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OPTIMAL INEQUALITIES FOR THE POWER, HARMONIC AND LOGARITHMIC MEANS

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Abstract. For all $a, b > 0$, the following two optimal inequalities are presented: $H^{\alpha}(a,b)L^{1-\alpha}(a,b) \geq M_{\frac{1-4\alpha}{3}}(a,b)$ for $\alpha \in [\frac{1}{4}, 1)$, and $H^{\alpha}(a,b)L^{1-\alpha}(a,b) \leq M_{\frac{1-4\alpha}{3}}(a,b)$ for $\alpha \in (0, \frac{3\sqrt{5}-5}{40}]$. Here, $H(a,b)$, $L(a,b)$, and $M_p(a,b)$ denote the harmonic, logarithmic, and power means of order p of two positive numbers a and b, respectively.

1. Introduction

For $p \in \mathbb{R}$ the power mean $M_p(a,b)$ of order p and logarithmic mean $L(a,b)$ of two positive numbers a and b are defined by

$$M_p(a,b) = \begin{cases} \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}}, & p \neq 0, \\ \sqrt{ab}, & p = 0 \end{cases} \tag{1.1}$$

and

$$L(a,b) = \begin{cases} \frac{b-a}{\log b - \log a}, & a \neq b, \\ a, & a = b, \end{cases} \tag{1.2}$$

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respectively.

It is well-known that $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Recently both mean values have been the subject of intensive research. In particular, many remarkable inequalities for $M_p(a,b)$ and $L(a,b)$ can be found in the literature [2, 3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18].

Let $I(a,b) = \frac{1}{e}(a^a/b^b)^{1/(a-b)}$, $A(a,b) = (a+b)/2$, $G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ be the identric, arithmetic, geometric, and harmonic means of two positive real numbers a and b with $a \neq b$, respectively. Then

$$\begin{aligned}(1.3)\quad \min\{a,b\} < H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) < L(a,b) \\ < I(a,b) < A(a,b) = M_1(a,b) < \max\{a,b\}.\end{aligned}$$

In [3], Alzer and Janous established the following sharp double inequality (see also [6, p.350]):

$$M_{\frac{\log 2}{\log 3}}(a,b) \leq \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b) \leq M_{\frac{2}{3}}(a,b)$$

for all $a, b > 0$.

For any $\alpha \in (0,1)$, Janous [11] found the greatest value p and the least value q such that

$$M_p(a,b) \leq \alpha A(a,b) + (1-\alpha)G(a,b) \leq M_q(a,b)$$

for all $a, b > 0$.

In [15], Sándor proved

$$\frac{A(a,b)G(a,b)}{I(a,b)} < \sqrt{G(a,b)I(a,b)} < L(a,b)$$

and

$$L(a,b) < A(a,b) + G(a,b) - I(a,b) < \frac{1}{2}(G(a,b) + I(a,b))$$

for all $a, b > 0$ with $a \neq b$.

The following companion of (1.3) provides inequalities for the geometric and arithmetic means of L and I, the proof can be found in [1].

$$\begin{aligned}G(a,b)^{\frac{1}{2}}A(a,b)^{\frac{1}{2}} \leq L(a,b)^{\frac{1}{2}}I^{\frac{1}{2}}(a,b) \leq \frac{1}{2}L(a,b) + \frac{1}{2}I(a,b) \\ \leq \frac{1}{2}G(a,b) + \frac{1}{2}A(a,b)\end{aligned}$$

for all $a, b > 0$.

Burk [7] established

$$L(a,b) < \left(\frac{a^{\frac{1}{3}} + b^{\frac{1}{3}}}{2}\right)^3$$

for all $a, b > 0$ with $a \neq b$.

Alzer and Qiu [4] proved

$$M_c(a,b) \leq \frac{1}{2}L(a,b) + \frac{1}{2}I(a,b)$$

for all a, $b > 0$ with the best possible parameter $c = \frac{\log 2}{1+\log 2}$.

$$[G(a,b)]^{A(a,b)} < [L(a,b)]^{I(a,b)} < [A(a,b)]^{G(a,b)}$$

for $a, b \geq e$, and

$$[A(a,b)]^{G(a,b)} < [I(a,b)]^{L(a,b)} < [G(a,b)]^{A(a,b)}$$

for $0 < a, b < e$.

The main purpose of this paper is to present the optimal bounds for $H^{\alpha}(a,b)L^{1-\alpha}(a,b)$ in terms of the power mean $M_p(a,b)$ for some $\alpha \in (0,1)$.

2. **Set up**

In order to establish our main results we need a lemma, which we present in this section.

Lemma 2.1. *Let* $r \in (0,1)$, $p = \frac{1-4r}{3}$ *and* $g(t) = [rt^{p+2} - t^{p+1} + (r-1)t^p - (1-r)t^2 - t + r]\log t + (1-r)(t^{p+2} - t^p + t^2 - 1)$. *Then the following statements are true:*

(1) If $r \in (\frac{1}{4}, 1)$, *then* $g(t) < 0$ *for* $t \in (1, \infty)$;

(2) If $r \in (0, \frac{3\sqrt{5}-5}{40}]$, *then* $g(t) > 0$ *for* $t \in (1, \infty)$.

Proof. Let $g_1(t) = t^{1-p}g'(t)$, $g_2(t) = t^p g_1'(t)$, $g_3(t) = t^{1-p}g_2'(t)$, $g_4(t) = t^p g_3'(t)$, $g_5(t) = t^{1-p}g_4'(t)$, $g_6(t) = t^{3+p}g_5'(t)$, $g_7(t) = t^{1-p}g_6'(t)$, $g_8(t) =$

$t^{p-1}g_7'(t)$, $g_9(t) = t^2 g_8'(t)$, and $g_{10}(t) = t^{1-p}g_9'(t)$. Then simple computations lead to

$$g(1) = 0, \tag{2.1}$$

$$\begin{aligned} g_1(t) = {} & [r(p+2)t^2 - (p+1)t - 2(1-r)t^{2-p} - t^{1-p} \\ & - p(1-r)]\log t + (p - pr + 2 - r)t^2 - t \\ & + (1-r)t^{2-p} + rt^{-p} - t^{1-p} - (p+1)(1-r), \end{aligned}$$

$$g_1(1) = 0, \tag{2.2}$$

$$\begin{aligned} g_2(t) = {} & [2r(p+2)t^{1+p} - (p+1)t^p + 2(p-2)(1-r)t \\ & + p - 1]\log t + (2p + 4 - pr)t^{1+p} - (p+2)t^p \\ & - p(1-r)t^{p-1} - p(1-r)t - prt^{-1} + p - 2, \end{aligned}$$

$$g_2(1) = 0, \tag{2.3}$$

$$\begin{aligned} g_3(t) = {} & [2r(p+1)(p+2)t + 2(p-2)(1-r)t^{1-p} \\ & - p(p+1)]\log t + (2p^2 - p^2 r + pr + 6p + 4r + 4)t \\ & - p(p-1)(1-r)t^{-1} + prt^{-1-p} + (p-1))t^{-p} \\ & + (1-r)(p-4)t^{1-p} - (p^2 + 3p + 1), \end{aligned}$$

$$g_3(1) = 6p + 8r - 2 = 0, \tag{2.4}$$

$$\begin{aligned} g_4(t) = {} & 2[r(p+1)(p+2)t^p - (p-1)(p-2)(1-r)]\log t \\ & + p(p-1)(1-r)t^{p-2} - p(p+1)t^{p-1} - p(p-1)t^{-1} \\ & - p(p+1)rt^{-2} + (2p^2 + p^2 r + 7pr + 6p + 8r + 4)t^p \\ & - (p^2 - 7p + 8)(1-r), \end{aligned}$$

$$g_4(1) = 12p + 16r - 4 = 0, \tag{2.5}$$

$$\begin{aligned} g_5(t) = {} & 2pr(p+1)(p+2)\log t - 2(p-1)(p-2)t^{-p} \\ & + p(p-1)t^{-1-p} - p(p+1)(p-1)t^{-1} \\ & + p(p-1)(p-2)(1-r)t^{-2} + 2p^3 + p^3 r + 9p^2 r \\ & + 6p^2 + 14pr + 4p + 4r, \end{aligned}$$

$$\begin{aligned} g_5(1) &= 2p^3 + 16rp^2 + 2p^2 + 8pr + 12p + 8r - 4 \\ &= \frac{8(r-1)}{27}(80r^2 + 20r - 1), \end{aligned} \tag{2.6}$$

$$\begin{aligned} g_6(t) = {} & p[2(p-1)(p-2)(1-r)t^2 - (p-1)(p+1)t \\ & - 2r(p+1)(p+2) + 2r(p+1)(p+2)t^{2+p} \\ & + (p-1)(p+1)t^{1+p} - 2(p-1)(p-2)(1-r)t^p], \end{aligned}$$

$$g_6(1) = 0, \tag{2.7}$$

$$\begin{aligned} g_7(t) &= p[2r(p+1)(p+2)^2t^2 + (p-1)(p+1)^2 t \\ &\quad - (p-1)(p+1)t^{1-p} + 4(p-1)(p-2)(1-r)t^{2-p} \\ &\quad - 2p(p-1)(p-2)(1-r)], \end{aligned}$$

$$\begin{aligned} g_7(1) &= p[4rp^3 - p^3 + 10p^2 + 32pr - 17p - 8] \\ &= \frac{4(1-4r)}{81}(-64r^4 + 64r^3 - 192r^2 + 169r + 23), \end{aligned} \tag{2.8}$$

$$\begin{aligned} g_8(t) &= p[4r(p+1)(p+2)^2t^p + (p-1)(p+1)^2t^{p-1} \\ &\quad + (p-1)^2(p+1)t^{-1} - 4(p-1)(p-2)^2(1-r)], \end{aligned}$$

$$\begin{aligned} g_8(1) &= 2p[4rp^3 - p^3 + 10p^2 + 32pr - 17p - 8] \\ &= \frac{8(1-4r)}{81}(-64r^4 + 64r^3 - 192r^2 + 169r + 23), \end{aligned} \tag{2.9}$$

$$\begin{aligned} g_9(t) &= p[4rp(p+1)(p+2)^2t^{p+1} + (p-1)^2(p+1)^2t^p \\ &\quad - (p+1)(p-1)^2], \end{aligned}$$

$$g_9(1) = p^2(p+1)[4r(p+2)^2 + (p-1)^2] > 0, \tag{2.10}$$

$$g_{10}(t) = p^2(p+1)^2[4r(p+2)^2t + (p-1)^2], \tag{2.11}$$

$$g_{10}(1) = p^2(p+1)^2[4r(p+2)^2 + (p-1)^2] > 0. \tag{2.12}$$

(1) If $r \in (\frac{1}{4}, 1)$, then from (2.6), (2.8) and (2.9) we have

$$g_8(1) < 0, \tag{2.13}$$

$$g_7(1) < 0, \tag{2.14}$$

$$g_5(1) < 0, \tag{2.15}$$

$$\lim_{t\to+\infty} g_8(t) < 0, \tag{2.16}$$

$$\lim_{t\to+\infty} g_7(t) = -\infty. \tag{2.17}$$

Equation (2.11) implies that $g_{10}(t)$ is strictly increasing in $(1,\infty)$, then from (2.10), (2.12) and the monotonicity of $g_{10}(t)$ we clearly see that $g_9(t) > 0$ for all $t \in (1,\infty)$. Hence we get that $g_8(t)$ is strictly increasing in $(1,\infty)$.

From (2.13) and (2.16) together with the monotonicity of $g_8(t)$ we see that $g_8(t) < 0$ in $(1,\infty)$. Hence we obtain that $g_7(t)$ is strictly decreasing in $(1,\infty)$. From (2.14) and (2.17) together with the monotonicity of $g_7(t)$

we clearly see that $g_7(t) < 0$ in $(1, \infty)$. So we get that $g_6(t)$ is strictly decreasing in $(1, \infty)$.

Therefore, Lemma 2.1(1) follows from the monotonicity of $g_6(t)$, (2.7), (2.15) and (2.1)-(2.5).

(2) If $r \in (0, \frac{3\sqrt{5}-5}{40}]$, then from (2.6), (2.8) and (2.9) we have

$$g_8(1) > 0, \tag{2.18}$$

$$g_7(1) > 0, \tag{2.19}$$

$$g_5(1) \geq 0. \tag{2.20}$$

Equation (2.11) implies that $g_{10}(t)$ is strictly increasing in $(1, \infty)$. Therefore, Lemma 2.1(2) follows from (2.12), (2.10), (2.18), (2.19), (2.7), (2.20) and (2.1)-(2.5) together with the monotonicity of $g_{10}(t)$. □

3. The main result

Theorem 3.1. *For all $a, b > 0$ we*

$$M_{\frac{1-4\alpha}{3}}(a,b) \leq H^{\alpha}(a,b)L^{1-\alpha}(a,b)$$

for $\alpha \in (\frac{1}{4}, 1)$, and

$$H^{\alpha}(a,b)L^{1-\alpha}(a,b) \leq M_{\frac{1-4\alpha}{3}}(a,b)$$

for $\alpha \in (0, \frac{3\sqrt{5}-5}{40}]$. Each inequality becomes equality if and only if $a = b$, and the parameter $\frac{1-4\alpha}{3}$ in each inequality is the best possible.

Proof. If $a = b$, then from (1.1) we clearly see that

$$a = M_0(a,b) = H^{\alpha}(a,b)L^{1-\alpha}(a,b) = M_{\frac{1-4\alpha}{3}}(a,b) = b.$$

Without loss of generality, we assume that $a > b$. Let $t = \frac{a}{b} > 1$. Then (1.1) and (1.2) lead to

$$\begin{aligned} & M_p(a,b) - H^{\alpha}(a,b)L^{1-\alpha}(a,b) \\ = \quad & b\left[\left(\frac{t^p+1}{2}\right)^{\frac{1}{p}} - \left(\frac{2t}{t+1}\right)^{\alpha}\left(\frac{t-1}{\log t}\right)^{1-\alpha}\right]. \end{aligned} \tag{3.1}$$

Let

$$\begin{aligned} f(t) \quad = \quad & \frac{1}{p}\log\frac{1+t^p}{2} - \alpha\log\left(\frac{2t}{t+1}\right) - (1-\alpha)[\log(t-1) \\ & + \log(\log t)]. \end{aligned} \tag{3.2}$$

Then simple computations yield

$$\lim_{t\to 1^+} f(t) = 0, \tag{3.3}$$

$$f'(t) = \frac{g(t)}{t(t-1)(t+1)(t^p+1)\log t}, \tag{3.4}$$

where

$$g(t) = [\alpha t^{p+2} - t^{p+1} + (\alpha-1)t^p - (1-\alpha)t^2 - t + \alpha]\log t$$
$$+(1-\alpha)(t^{p+2} - t^p + t^2 - 1).$$

(1) If $\alpha \in (\frac{1}{4}, 1)$, then from Lemma 2.1(1) and (3.4) we get

$$f'(t) < 0 \tag{3.5}$$

for all $t \in (1, \infty)$.

Therefore, from (3.1)-(3.3) and (3.5) we see that the inequality

$$H^{\alpha}(a,b)L^{1-\alpha}(a,b) > M_{\frac{1-4\alpha}{3}}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

(2) If $\alpha \in (0, \frac{3\sqrt{5}-5}{40}]$, from Lemma 2.1(2) and (3.4) we have

$$f'(t) > 0 \tag{3.6}$$

for all $t \in (1, \infty)$.

Therefore, from (3.1)-(3.3) and (3.6) we obtain that inequality

$$H^{\alpha}(a,b)L^{1-\alpha}(a,b) < M_{\frac{1-4r}{3}}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

Next, we prove that the parameter $\frac{1-4\alpha}{3}$ in each inequality is optimal.

Case A. If $\alpha \in [\frac{1}{4}, 1)$, then for any $0 < \varepsilon < (4\alpha - 1)/3$ and $x > 0$ one has

$$\begin{aligned} &M_{\frac{1-4\alpha}{3}+\varepsilon}(x+1,1) - H^{\alpha}(x+1,1)L^{1-\alpha}(x+1,1) \\ &= \left[\frac{(x+1)^{\frac{1-4\alpha}{3}+\varepsilon}+1}{2}\right]^{\frac{1}{\frac{1-4\alpha}{3}+\varepsilon}} - \left(\frac{x+1}{1+\frac{x}{2}}\right)^{\alpha} \cdot \left[\frac{x}{\log(x+1)}\right]^{1-\alpha} \\ &= \frac{f_1(x)}{\left(1+\frac{x}{2}\right)^{\alpha}[\log(1+x)]^{1-\alpha}}, \end{aligned} \tag{3.7}$$

where

$$f_1(x) = \left[\frac{(x+1)^{\frac{1-4\alpha}{3}+\varepsilon}+1}{2}\right]^{\frac{1-4\alpha}{3}+\varepsilon}[\log(1+x)]^{1-\alpha}\left(1+\frac{x}{2}\right)^{\alpha} - (1+x)^{\alpha}x^{1-\alpha}.$$

Letting $x\to 0$ and making use of Taylor expansion we get

$$f_1(x) = \left[\frac{\varepsilon}{8}x^2+o(x^2)\right]x^{1-\alpha}. \tag{3.8}$$

Equations (3.7) and (3.8) imply that for any $\alpha\in[\frac{1}{4},1)$ and $0<\varepsilon<\frac{4\alpha-1}{3}$ there exists $\delta_1(\varepsilon,\alpha)>0$, such that

$$H^{\alpha}(x+1,1)L^{1-\alpha}(x+1,1) < M_{\frac{1-4\alpha}{3}+\varepsilon}(x+1,1)$$

for $x\in(0,\delta_1)$.

Case B. If $\alpha\in(0,\frac{3\sqrt{5}-5}{40}]$, then for any $0<\varepsilon<\frac{1-4\alpha}{3}$ and $x>0$ one has

$$\begin{aligned} & H^{\alpha}(x+1,1)L^{1-\alpha}(x+1,1) - M_{\frac{1-4\alpha}{3}-\varepsilon}(x+1,1) \\ &= \left(\frac{x+1}{1+\frac{x}{2}}\right)^{\alpha}\left[\frac{x}{\log(x+1)}\right]^{1-\alpha} - \left[\frac{(x+1)^{\frac{1-4\alpha}{3}-\varepsilon}+1}{2}\right]^{\frac{1}{\frac{1-4\alpha}{3}-\varepsilon}} \\ &= \frac{f_2(x)}{\left(1+\frac{x}{2}\right)^{\alpha}[\log(1+x)]^{1-\alpha}}, \end{aligned} \tag{3.9}$$

where

$$f_2(x) = (1+x)^{\alpha}\cdot x^{1-\alpha} - \left[\frac{(x+1)^{\frac{1-4\alpha}{3}-\varepsilon}+1}{2}\right]^{\frac{1-4\alpha}{3}-\varepsilon}[\log(1+x)]^{1-\alpha} \times\left(1+\frac{x}{2}\right)^{\alpha}.$$

Letting $x\to 0$ and making use of Taylor expansion we have

$$f_2(x) = \left[\frac{\varepsilon}{8}x^2+o(x^2)\right]x^{1-\alpha}. \tag{3.10}$$

Equations (3.9) and (3.10) imply that for any $\alpha\in(0,\frac{3\sqrt{5}-5}{40}]$ and $0<\varepsilon<\frac{1-4\alpha}{3}$ there exists $\delta_2(\varepsilon,\alpha)>0$, such that

$$H^{\alpha}(1+x,1)L^{1-\alpha}(1+x,1) > M_{\frac{1-4\alpha}{3}-\varepsilon}(1+x,1)$$

for $x\in(0,\delta_2)$. □

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References

[1] H. Alzer, Ungleichungen für Mittelwerte, *Arch. Math. (Basel)* **47** (1986), no. 5, 422–426.
[2] H. Alzer, A power mean inequality for the gamma function, *Monatsh. Math.* **131** (2000), no. 3, 179–188.
[3] H. Alzer and W. Janous, Solution of problem 8*, *Crux. Math.* **13** (1987) 173–178.
[4] H. Alzer and S. L. Qiu, Inequalities for means in two variables, *Arch. Math. (Basel)* **80** (2003), no. 2, 201–215.
[5] T. J. Bukor and L. Zsilinszky, The logarithmic mean and the power mean of positive numbers, *Octogon Math. Mag.* **2** (1994), no. 1, 19–24.
[6] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, Means and Their Inequalities, D. Reidel Publishing Co., Dordrecht, 1988.
[7] F. Burk, The geometric, logarithmic, and arithmetic mean inequality, *Amer. Math. Monthly* **94** (1987), no. 6, 527–528.
[8] Y. M. Chu and W. F. Xia, Two optimal double inequalities between power mean and logarithmic mean, *Comput. Math. Appl.* **60** (2010), no. 1, 83–89.
[9] P. A. Hästö, Optimal inequalities between Seiffert's mean and power means, *Math. Inequal. Appl.* **7** (2004), no. 1, 47–53.
[10] C. O. Imoru, The power mean and the logarithmic mean, *Internat. J. Math. Math. Sci.* **5** (1982), no. 2, 337–343.
[11] W. Janous, A note on generalized Heronian means, *Math. Inequal. Appl.* **4** (2001), no. 3, 369–375.
[12] T. P. Lin, The power mean and the logarithmic mean, *Amer. Math. Monthly* **81** (1974) 879–883.
[13] B. Long, W. Xia and Y. Chu, An optimal inequality for power mean, geometric mean and harmonic mean, *Int. J. Mod. Math.* **5** (2010), no. 2, 149–155.
[14] F. Qi and B. N. Guo, An inequality between ratio of the extended logarithmic means and ratio of the exponential means, *Taiwanese J. Math.* **7** (2003), no. 2, 229–237.
[15] J. Sándor, A note on some inequalities for means, *Arch. Math. (Basel)* **56** (1991), no. 5, 471–473.
[16] T. Trif, On certain inequalities involving the identric mean in n variables, *Studia Univ. Babeş-Bolyai Math.* **46** (2001), no. 4, 105–114.
[17] M. K. Wang, Y. F. Qiu and Y. M. Chu, Sharp bounds for Seiffert means in terms of Lehmer means, *J. Math. Inequal.* **4** (2010), no. 4, 581–586.

[18] S. H. Wu, Generalization and sharpness of the power means inequality and their applications, *J. Math. Anal. Appl.* **312** (2005), no. 2, 637–652.

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