved that the mappings F in $H^1_{p,1}$ is sense-preserving, univalent and starlike in $\mathbb{D}$ (cf. [16, Theorems 3.1 and 3.2]).

In Section 2, we will introduce three classes of p-harmonic mappings: $H^0_{p,q}$, $H^1_{p,q}$ and TH^*_p. When $p=1$, $H^0_{p,q}$, and TH^*_p coincide with $H^0_{1,q}$ and TH^*_1, respectively, and when $q=1$, $H^1_{p,q}$ is $H^1_{p,1}$.

The first aim of this paper is to discuss the starlikeness and convexity of p-harmonic mappings in $H^0_{p,q}$. Our results are Theorems 3.3 and 3.5, where Theorem 3.3 extends [9, Theorems 1 and 4] to the setting of p-harmonic mappings, and Theorem 3.5 is a generalization of [9, Theorems 2 and 3]). Also we consider the univalence, starlikeness and convexity of mappings belonging to $H^1_{p,q}$ with $q\in(0,1]$. Our result is Theorem 3.7 which is a generalization of [16, Theorems 3.1 and 3.2]. The proofs of the mentioned theorems will be presented in Section 3.

As the second aim of this paper, we investigate the covering theorem for mappings in $H^1_{p,q}$. Our result is Theorem 4.1 which is a generalization of [9, Theorem 5]. We will prove this theorem in Section 4.

Finally, we get a necessary and sufficient condition for a p-harmonic mapping to be in TH^*_p and then determine the extreme points of TH^*_p. Our main results are Theorems 5.1 and 5.2, where Theorems 5.1 and 5.2 are generalizations of [18, Theorem 2] and [18, Theorem 4(a)], respectively. Theorems 5.1, 5.2 proved in Section 5.

2. **Necessary notions and notations**

Let

$$
\begin{aligned}
(2.1)\qquad F(z) &= \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z) \\
&= \sum_{k=1}^{p} |z|^{2(k-1)} \big(h_{p-k+1} + \overline{g}_{p-k+1}\big) \\
&= \sum_{k=1}^{p} |z|^{2(k-1)} \big(\sum_{j=1}^{\infty} a_{j,p-k+1} z^j + \sum_{j=1}^{\infty} \overline{b}_{j,p-k+1} \bar{z}^j\big)
\end{aligned}
$$

be a p-harmonic mapping with $a_{1,p} = 1$ and $b_{1,p} = 0$.

We denote by $H^0_{p,q}$ with $q > 0$ the class of all univalent mappings satisfying the form (2.1) and the following condition:

$$
(2.2)\qquad \sum_{k=1}^{p}\sum_{j=1}^{\infty} j^q \big(|a_{j,p-k+1}| + |b_{j,p-k+1}|\big) \leq 2,
$$

and the class $H^1_{p,q}$ with $q > 0$ the class of all mappings satisfying the form (2.1) and the following condition:

$$
(2.3)\qquad \sum_{k=1}^{p}\sum_{j=1}^{\infty} (2(k-1)+j)^q \big(|a_{j,p-k+1}| + |b_{j,p-k+1}|\big) \leq 2.
$$

Proposition 2.1. *If $f \in H^1_{p,q}$ and f is univalent, then $f \in H^0_{p,q}$.*

We use J_F to denote the Jacobian of F, that is,

$$
J_F = |F_z|^2 - |F_{\bar{z}}|^2.
$$

Then it is known that F is sense-preserving and locally univalent if $J_F > 0$.

Definition 2.2. *We say that a univalent p-harmonic mapping F with $F(0) = 0$ is starlike of order $\alpha \in [0,1)$ with respect to the origin if the curve $F(re^{i\theta})$ is starlike of order α with respect to the origin for each $r \in (0,1)$. In other words, F is starlike of order α if $\frac{\partial}{\partial\theta}(argF(re^{i\theta})) \geq \alpha$ for all $z = re^{i\theta} \neq 0$.*

Definition 2.3. *A univalent p-harmonic mapping F with $F(0) = 0$ and $\frac{\partial}{\partial\theta}F(re^{i\theta}) \neq 0$ whenever $0 < r < 1$ is said to be convex of order $\beta \in [0, 1)$ if the curve $F(re^{i\theta})$ is convex of order β for each $r \in (0, 1)$. In other words, F is convex of order β if $\frac{\partial}{\partial\theta}(arg\frac{\partial}{\partial\theta}F(re^{i\theta})) \geq \beta$ for all $z = re^{i\theta} \neq 0$.*

Definition 2.4. *Let X be a topological vector space over the field of complex numbers, and let D be a subset of X. A point $x \in D$ is called an extreme point of D if it has no representation of the form $x = ty+(1-t)z$ ($t \in (0, 1)$) as a proper convex combination of two distinct points y and z in D.*

Furthermore, we introduce following notions and notations.

Let TH_p^* denote the class of all p-harmonic mappings F which are univalent, starlike and has the form (2.1), where $a_{1,p} = 1$ and $b_{1,p} = 0$, with an additional restriction that all the other coefficients are nonpositive.

3. Starlikeness and convexity

We start this section with two lemmas which will be useful for the following proofs.

Lemma 3.1. *Let*

$$F(z) = \sum_{k=1}^{p}\sum_{j=1}^{\infty} |z|^{2(k-1)}\big(a_{j,p-k+1}z^j + \overline{b}_{j,p-k+1}\overline{z}^j\big)$$

be a univalent p-harmonic mapping with $a_{1,p} = 1$ and $b_{1,p} = 0$. If

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}\big(\frac{j-\alpha}{1-\alpha}|a_{j,p-k+1}| + \frac{j+\alpha}{1-\alpha}|b_{j,p-k+1}|\big) \leq 2 \tag{3.1}$$

for some $\alpha \in [0, 1)$, then F is starlike of order α.

Proof. Note that

$$\frac{\partial}{\partial\theta}(\arg F(re^{i\theta})) = \mathrm{Re}\Big\{\frac{z\frac{\partial}{\partial z}F(z) - \overline{z}\frac{\partial}{\partial \overline{z}}F(z)}{F(z)}\Big\} = \mathrm{Re}\frac{1 + A(z)}{1 + B(z)}$$

for $r \neq 0$, where

$$A(z) = -1 + \sum_{k=1}^{p}\sum_{j=1}^{\infty} j|z|^{2(k-1)}(a_{j,p-k+1}z^j - \overline{b}_{j,p-k+1}\overline{z}^j)$$

and

$$B(z) = -1 + \sum_{k=1}^{p}\sum_{j=1}^{\infty} |z|^{2(k-1)}(a_{j,p-k+1}z^j + \overline{b}_{j,p-k+1}\overline{z}^j).$$

Let

$$w_1(z) = \frac{A(z) - B(z)}{2 - 2\alpha + A(z) + (1 - 2\alpha)B(z)}.$$

Then

$$\frac{1 + A(z)}{1 + B(z)} = \frac{1 + (1 - 2\alpha)w_1(z)}{1 - w_1(z)}.$$

An elementary calculation shows that

$$\mathrm{Re}\frac{1 + A(z)}{1 + B(z)} = \mathrm{Re}\frac{1 + (1 - 2\alpha)w_1(z)}{1 - w_1(z)} \geq \alpha$$

if and only if

$$|w_1(z)| \leq 1.$$

Obviously, a sufficient condition of

$$|w_1(z)| \leq 1$$

is

$$\begin{aligned}
&\sum_{k=1}^{p}\sum_{j=1}^{\infty} \Big((j - 1)|a_{j,p-k+1}| + (j + 1)|b_{j,p-k+1}|\Big) \\
&\leq 4 - 4\alpha - \sum_{k=1}^{p}\sum_{j=1}^{\infty} \Big((j + 1 - 2\alpha)|a_{j,p-k+1}| + (j - 1 + 2\alpha)|b_{j,p-k+1}|\Big),
\end{aligned}$$

which is equivalent to (3.1).

The proof of the lemma is complete. □

Lemma 3.2. *Let*

$$F(z) = \sum_{k=1}^{p}\sum_{j=1}^{\infty} |z|^{2(k-1)}\big(a_{j,p-k+1}z^j + \overline{b}_{j,p-k+1}\overline{z}^j\big)$$

be a univalent p-harmonic mapping with $a_{1,p} = 1$ and $b_{1,p} = 0$. If

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty} \big(\frac{j(j - \beta)}{1 - \beta}|a_{j,p-k+1}| + \frac{j(j + \beta)}{1 - \beta}|b_{j,p-k+1}|\big) \leq 2 \tag{3.2}$$

for some $\beta \in [0, 1)$, then F is convex of order β.

Proof. Note that

$$\frac{\partial}{\partial\theta}(\arg\frac{\partial}{\partial\theta}F(re^{i\theta})) = \mathrm{Re}\frac{1+P(z)}{1+Q(z)}$$

for $r \neq 0$, where

$$\begin{aligned} P(z) = &\\ & z\frac{\partial}{\partial z}F(z) + z^2\frac{\partial^2}{\partial z^2}F(z) - 2|z|^2\frac{\partial^2}{\partial z\partial\overline{z}}F(z) + \overline{z}\frac{\partial}{\partial\overline{z}}F(z) \\ & + \overline{z}^2\frac{\partial^2}{\partial\overline{z}^2}F(z) - 1 \\ = & -1 + \sum_{k=1}^{p}\sum_{j=1}^{\infty} j^2|z|^{2(k-1)}(a_{j,p-k+1}z^j + \overline{b}_{j,p-k+1}\overline{z}^j) \end{aligned}$$

and

$$\begin{aligned} Q(z) = & z\frac{\partial}{\partial z}F(z) - \overline{z}\frac{\partial}{\partial\overline{z}}F(z) - 1 \\ = & -1 + \sum_{k=1}^{p}\sum_{j=1}^{\infty} j|z|^{2(k-1)}(a_{j,p-k+1}z^j - \overline{b}_{j,p-k+1}\overline{z}^j). \end{aligned}$$

Let

$$w_2(z) = \frac{P(z) - Q(z)}{2 - 2\beta + P(z) + (1-2\beta)Q(z)}.$$

Then

$$\frac{1+P(z)}{1+Q(z)} = \frac{1+(1-2\beta)w_2(z)}{1-w_2(z)}.$$

It is easy to deduce that

$$\mathrm{Re}\frac{1+P(z)}{1+Q(z)} = \mathrm{Re}\frac{1+(1-2\beta)w_2(z)}{1-w_2(z)} \geq \beta$$

if and only if

$$|w_2(z)| \leq 1.$$

Obviously, a sufficient condition of

$$|w_2(z)| \leq 1$$

is

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}\left((j^2-j)|a_{j,p-k+1}|+(j^2+j)|b_{j,p-k+1}|\right)$$
$$\leq 4-4\beta-\sum_{k=1}^{p}\sum_{j=1}^{\infty}\left((j^2+j-2\beta j)|a_{j,p-k+1}|+(j^2-j+2\beta j)|b_{j,p-k+1}|\right),$$

which is equivalent to (3.2). □

Now we are ready to state and prove the results concerning the geometric properties of mappings in $H^0_{p,q}$.

Theorem 3.3. *Suppose $F\in H^0_{p,q}$ and $b_{1,p-k+1}=0$ for $k\in\{2,\cdots,p\}$.*

(1) *If $q\in[1,2)$, then F is starlike of order $\alpha(q)$, where $\alpha(q)=\frac{2^q-2}{2^q+1}$;*
(2) *If $q\in[2,+\infty)$, then F is convex of order $\beta(q)$, where $\beta(q)=\frac{2^{q-1}-2}{2^{q-1}+1}$.*

Proof. By Lemma 3.1, for a fixed $q\in[1,2)$ and any $j\in\{2,3,\cdots\}$, we know that F is p-harmonic starlike of order $\alpha=\alpha(q)$ if

$$j^q\geq\frac{j+\alpha}{1-\alpha},$$

which is equivalent to

$$\alpha\leq\frac{j^q-j}{j^q+1}.$$

Since $\{S_q(j)=\frac{j^q-j}{j^q+1}\}$ is an increasing sequence about j for any fixed $q\in[1,2)$, it follows that

$$\frac{j^q-j}{j^q+1}\geq\frac{2^q-2}{2^q+1}=S_q(2)=\alpha,$$

which proves (1).

Next, we prove (2). By Lemma 3.2, for a fixed $q\in[2,+\infty)$, F will be p-harmonic and convex of order $\beta=\beta(q)$ if

$$j^q\geq\frac{j(j+\beta)}{1-\beta},$$

which is equivalent to

$$\beta\leq\frac{j^q-j^2}{j^q+j}.$$

It is easy to know that $\{T_q(j)=\frac{j^q-j^2}{j^q+j}\}$ is an increasing sequence about j for any fixed $q\in[2,\infty)$. Hence

$$\frac{j^q-j^2}{j^q+j}\geq\frac{2^{q-1}-2}{2^{q-1}+1}=T_q(2)=\beta(q),$$

which shows that (2) holds. □

Corollary 3.4. *If $F\in H^0_{p,1}$ (respectively $F\in H^0_{p,2}$), then F is starlike (respectively convex) in $\mathbb{D}$.*

By taking $\alpha=0$ (respectively $\beta=0$), Lemma 3.1 (respectively Lemma 3.2) implies that if $F\in H^0_{p,q}$ with $q\geq 1$ (respectively $q\geq 2$), then F is starlike (respectively convex) in $\mathbb{D}$. However, when $q\in(0,1)$ (respectively $q\in(0,2)$), $F\in H^0_{p,q}$ need not be starlike (respectively convex). For instance, the harmonic polynomials

$$f^*_q(z)=z-2^{-q}\overline{z}^2\ \ \text{(respectively } f^c_q(z)=z+2^{-q}\overline{z}^2)$$

with $q\in(0,1)$ (respectively $q\in(0,2)$). Upon choosing the value of z in the interval $z\in(-1,-2^{q-1})$ (respectively $z\in(-1,-2^{q-2})$) , it is easy to know that

$$\frac{\partial}{\partial\theta}\big(\arg f^*_q(re^{i\theta})\big)<0\ \ \text{(respectively } \frac{\partial}{\partial\theta}\arg\big(\frac{\partial}{\partial\theta}f^c_q(re^{i\theta})\big)<0).$$

By replacing $\mathbb{D}$ by some subdisk, in this case, we can prove the following result.

Theorem 3.5. *Suppose $F\in H^0_{p,q}$ for $k\in\{2,\cdots,p\}$.*

(1) *If $q\in(0,1]$, then F is starlike in $\mathbb{D}_{\frac{1}{2^{1-q}}}$;*

(2) *If $q\in(0,2]$, then F is convex in $\mathbb{D}_{\frac{1}{2^{2-q}}}$.*

And the results are sharp with extremal functions

$$F_1(z)=z+2^{-q}\alpha\overline{z}^2\ \ \text{and}\ \ F_2(z)=z+2^{-q}\beta\overline{z}^2,$$

respectively, where α, β are constants with $|\alpha|=|\beta|=1$.

Proof. Let

$$F^*(z)=2^{1-q}F\big(\frac{z}{2^{1-q}}\big).$$

Then

$$F^*(z)=\sum_{k=1}^{p}\sum_{j=1}^{\infty}|z|^{2(k-1)}\Big(\frac{a_{j,p-k+1}}{2^{(1-q)(2k+j-3)}}z^j+\frac{\overline{b}_{j,p-k+1}}{2^{(1-q)(2k+j-3)}}\overline{z}^j\Big).$$

By (2.2) and the inequality

$$\frac{j}{2^{(1-q)(2k+j-3)}}\cdot\frac{1}{j^q}\le\frac{(2j)^{1-q}}{(2^j)^{1-q}}\le 1$$

for any $j\in\{1,2,\cdots\}$, $k\in\{1,\cdots,p\}$ and fixed $q\in(0,1)$, it follows that

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}j\Big(\frac{|a_{j,p-k+1}|+|b_{j,p-k+1}|}{2^{(1-q)(2k+j-3)}}\Big)\le\sum_{k=1}^{p}\sum_{j=1}^{\infty}j^q(|a_{j,p-k+1}|+|b_{j,p-k+1}|)\le 2.$$

Then Lemma 3.1 implies that F^* is starlike in $\mathbb{D}$, which shows that F is starlike in $\mathbb{D}_{\frac{1}{2^{1-q}}}$.

Let

$$F^c(z)=2^{2-q}F\big(\frac{z}{2^{2-q}}\big).$$

By similar arguments as in the proof of (1), we know that (2) holds. □

Corollary 3.6. *If $F\in H^0_{p,1}$, then F maps the disk $\mathbb{D}_{\frac{1}{2}}$ onto a convex domain.*

Next, we consider the starlikeness and convexity of $F\in H^1_{p,q}$ and prove

Theorem 3.7. *If $F\in H^1_{p,q}$ is a p-harmonic mapping with $q\in(0,1]$, then F is sense-preserving and univalent in $\mathbb{D}_{\frac{1}{2^{1-q}}}$. Moreover, F is starlike in $\mathbb{D}_{\frac{1}{2^{1-q}}}$ and convex in $\mathbb{D}_{\frac{1}{2^{2-q}}}$, and the extremal functions are*

$$F_3(z)=z+2^{-q}\alpha_1\overline{z}^2\ \text{ and }\ F_4(z)=z+2^{-q}\beta_1\overline{z}^2,$$

respectively, where α_1, β_1 are constants with $|\alpha_1|=|\beta_1|=1$.

Proof. First, we prove that F is sense-preserving in $\mathbb{D}_{\frac{1}{2^{1-q}}}$. Let $0\le|z|=r<\frac{1}{2^{1-q}}$. Then

$$\begin{aligned}\Big|\frac{\partial}{\partial z}F(z)\Big|-\Big|\frac{\partial}{\partial\overline{z}}F(z)\Big|\ge&2-\sum_{k=1}^{p}\sum_{j=1}^{\infty}\big(2(k-1)+j\big)r^{2k+j-3}(|a_{j,p-k+1}|\\&+|b_{j,p-k+1}|)\\>&2-\sum_{k=1}^{p}\sum_{j=1}^{\infty}\frac{2(k-1)+j}{2^{(1-q)(2k+j-3)}}(|a_{j,p-k+1}|+|b_{j,p-k+1}|)\\\ge&0,\end{aligned}$$

since

$$\frac{2(k-1)+j}{2^{(1-q)(2k+j-3)}} \leq (2(k-1)+j)^q$$

for $j \in \{1, \cdots\}$, $k \in \{1, \cdots, p\}$ and $q \in (0,1]$. Therefore, F is sense-preserving in $\mathbb{D}_{\frac{1}{2^{1-q}}}$.

Next, we show that $F(z_1) \neq F(z_2)$ if $z_1 \neq z_2$. Suppose $z_1, z_2 \in \mathbb{D}_{\frac{1}{2^{1-q}}}$ such that $z_1 \neq z_2$ and $|z_1| \geq |z_2|$. Then

$$\begin{aligned}
\Big|\frac{F(z_1)-F(z_2)}{z_1-z_2}\Big| &\geq 2 - \sum_{k=1}^{p}\sum_{j=1}^{\infty}\big(2(k-1)+j\big)(|a_{j,p-k+1}| \\
&\quad + |b_{j,p-k+1}|)|z_1|^{2k+j-3} \\
&> 2 - \sum_{k=1}^{p}\sum_{j=1}^{\infty}\frac{2(k-1)+j}{2^{(1-q)(2k+j-3)}}(|a_{j,p-k+1}| + |b_{j,p-k+1}|) \\
&\geq 2 - \sum_{k=1}^{p}\sum_{j=1}^{\infty}\big(2(k-1)+j\big)^q(|a_{j,p-k+1}| + |b_{j,p-k+1}|)| \\
&\geq 0.
\end{aligned}$$

Hence F is univalent in $\mathbb{D}_{\frac{1}{2^{1-q}}}$.

The remaining part of the proof easily follows from the similar reasoning as in Theorem 3.5. □

4. **Covering theorem**

Theorem 4.1. *Let $F \in H^1_{p,q}$ be a p-harmonic mapping with $q \in (0,\infty)$. Then*

(1) $\{\omega : |\omega| < \frac{1}{2^{2-q}}\} \subseteq F(\mathbb{D}_{\frac{1}{2^{1-q}}}) \subseteq \{\omega : |\omega| < \frac{3}{2^{2-q}}\}$ *if* $q \in (0,1]$;

(2) $\{\omega : |\omega| < 1 - \frac{1}{2^q}\} \subseteq F(\mathbb{D}) \subseteq \{\omega : |\omega| < 1 + \frac{1}{2^q}\}$ *if* $q \in [1,\infty)$.

Proof. Since $F \in H^1_{p,q}$, it is easy to show that

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}(|a_{j,p-k+1}| + |b_{j,p-k+1}|) \leq 1 + \frac{1}{2^q}.$$

Then for $0 < r < 1$, we have

$$|F(re^{i\theta})| \geq 2r - \sum_{k=1}^{p}\sum_{j=1}^{\infty} r^{2(k-1)+j}(|a_{j,p-k+1}| + |b_{j,p-k+1}|)$$
$$\geq r + r^2 - r^2 \sum_{k=1}^{p}\sum_{j=1}^{\infty}(|a_{j,p-k+1}| + |b_{j,p-k+1}|)$$
$$\geq r - \frac{1}{2^q} r^2$$

and

$$|F(re^{i\theta})| \leq \sum_{k=1}^{p}\sum_{j=1}^{\infty} r^{2(k-1)+j}(|a_{j,p-k+1}| + |b_{j,p-k+1}|)$$
$$\leq r - r^2 + r^2 \sum_{k=1}^{p}\sum_{j=1}^{\infty}(|a_{j,p-k+1}| + |b_{j,p-k+1}|)$$
$$\leq r + \frac{1}{2^q} r^2.$$

Hence

$$r - \frac{1}{2^q} r^2 \leq |F(re^{i\theta})| \leq r + \frac{1}{2^q} r^2.$$

By Theorem 3.7, if $0 < q \leq 1$, then F is univalent in $\mathbb{D}_{\frac{1}{2^{1-q}}}$. Letting $r \to \frac{1}{2^{1-q}}$ in the above inequality gives (1). By [16, Theorem 3.1], if $q \geq 1$, then F is univalent in $\mathbb{D}$. By letting $r \to 1$ in the above inequality, (2) easily follows. These complete the proof. □

5. **Extreme points of** TH_p^*

In this section, we consider the mappings in TH_p^*. First, we give a characterization for a p-harmonic mapping to be in TH_p^*.

Theorem 5.1. *Let F be a p-harmonic mapping with the form* (2.3). *Then $F \in TH_p^*$ if and only if*

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}(2(k-1)+j)(|a_{j,p-k+1}| + |b_{j,p-k+1}|) \leq 2.$$

Proof. The sufficiency easily follows from [16, Theorems 3.1 and 3.2]. To prove the necessity, it suffices to show that $F \notin TH_p^*$ if

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}(2(k-1)+j)(|a_{j,p-k+1}|+|b_{j,p-k+1}|) > 2.$$

Under this assumption, it suffices to prove that F is not univalent. Setting $z = r > 0$ gives

$$F(r) = 2r - \sum_{k=1}^{p}\sum_{j=1}^{\infty} r^{2(k-1)+j}(|a_{j,p-k+1}|+|b_{j,p-k+1}|)$$

and

$$F'(r) = 2 - \sum_{k=1}^{p}\sum_{j=1}^{\infty}(2(k-1)+j)r^{2k-3+j}(|a_{j,p-k+1}|+|b_{j,p-k+1}|).$$

Since $F'(0) = 1$ and $F'(1) < 0$, there must exist some r_0 with $r_0 < 1$ such that $F'(r_0) = 0$. Hence $F(r)$ is not one-to-one on the real interval (0, 1) which implies $F \notin TH_p^*$. □

From Theorem 5.1, we know that TH_p^* is closed under the convex combination. Now we use Theorem 5.1 to determine the extreme points in TH_p^*.

Theorem 5.2. *Let*

$$h_{1,p}(z) = z, \quad h_{j,p-k+1}(z) = z - \frac{|z|^{2(k-1)}z^j}{2(k-1)+j}$$

and

$$g_{1,p}(z) = 0 \text{ and } g_{j,p-k+1}(z) = z - \frac{|z|^{2(k-1)}\overline{z}^j}{2(k-1)+j},$$

where $j \in \{1, \cdots\}$, $k \in \{1, \cdots, p\}$ *and* $|j-1|+|k-1| \neq 0$. *Then*

(1) $F \in TH_p^*$ *if and only if it can be expressed in the form*

$$F(z) = \sum_{k=1}^{p}\sum_{j=1}^{\infty}\big(\lambda_{j,p-k+1}h_{j,p-k+1}(z) + \gamma_{j,p-k+1}g_{j,p-k+1}(z)\big),$$

where $\lambda_{j,p-k+1} \geq 0$, $\gamma_{j,p-k+1} \geq 0$, $\gamma_{1,p} = 0$ *and* $\sum_{k=1}^{p}\sum_{j=1}^{\infty}(\lambda_{j,p-k+1}+\gamma_{j,p-k+1}) = 1$.

(2) *The set of all extreme points of* TH_p^* *are the union of the sets* $\{h_{j,p-k+1}\}$ *and* $\{g_{j,p-k+1}\}$.

Proof. Suppose

$$F(z) = \sum_{k=1}^{p} \sum_{j=1}^{\infty} \big(\lambda_{j,p-k+1} h_{j,p-k+1}(z) + \gamma_{j,p-k+1} g_{j,p-k+1}(z)\big)$$

$$=(1+\lambda_{1,p})z - \sum_{k=1}^{p} \sum_{j=1}^{\infty} |z|^{2(k-1)} \Big(\frac{\lambda_{j,p-k+1} z^j}{2(k-1)+j} + \frac{\gamma_{j,p-k+1} \overline{z}^j}{2(k-1)+j}\Big).$$

Since $\lambda_{j,p-k+1} \geq 0$, $\gamma_{j,p-k+1} \geq 0$, $\gamma_{1,p} = 0$ and $\sum_{k=1}^{p} \sum_{j=1}^{\infty} (\lambda_{j,p-k+1} + \gamma_{j,p-k+1}) = 1$, the starlikeness of F follows from Theorem 5.1. Hence, $F \in TH_p^*$.

Conversely, if $F \in TH_p^*$, then by Theorem5.1,

$$|a_{j,p-k+1}| \leq \frac{1}{2(k-1)+j} \text{ and } |b_{j,p-k+1}| \leq \frac{1}{2(k-1)+j}.$$

Set

$$\lambda_{j,p-k+1} = -(2(k-1)+j) a_{j,p-k+1}, \quad \gamma_{j,p-k+1} = -(2(k-1)+j) b_{j,p-k+1},$$

$$\lambda_{1,p} = 1 - \sum_{j \cdot k > 1} (\lambda_{j,p-k+1} + \gamma_{j,p-k+1})$$

and

$$\gamma_{1,p} = 0.$$

Then

$$F(z) = \sum_{k=1}^{p} \sum_{j=1}^{\infty} (\lambda_{j,p-k+1} h_{j,p-k+1}(z) + \gamma_{j,p-k+1} g_{j,p-k+1}(z)).$$

Hence (1) holds.

The proof of (2) easily follows from (1). Hence we complete the proof of Theorem 5.2. □

Acknowledgments

The author wishes to thank the referee for the valuable suggestions which improve the presentation of the paper.

The research was partly supported by "NSF of China (No. 11071063)" and "the program for Science and Technology Innovative Research Team in Higher Educational Institutions of Hunan Province".

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