Bulletin of the Iranian Mathematical Society Vol. 38 No. 3 (2012), pp 625-645.

SOME RESULTS ON WEAKLY CONTRACTIVE MAPS

S. RADENOVIĆ, Z. KADELBURG*, D. JANDRLIĆ AND A. JANDRLIĆ

Communicated by Behzad Djafari-Rouhani

Abstract. In this paper direct proofs of some common fixed point results for two and three mappings under weak contractive conditions are given. Some of these results are improved by using different arguments of control functions. Examples are presented showing that some generalizations can not be obtained and also that our results are distinct from the existing ones.

1. Introduction and preliminaries

1.1. **Quasicontractive and generalized quasicontractive conditions.** It is well known that the Banach contraction principle is a fundamental result in the Fixed Point Theory, which has been used and extended in many different directions. Let (X, d) be a metric space. A map $f : X \to X$ is a contraction if there exists a constant $\lambda \in [0, 1)$ such that for each $x, y \in X$

(1.1) $$d(fx, fy) \leq \lambda d(x, y).$$

MSC(2010): Primary: 47H10; Secondary: 54H25.
Keywords: Common fixed point, generalized weak contraction, generalized quasicontraction, coincidence point.
Received: 11 March 2011, Accepted: 30 March 2011.
*Corresponding author
© 2012 Iranian Mathematical Society.

One possibility is to replace the term $d(x,y)$ on the right-hand side of (1.1) by the maximum $m(x,y)$ of one of the sets

$$M_f^5(x,y) = \{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\},$$
$$M_f^4(x,y) = \left\{d(x,y), d(x,fx), d(y,fy), \tfrac{1}{2}(d(x,fy)+d(y,fx))\right\},$$
$$M_f^3(x,y) = \left\{d(x,y), \tfrac{1}{2}(d(x,fx)+d(y,fy)), \tfrac{1}{2}(d(x,fy)+d(y,fx))\right\}.$$

In this case the proof of existence and uniqueness of a fixed point of f proceeds by the usual construction of a Picard sequence $x_{n+1} = fx_n$, $x_0 \in X$. Results of this kind where first obtained by Ćirić [7].

When two mappings $f, g : X \to X$ are given, and a (unique) common fixed point of f and g is searched for, one can consider the condition

$$d(fx, fy) \le \lambda m(x,y), \tag{1.2}$$

where $m(x,y)$ is the maximum of one of the sets

$$M_{f,g}^5(x,y) = \{d(gx,gy), d(gx,fx), d(gy,fy), d(gx,fy), d(gy,fx)\},$$
$$M_{f,g}^4(x,y) = \left\{d(gx,gy), d(gx,fx), d(gy,fy), \tfrac{1}{2}(d(gx,fy)+d(gy,fx))\right\},$$
$$M_{f,g}^3(x,y) = \Big\{d(gx,gy), \tfrac{1}{2}(d(gx,fx)+d(gy,fy)), \tfrac{1}{2}(d(gx,fy)+d(gy,fx))\Big\}.$$

Obviously, $M_{f,i}^k(x,y) = M_f^k(x,y)$ for $k \in \{3,4,5\}$, where $i : X \to X$ is the identity mapping. In this case, the assumption that $f(X) \subset g(X)$ is usually present. Then the proof uses a Jungck sequence $y_n = fx_n = gx_{n+1}$, $x_0 \in X$. Also, some condition of compatibility of f and g must be present. This approach was used by Das and Naik [8].

Alternatively one can use mappings f and g at some places in the M-sets:

$$N_{f,g}^5(x,y) = \{d(x,y), d(x,fx), d(y,gy), d(x,gy), d(y,fx)\},$$
$$N_{f,g}^4(x,y) = \left\{d(x,y), d(x,fx), d(y,gy), \tfrac{1}{2}(d(x,gy)+d(y,fx))\right\},$$
$$N_{f,g}^3(x,y) = \left\{d(x,y), \tfrac{1}{2}(d(x,fx)+d(y,gy)), \tfrac{1}{2}(d(x,gy)+d(y,fx))\right\}.$$

In this case (1.2) is usually replaced by:

$$d(fx, gy) \le \lambda m(x,y), \tag{1.3}$$

where $m(x,y)$ is the maximum of one of the N-sets. In these cases, the construction of Jungck sequence follows the relations $x_{2n+1} = fx_{2n}$, $x_{2n+2} = gx_{2n+1}$, $x_0 \in X$.

A further generalization is to consider three or four mappings and search for their common fixed point. In the case of four mappings $f, g, S, T : X \to X$, the N-sets take the form

$$N^5_{f,g,S,T}(x,y) = \{d(Sx,Ty), d(Sx,fx), d(Ty,gy), d(Sx,gy), d(Ty,fx)\},$$
$$N^4_{f,g,S,T}(x,y) = \{d(Sx,Ty), d(Sx,fx), d(Ty,gy), \tfrac{1}{2}(d(Sx,gy)+d(Ty,fx))\},$$
$$N^3_{f,g,S,T}(x,y) = \{d(Sx,Ty), \tfrac{1}{2}(d(Sx,fx)+d(Ty,gy)), \tfrac{1}{2}(d(Sx,gy)+d(Ty,fx))\},$$

and the contractive condition is in the form (1.3).

In the case of three mappings $f, g, S : X \to X$, one takes $T = S$ and considers the sets

$$N^k_{f,g,S}(x,y) = N^k_{f,g,S,S}(x,y), \quad k \in \{3,4,5\},$$

and the contractive condition is again in the form (1.3).

Let us mention also that sometimes some of the subsets of M-sets or N-sets are used in respective contractive conditions.

1.2. **Weak and generalized weak contractive conditions.** Consider the following two sets of real functions:

$$\Psi = \{\, \psi : [0,+\infty) \to [0,+\infty) \mid \psi \text{ is continuous nondecreasing, and } \psi^{-1}(\{0\}) = \{0\}\,\},$$
$$\Phi = \{\, \varphi : [0,+\infty) \to [0,+\infty) \mid \varphi \text{ is lower semi-continuous, and } \varphi^{-1}(\{0\}) = \{0\}\,\}.$$

The concept of a weak contraction was introduced in 1997 by Alber and Guerre-Delabriere [3]. A map $f : X \to X$ is called a φ-weak contraction if there exists a function $\varphi \in \Phi$ such that for each $x, y \in X$

$$d(fx,fy) \le d(x,y) - \varphi(d(x,y)).$$

They defined this concept for maps on Hilbert spaces and proved the existence of fixed points. In 2001 Rhoades showed that most of the results from [3] are still true for any Banach space. In particular, he proved the following theorem which is obviously one of the generalizations of the Banach contraction principle because it contains contraction as a special case ($\varphi(t) = (1-k)t$).

Theorem 1.1. [17] *Let (X, d) be a complete metric space, and f be a φ-weak contraction on X for some $\varphi \in \Phi$. Then f has a unique fixed point.*

Introducing a new generalization of the contraction principle, Dutta and B.S. Choudhury proved the following theorem.

Theorem 1.2. [11] *Let (X, d) be a complete metric space and let $f : X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \varphi(d(x, y)),$$

for some $\psi \in \Psi$ and $\varphi \in \Phi$ and all $x, y \in X$. Then f has a unique fixed point.

In generalizing this theorem of Dutta and Choudhury, Beg and Abbas proved the following result.

Theorem 1.3. [4] *Let (X, d) be a metric space and let f be a weakly contractive mapping with respect to g, that is,*

$$\psi(d(fx, fy)) \leq \psi(d(gx, gy)) - \varphi(d(gx, gy)),$$

for some $\psi \in \Psi$ and $\varphi \in \Phi$ and all $x, y \in X$. If $fX \subset gX$, and gX is a complete subspace of X, then f and g have coincidence point in X.

Further, Zhang and Song used the generalized φ-weak contraction which is defined for two mappings and gave conditions for the existence of a common fixed point.

Theorem 1.4. [18] *Let (X, d) be a complete metric space, and $f, g : X \to X$ two mappings such that for all $x, y \in X$*

$$d(fx, gy) \leq m(x, y) - \varphi(m(x, y)),$$

where $\varphi \in \Phi$, and $m(x, y) = \max N^4_{f,g}(x, y)$. Then there exists a unique common fixed point of f and g.

Recently, Djorić extended the result of Zhang and Song to a pair of (ψ, φ)-weak contractive mappings. He proved the following theorem.

Theorem 1.5. [10] *Let (X, d) be a complete metric space, and let $f, g : X \to X$ be two self-mappings such that for all $x, y \in X$*

$$\psi(d(fx, gy)) \leq \psi(m(x, y)) - \varphi(m(x, y)), \tag{1.4}$$

for some $\varphi \in \Phi$, $\psi \in \Psi$, and $m(x, y) = \max N^4_{f,g}(x, y)$. Then there exists a unique common fixed point of f and g.

Very recently, Choudhury et al. proved the following two theorems.

Theorem 1.6. [6] *Let (X, d) be a complete metric space, and let $f : X \to X$ be such that*

$$\psi(d(fx, fy)) \leq \psi(m(x, y)) - \varphi(\max\{d(x, y), d(y, fy)\}),$$

for some $\varphi \in \Phi$, $\psi \in \Psi$, and $m(x, y) = \max M_f^4(x, y)$. Then f has a unique fixed point.

Theorem 1.7. [6] *Let (X, d) be a complete metric space. Let $f, g : X \to X$ be self-mappings such that for all $x, y \in X$,*

$$\psi(d(fx, gy)) \leq \psi(m(x, y)) - \varphi(\max\{d(x, y), d(x, fx), d(y, gy)\}),$$

where $\varphi \in \Phi$, $\psi \in \Psi$, and $m(x, y) = \max N_{f,g}^4(x, y)$. Then f and g have a unique common fixed point. Moreover, any fixed point of f is a fixed point of g and conversely.

The following weak-contractive common fixed point result for four mappings was recently obtained by Abbas and Djorić.

Theorem 1.8. [2] *Suppose that f, g, S, T are self-mappings on a complete metric space (X, d), $fX \subset TX$, $gX \subset SX$, and let the pairs (f, S) and (g, T) be weakly compatible. If for some $\varphi \in \Phi$, $\psi \in \Psi$ and for all $x, y \in X$,*

$$\psi(d(fx, gy)) \leq \psi(m(x, y)) - \varphi(m(x, y))$$

holds, where $m(x, y) = \max N_{f,g,S,T}^4(x, y)$, then f, g, S, T have a unique common fixed point, provided one of the ranges fX, gX, SX, TX is closed.

Let us mention also an important recent paper of Jachymski [12] who showed that some of the results involving two functions $\varphi \in \Phi$ and $\psi \in \Psi$ can be reduced to the case of one function $\varphi' \in \Phi$. It was also shown by Popescu in [16] that conditions on functions ψ and φ can be weakened.

Some other works related to the concept of weak contractions are [1, 5, 9, 14].

In this paper direct proofs of some common fixed point results for two and three mappings under weak contractive conditions are given. Some of these results are slightly improved by using different arguments of control functions. Examples are presented showing that some generalizations can not be obtained and also that our results are distinct from the existing ones. Also, some improvements of results of weak Hardy-Rogers type are given.

2. **An auxiliary result**

Assertions similar to the following lemma were used (and proved) in the course of proofs of several fixed point results in various papers.

Lemma 2.1. *Let (X, d) be a metric space and let $\{y_n\}$ be a sequence in X such that $d(y_{n+1}, y_n)$ is nonincreasing and that*

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = 0. \tag{2.1}$$

If $\{y_{2n}\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \to \infty$:

(2.2)

$$d(y_{2m_k}, y_{2n_k}), \quad d(y_{2m_k}, y_{2n_k+1}), \quad d(y_{2m_k-1}, y_{2n_k}), \quad d(y_{2m_k-1}, y_{2n_k+1}).$$

Proof. If $\{y_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that

$$n_k > m_k > k, \quad d(y_{2m_k}, y_{2n_k-2}) < \varepsilon, \quad d(y_{2m_k}, y_{2n_k}) \geq \varepsilon$$

for all positive integers k. Then

$$\begin{aligned}\varepsilon \leq d(y_{2m_k}, y_{2n_k}) &\leq d(y_{2m_k}, y_{2n_k-2}) + d(y_{2n_k-2}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2n_k}) \\ &< \varepsilon + d(y_{2n_k-2}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2n_k}).\end{aligned}$$

Using (2.1) we conclude that

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \varepsilon. \tag{2.3}$$

Further,

$$d(y_{2m_k}, y_{2n_k}) \leq d(y_{2m_k}, y_{2n_k+1}) + d(y_{2n_k+1}, y_{2n_k}),$$

and

$$d(y_{2m_k}, y_{2n_k+1}) \leq d(y_{2m_k}, y_{2n_k}) + d(y_{2n_k}, y_{2n_k+1}).$$

Passing to the limit when $k \to \infty$ and using (2.1) and (2.3), we obtain that

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k+1}) = \varepsilon.$$

That the remaining two sequences in (2.2) tend to ε can be proved similarly. $\square$

3. **Weak contractions for three mappings**

In this section we prove the existence of coincidence point and common fixed point for three mappings satisfying the generalized (ψ, φ)-contractive condition. Our main result is the following theorem.

Theorem 3.1. *Let (X, d) be a metric space, and let $f, g, S : X \to X$ be three mappings such that for all $x, y \in X$*

$$\psi(d(fx, gy)) \leq \psi(m(x, y)) - \varphi(m_1(x, y)), \tag{3.1}$$

for some $\varphi \in \Phi$, $\psi \in \Psi$, where $m(x, y) = \max N^4_{f,g,S}(x, y)$ and

$$m_1(x, y) = \max\{d(Sx, Sy), d(fx, Sx), d(gy, Sy)\}.$$

If $fX \cup gX \subset SX$ and SX is a complete subspace of X, then f, g and S have a unique point of coincidence. Moreover, if (f, S) and (g, S) are weakly compatible, then f, g and S have a unique common fixed point.

Recall that a point $u \in X$ is called a coincidence point of the pair (f, g) and v is its point of coincidence if $fu = gu = v$. The pair (f, g) is said to be weakly compatible if for each $x \in X$, $fx = gx$ implies $fgx = gfx$.

Proof. Let $x_0 \in X$ be arbitrary. Using the condition $fX \cup gX \subset SX$, choose a sequence $\{x_n\}$ such that $Sx_{2n+1} = fx_{2n}$ and $Sx_{2n+2} = gx_{2n+1}$ for all $n \in \mathbb{N}_0$. Applying the contractive condition (3.1) we obtain that

$$\begin{aligned}
\psi(d(Sx_{2n+1}, Sx_{2n+2})) &= \psi(d(fx_{2n}, gx_{2n+1})) \\
&\leq \psi(m(x_{2n}, x_{2n+1})) - \varphi(m_1(x_{2n}, x_{2n+1})) \\
&\leq \psi(m(x_{2n}, x_{2n+1})),
\end{aligned}$$

which implies that $d(Sx_{2n+1}, Sx_{2n+2}) \leq m(x_{2n}, x_{2n+1})$. In a similar way one obtains that $d(Sx_{2n+2}, Sx_{2n+3}) \leq m(x_{2n+1}, x_{2n+2})$.

Now from the triangle inequality for metric d we have

$$\begin{aligned}
m(x_{2n}, x_{2n+1}) &= \max\{d(Sx_{2n}, Sx_{2n-1}), d(Sx_{2n+1}, Sx_{2n}), \\
&\qquad d(Sx_{2n}, Sx_{2n-1}), \tfrac{1}{2}(d(Sx_{2n-1}, Sx_{2n+1}) + d(Sx_{2n}, Sx_{2n}))\} \\
&= \max\{d(Sx_{2n}, Sx_{2n-1}), d(Sx_{2n+1}, Sx_{2n}), \tfrac{1}{2}d(Sx_{2n-1}, Sx_{2n+1})\} \\
&\leq \max\{d(Sx_{2n}, Sx_{2n-1}), d(Sx_{2n+1}, Sx_{2n}), \\
&\qquad \tfrac{1}{2}(d(Sx_{2n-1}, Sx_{2n}) + d(Sx_{2n}, Sx_{2n+1}))\} \\
&= \max\{d(Sx_{2n}, Sx_{2n-1}), d(Sx_{2n+1}, Sx_{2n})\},
\end{aligned}$$

and similarly,

$$m(x_{2n+1}, x_{2n+2}) \leq \max\{d(Sx_{2n+1}, Sx_{2n}), d(Sx_{2n+2}, Sx_{2n+1})\}.$$

If $d(Sx_{2n+1}, Sx_{2n}) > d(Sx_{2n}, Sx_{2n-1})$, then $m(x_{2n}, x_{2n+1}) = d(Sx_{2n+1}, Sx_{2n}) > 0$, and also $m_1(x_{2n}, x_{2n+1}) = d(Sx_{2n+1}, Sx_{2n}) > 0$. Contractivity condition (3.1) would imply that

$$\psi(d(Sx_{2n+1}, Sx_{2n})) \leq \psi(d(Sx_{2n+1}, Sx_{2n})) - \varphi(d(Sx_{2n+1}, Sx_{2n})),$$

and, since $\varphi \in \Phi$, $d(Sx_{2n+1}, Sx_{2n}) = 0$. A contradiction.

So, we have

$$d(Sx_{2n+1}, Sx_{2n}) \leq m(x_{2n}, x_{2n+1}) \leq d(Sx_{2n}, Sx_{2n-1}). \tag{3.2}$$

Similarly, we obtain that

$$d(Sx_{2n+2}, Sx_{2n+1}) \leq m(x_{2n+1}, x_{2n+2}) \leq d(Sx_{2n+1}, Sx_{2n}). \tag{3.3}$$

Therefore, the sequence $\{d(Sx_{n+1}, Sx_n)\}$ is nonincreasing and bounded from below. So,

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = \lim_{n\to\infty} m(x_{n+1}, x_n) = \lim_{n\to\infty} m_1(x_{n+1}, x_n) = r \geq 0.$$

Letting $n \to \infty$ in inequality

$$\psi(d(Sx_{n+1}, Sx_n)) \leq \psi(m(x_{n+1}, x_n)) - \varphi(m_1(x_{n+1}, x_n))$$

we obtain $\psi(r) \leq \psi(r) - \varphi(r)$ which is a contradiction unless $r = 0$. Hence,

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = 0. \tag{3.4}$$

We next prove that $\{Sx_n\}$ is a Cauchy sequence. According to (3.4), it is sufficient to show that the subsequence $\{Sx_{2n}\}$ is a Cauchy sequence. Suppose that this is not the case. Applying Lemma 2.1 to the sequence $y_n = Sx_n$ we obtain that there exist $\varepsilon > 0$ and two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that the sequences

$$\begin{aligned} d(Sx_{2m_k}, Sx_{2n_k}), &\quad d(Sx_{2m_k}, Sx_{2n_k+1}),\\ d(Sx_{2m_k-1}, Sx_{2n_k}), &\quad d(Sx_{2m_k-1}, Sx_{2n_k+1}). \end{aligned}$$

all tend to ε when $k \to \infty$.

Now, from the definition of $m(x,y)$ and from (3.2), (3.3) and the obtained limits, we have

$$\lim_{k\to\infty} m(x_{2m_k-1}, x_{2n_k}) = \lim_{k\to\infty} m_1(x_{2m_k-1}, x_{2n_k}) = \varepsilon. \tag{3.5}$$

Putting $x = x_{2n_k}$, $y = x_{2m_k-1}$ in (3.1) we have

$$\begin{aligned}\psi(d(Sx_{2n_k+1}, Sx_{2m_k})) &= \psi(d(fx_{2n_k}, g_{2m_k-1})) \\ &\le \psi(m(x_{2n_k}, x_{2m_k-1})) - \varphi(m_1(x_{2n_k}, x_{2m_k-1})).\end{aligned}$$

Letting $k \to \infty$, utilizing (3.5) and the obtained limits, we get

$$\psi(\varepsilon) \le \psi(\varepsilon) - \varphi(\varepsilon),$$

which is a contradiction if $\varepsilon > 0$.

This shows that $\{Sx_{2n}\}$ is a Cauchy sequence and hence $\{Sx_n\}$ is a Cauchy sequence.

Since the subspace SX is complete, there exist $u, v \in X$ such that $Sx_n \to v = Su$ $(n \to \infty)$. We shall prove that $Su = fu = gu$.

To see it, we have

$$\begin{aligned}\psi(d(fu, Sx_{2n+2})) &= \psi(d(fu, gx_{2n+1})) \\ &\le \psi(m(u, x_{2n+1})) - \varphi(m_1(u, x_{2n+1})),\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}m(u, x_{2n+1}) &= \max\{d(Su, Sx_{2n+1}), d(fu, Su), d(gx_{2n+1}, Sx_{2n+1}), \\ &\qquad \tfrac{1}{2}(d(fu, Sx_{2n+1}) + d(gx_{2n+1}, Su))\} \\ &\to \max\{0, d(fu, Su), 0, \tfrac{1}{2}d(fu, Su)\} = d(fu, Su),\end{aligned}$$

as $n \to \infty$, and also

$$\begin{aligned}m_1(u, x_{2n+1}) &= \max\{d(Su, Sx_{2n+1}), d(fu, Su), d(gx_{2n+1}, Sx_{2n+1})\} \\ &\to d(fu, Su).\end{aligned}$$

Letting $n \to \infty$ in (3.6) we obtain

$$\psi(d(fu, Su)) \le \psi(d(fu, Su)) - \varphi(d(fu, Su))$$

which implies $\varphi(d(fu, Su)) = 0$. Hence, $d(fu, Su) = 0$, i.e., $fu = Su = v$.

Similarly, using that

$$\begin{aligned}\psi(d(Sx_{2n+1}, gu)) &= \psi(d(fx_{2n}, gu)) \\ &\le \psi(m(x_{2n}, u)) - \varphi(m_1(x_{2n}, u)),\end{aligned}$$

where

$$\begin{aligned}m(x_{2n}, u) &= \max\{d(Sx_{2n}, Su), d(fx_{2n}, Sx_{2n}), d(gu, Su), \\ &\qquad \tfrac{1}{2}(d(fx_{2n}, Su) + d(gu, Sx_{2n}))\} \\ &\to \max\{0, 0, d(gu, Su), \tfrac{1}{2}(d(gu, Su))\} = d(gu, Su),\end{aligned}$$

and $m_1(x,y) \to d(gu, Su)$, it can be deduced that $gu = Su = v$. It follows that v is a common point of coincidence for f, g and S, i.e.,

$$v = fu = gu = Su.$$

Now we prove that the point of coincidence of f, g, S is unique. Suppose that there is another point $v_1 \in X$ such that $v_1 = fu_1 = gu_1 = Su_1$ for some $u_1 \in X$. Using condition (3.1) we obtain that

$$\psi(d(v,v_1)) = \psi(d(fu, gu_1)) \leq \psi(m(u,u_1)) - \varphi(m_1(u,u_1)), \tag{3.7}$$

where

$$\begin{aligned} m(u,u_1) &= \max\{d(Su,Su_1), d(fu,Su), d(gu_1,Su_1), \\ &\qquad \tfrac{1}{2}(d(fu,Su_1) + d(gu_1,Su))\} \\ &= \max\{d(v,v_1), 0, 0, \tfrac{1}{2}(d(v,v_1) + d(v_1,v))\} = d(v,v_1), \end{aligned}$$

and $m_1(u,u_1) = \max\{d(Su,Su_1), d(fu,Su), d(gu_1,Su_1)\} = d(v,v_1)$. Now, (3.7) implies that $\varphi(d(v,v_1)) = 0$, i.e., $d(v,v_1) = 0$. Hence, $v = v_1$.

Using weak compatibility of the pairs (f,S) and (g,S) and a classical result of G. Jungck, we conclude that the mappings f, g, S have a unique common fixed point, i.e., $fv = gv = Sv = v$. The proof of the theorem is complete. □

By setting $S = i_X$ (the identity mapping of X) in Theorem 3.1, we obtain Theorem 1.7 of Choudhury et al., which is a variation of Theorem 1.5 of Djorić.

By setting $T = S$ in Theorem 1.8 of Abbas and Djorić, one obtains a slightly weaker variant of our Theorem 3.1.

Now, we give an example of the case when Theorem 3.1 can be applied, while the known results are not applicable.

Example 3.2. *Let $X = [0,1]$ be endowed with the Euclidean metric $d(x,y) = |x-y|$ and let $f(x) = g(x) = \frac{1}{3}x$, $S(x) = \frac{2}{3}x$ for each $x \in X$.*

Then $d(fx, gy) = \frac{1}{3}|x - y|$ *and*

$$\begin{aligned} m(x,y) =& \max\{d(Sx,Sy), d(Sx,fx), d(Sy,gy), \tfrac{1}{2}(d(Sy,fx) + d(Sx,gy))\} \\ &= \max\{\tfrac{2}{3}|x-y|, \tfrac{1}{3}x, \tfrac{1}{3}y, \tfrac{1}{6}(|x-2y| + |y-2x|)\} \\ &= \begin{cases} \frac{2}{3}(x-y), & 0 \le y \le \frac{x}{2} \\ \frac{x}{3}, & \frac{x}{2} \le y \le x \\ \frac{y}{3}, & x \le y \le 2x \\ \frac{2}{3}(y-x), & 2x \le y \le 1 \end{cases} \\ &= m_1(x,y). \end{aligned}$$

For $\psi(t) = 3t$ *and* $\varphi(t) = t$ *we have* $\psi(d(fx, gy)) = \psi(\frac{1}{3}|x - y|) = |x - y|$ *and*

$$\psi(m(x,y)) - \varphi(m_1(x,y)) = \begin{cases} \frac{4}{3}(x-y), & 0 \le y \le \frac{x}{2} \\ \frac{2x}{3}, & \frac{x}{2} \le y \le x \\ \frac{2y}{3}, & x \le y \le 2x \\ \frac{4}{3}(y-x), & 2x \le y \le 1 \end{cases}$$

Now we easily conclude that the mappings f, g *and* $S \neq i_X$ *satisfy the relation* (3.1). *Hence, the existence of a common fixed point of these three mappings under weak contractive conditions follows from Theorem 3.1.*

The proof of the following theorem is similar to that of Theorem 3.1.

Theorem 3.3. *Let* (X, d) *be a complete metric space, and* $f, g, S : X \to X$ *three mappings such that for all* $x, y \in X$

$$\psi(d(fx, gy)) \le \psi(m(x, y)) - \varphi(m_1(x, y)), \tag{3.8}$$

where $m(x, y) = \max N^3_{f,g,S}(x, y)$ *and*

$$m_1(x, y) = \max\{d(Sx, Sy), d(fx, Sx), d(gy, Sy)\},$$

and where $\psi \in \Psi$ *and* $\varphi \in \Phi$. *If* $fX \cup gX \subset SX$ *and* SX *is a complete subspace of* X, *then* f, g *and* S *have a unique point of coincidence. Moreover, if* (f, S) *and* (g, S) *are weakly compatible, then* f, g *and* S *have a unique common fixed point.*

The following example shows that the same conclusion may not hold if one takes $m(x, y) = \max N^5_{f,g,S}(x, y)$.

Example 3.4. *Let $X = \{p, q, r, s\}$, where $p = (0, 0, 0)$, $q = (4, 0, 0)$, $r = (2, 2, 0)$, $s = (2, -2, 1)$, and let d be the Euclidean metric in $\mathbb{R}^3$. Consider the mappings: $S = i_X$, identity mapping of X, $fp = fs = r$, $fq = fr = s$, $gp = gs = q$, $gq = gr = p$. By a careful computation it is easy to obtain that*

$$d(fx, gy) \leq \frac{3}{4} \max\{d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)\},$$

for all $x, y \in X$. This means that taking $\psi(t) = t$, $\varphi(t) = \frac{1}{4}t$, the condition (3.8) is satisfied, because it reduces to $d(fx, gy) \leq \frac{3}{4}M(x, y)$. Obviously, the mappings f, g and S have no fixed points.

Remark 3.5. *Note that if*

$$m(x, y) = \max\left\{d(Sx, Sy), d(fx, Sx), d(gy, Sy), d(fx, Sy), \tfrac{1}{2}d(gy, Sx)\right\},$$

then the result can again be obtained.

Finally, we state the following generalizations of Theorems 1.6 and 1.7.

Theorem 3.6. *Let (X, d) be a metric space, and let $f, S : X \to X$ be mappings such that for all $x, y \in X$*

$$\psi(d(fx, fy)) \leq \psi(m(x, y)) - \varphi(\max\{d(Sx, Sy), d(Sy, fy)\}),$$

where $\varphi \in \Phi$, $\psi \in \Psi$, and $m(x, y) \in M^4_{f,S}(x, y)$. If $fX \subset SX$ and SX is a complete subspace of X, then f and S have a unique point of coincidence. Moreover, if (f, S) is weakly compatible, then f and S have a unique common fixed point.

Theorem 3.7. *Let (X, d) be a metric space, and $f, g, S : X \to X$ are three self-mappings such that for all $x, y \in X$,*

$$\psi(d(fx, gy)) \leq \psi(m(x, y)) - \varphi(\max\{d(Sx, Sy), d(Sx, fx), d(Sy, gy)\}),$$

where $\varphi \in \Phi$, $\psi \in \Psi$, and $m(x, y) \in N^4_{f,g,S}(x, y)$. If $fX \cup gX \subset SX$ and SX is a complete subspace of X, then f, g and S have a unique point of coincidence. Moreover, if (f, S) and (g, S) are weakly compatible, then f, g and S have a unique common fixed point.

4. Weak contractions for two mappings

In this subsection we deduce another variation of Djorić's Theorem 1.5 on weak contractions for two mappings. The difference is that we do not use the maximum of the N-set, but its arbitrary element.

Theorem 4.1. *Let (X,d) be a complete metric space and $f,g : X \to X$ be two mappings such that for some $\varphi \in \Phi$, $\psi \in \Psi$ and for all $x, y \in X$ there exists $u(x,y) \in N^4_{f,g}(x,y)$ such that*

$$\psi(d(fx,gy)) \le \psi(u(x,y)) - \varphi(u(x,y)). \tag{4.1}$$

Then f and g have a unique common fixed point.

Proof. Let us prove first that the common fixed point of f and g is unique (if it exists). Suppose that $p \ne q$ are two distinct common fixed points of f and g. Then (4.1) implies that

$$\psi(d(p,q)) = \psi(d(fp,gq)) \le \psi(u(p,q)) - \varphi(u(p,q)),$$

where $u(p,q) \in N^4_{f,g}(p,q) = \{d(p,q),0,0,d(p,q)\} = \{0,d(p,q)\}$. Checking both possible cases, we readily obtain that $d(p,q) = 0$, i.e., $p = q$.

In order to prove the existence of a common fixed point, proceed in the usual way, constructing a Jungck sequence by $x_{2n+1} = fx_{2n}$, $x_{2n+2} = gx_{2n+1}$, for arbitrary $x_0 \in X$. Consider the two possible cases.

Suppose that $x_n = x_{n+1}$ for some $n \in \mathbb{N}$. Then $x_{n+1} = x_{n+2}$ and it follows that the sequence is eventually constant, and so convergent. Indeed, let, e.g., $n = 2k$ (in the case $n = 2k+1$ the proof is similar). Then, putting $x = x_{2k}$, $y = x_{2k+1}$ in (4.1), we get that there exists

$$\begin{aligned} u \in \{&d(x_{2k},x_{2k+1}), d(x_{2k},fx_{2k}), d(x_{2k+1},gx_{2k+1}),\\ &\tfrac{1}{2}(d(x_{2k},gx_{2k+1}) + d(x_{2k+1},fx_{2k}))\}\\ = \{&0, d(x_{2k+1},x_{2k+2}), \tfrac{1}{2}d(x_{2k},x_{2k+2})\},\end{aligned}$$

such that $\psi(d(x_{2k+1},x_{2k+2}) \le \psi(u) - \varphi(u)$. Consider the three possible cases:

1° $u = 0$; it trivially follows that $x_{2k} = x_{2k+1}$.

2° $u = d(x_{2k+1},x_{2k+2})$; it follows that

$$\psi(d(x_{2k+1},x_{2k+2})) \le \psi(d(x_{2k+1},x_{2k+2})) - \varphi(d(x_{2k+1},x_{2k+2})),$$

and by the properties of functions ψ and φ that $x_{2k} = x_{2k+1}$.

3° $u = \frac{1}{2}d(x_{2k},x_{2k+2})$; since $u \le \frac{1}{2}(d(x_{2k},x_{2k+1}) + d(x_{2k+1},x_{2k+2})) = \frac{1}{2}d(x_{2k+1},x_{2k+2})$, it follows that

$$\begin{aligned}\psi(d(x_{2k+1},x_{2k+2}) &\le \psi(\frac{1}{2}d(x_{2k+1},x_{2k+2})) - \varphi(\tfrac{1}{2}d(x_{2k},x_{2k+2}))\\ &\le \psi(\frac{1}{2}d(x_{2k+1},x_{2k+2})),\end{aligned}$$

implying that $d(x_{2k+1}, x_{2k+2} \leq \frac{1}{2}d(x_{2k+1}, x_{2k+2})$ which is only possible if $x_{2k} = x_{2k+1}$.

Now suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Putting $x = x_{2n}$, $y = x_{2n-1}$ in (4.1), we get that there exists

$$\begin{aligned} u \in \{&d(x_{2n}, x_{2n-1}), d(x_{2n}, fx_{2n}), d(x_{2n-1}, gx_{2n-1}), \\ &\tfrac{1}{2}(d(x_{2n}, gx_{2n-1}) + d(x_{2n-1}, fx_{2n}))\} \\ = \{&d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), \tfrac{1}{2}d(x_{2n-1}, x_{2n+1})\}, \end{aligned}$$

such that $\psi(d(x_{2n+1}, x_{2n}) \leq \psi(u) - \varphi(u)$. Consider the three possible cases:

1° $u = d(x_{2n}, x_{2n-1})$; it follows that

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n})) &\leq \psi(d(x_{2n}, x_{2n-1})) - \varphi(d(x_{2n}, x_{2n-1})) \\ &< \psi(d(x_{2n}, x_{2n-1})) \end{aligned}$$

and $d(x_{2n+1}, x_{2n}) < d(x_{2n}, x_{2n-1})$.

2° $u = d(x_{2n}, x_{2n+1})$; it follows that

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n})) &\leq \psi(d(x_{2n}, x_{2n+1})) - \varphi(d(x_{2n}, x_{2n+1})) \\ &< \psi(d(x_{2n}, x_{2n+1})), \end{aligned}$$

which is impossible.

3° $u = \frac{1}{2}d(x_{2n-1}, x_{2n+1})$; it follows that

$$\psi(d(x_{2n+1}, x_{2n})) \leq \psi(\frac{1}{2}d(x_{2n-1}, x_{2n+1})) - \varphi(\frac{1}{2}d(x_{2n-1}, x_{2n+1})).$$

By the properties of functions ψ and φ we obtain that
$d(x_{2n+1}, x_{2n}) \leq \frac{1}{2}(d(x_{2n-1}, x_{2n}) + d(x_{2n} + d_{2n+1}))$
and $d(x_{2n+1}, x_{2n}) \leq d(x_{2n}, x_{2n-1})$.

Hence, at any possible case, $d(x_{2n+1}, x_{2n}) \leq d(x_{2n}, x_{2n-1})$ and, similarly, $d(x_{2n+2}, x_{2n+1}) \leq d(x_{2n+1}, x_{2n})$. Thus, the sequence $\{d(x_n, x_{n+1})\}$ is nonincreasing; moreover,

$$d(x_{2n+2}, x_{2n+1}) \leq u(x_{2n+1}, x_{2n}) \leq d(x_{2n+1}, x_{2n}), \tag{4.2}$$

$$d(x_{2n+1}, x_{2n}) \leq u(x_{2n}, x_{2n-1}) \leq d(x_{2n}, x_{2n-1}). \tag{4.3}$$

Now we prove that

$$d(x_n, x_{n+1}) \to 0 \quad n \to \infty. \tag{4.4}$$

Indeed, passing to the limit in (4.2) and (4.3) when $n \to \infty$, we obtain that $d(x_{n+1}, x_n) \to r$ and $u(x_{n+1}, x_n) \to r$ $(n \to \infty)$ for some $r \geq 0$. If

$r > 0$, then passing to the limit in

$$\psi(d(x_{2n+1}, x_{2n+2})) \leq \psi(u(x_{2n}, x_{2n+1})) - \varphi(u(x_{2n}, x_{2n+1}))$$

we obtain that $\psi(r) \leq \psi(r) - \varphi(r)$ and $r = 0$ by the properties of functions $\psi \in \Psi$, $\varphi \in \Phi$. Hence, (4.4) holds.

We next prove that $\{x_n\}$ is a Cauchy sequence. According to (4.4), it is sufficient to show that the subsequence $\{x_{2n}\}$ is a Cauchy sequence. Suppose that this is not the case. Applying Lemma 2.1 we obtain that there exist $\varepsilon > 0$ and two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that the sequences

$$d(x_{2m_k}, x_{2n_k}), \quad d(x_{2m_k}, x_{2n_k+1}), \quad d(x_{2m_k-1}, x_{2n_k}), \quad d(x_{2m_k-1}, x_{2n_k+1}).$$

all tend to ε when $k \to \infty$.

Now, from (4.2), (4.3) and the obtained limits, we have

$$\lim_{k\to\infty} u(x_{2m_k-1}, x_{2n_k}) = \varepsilon, \tag{4.5}$$

for any $u(x_{2m_k-1}, x_{2n_k}) \in N^4_{f,g}(x_{2m_k-1}, x_{2n_k})$. Letting $k \to \infty$, utilizing (4.5) and the obtained limits, we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon),$$

which is a contradiction if $\varepsilon > 0$.

This shows that $\{x_{2n}\}$ is a Cauchy sequence and hence $\{x_n\}$ is a Cauchy sequence.

Since the space (X, d) is complete, there exists $p \in X$ such that $\lim_{n\to\infty} x_n = p$. Then also $x_{2n+1} = fx_{2n} \to p$ and $x_{2n} = gx_{2n-1} \to p$ $(n \to \infty)$. Putting $x = x_{2n}$ and $y = p$ in (4.1), we get $\psi(d(fx_{2n}, gp)) \leq \psi(u) - \varphi(u)$, where

$$u \in \{d(x_{2n}, p), d(x_{2n}, fx_{2n}), d(p, gp), \tfrac{1}{2}(d(x_{2n}, gp) + d(p, fx_{2n}))\}.$$

So, in this case we have four possibilities:

1° $\psi(d(fx_{2n}, gp)) \leq \psi(d(x_{2n}, p)) - \varphi(d(x_{2n}, p))$;

2° $\psi(d(fx_{2n}, gp)) \leq \psi(d(x_{2n}, fx_{2n})) - \varphi(d(x_{2n}, fx_{2n}))$;

3° $\psi(d(fx_{2n}, gp)) \leq \psi(d(p, gp)) - \varphi(d(p, gp))$;

4° $\psi(d(fx_{2n}, gp)) \leq \psi(\frac{1}{2}(d(x_{2n}, gp) + d(p, fx_{2n}))) - \varphi(\frac{1}{2}(d(x_{2n}, gp) + d(p, fx_{2n})))$.

Passing to the limit when $n \to \infty$ in these four relations, we obtain one of the next three inequalities:

$$\psi(d(p, gp)) \leq \psi(0) - \varphi(0), \quad \psi(d(p, gp)) \leq \psi(d(p, gp)) - \varphi(d(p, gp)),$$

$$\psi(d(p, gp)) \leq \psi(\tfrac{1}{2}d(p, gp)) + \varphi(\tfrac{1}{2}d(p, gp)) \leq \psi(d(p, gp)) + \varphi(\tfrac{1}{2}d(p, gp)).$$

In each of the cases it easily follows that $gp = p$.

Now, putting $x = y = p$ in (4.1), one gets

$$\psi(d(fp, gp)) \le \psi(u) - \varphi(u),$$

where $u \in \{0, d(p, fp), \frac{1}{2}d(p, fp)\}$ and in each of the possible three cases it easily follows that $fp = p$. Hence, p is a common fixed point of f and g. □

Putting $g = f$ in Theorem 4.1, one obtains

Corollary 4.2. *Let (X, d) be a complete metric space and $f : X \to X$ be such that for some $\varphi \in \Phi$, $\psi \in \Psi$ and for all $x, y \in X$ there exists $u(x, y) \in M_f^4(x, y)$ such that*

$$\psi(d(fx, fy)) \le \psi(u(x, y)) - \varphi(u(x, y)).$$

Then f has a unique common fixed point.

Remark 4.3. *If we use the contractivity condition in the form*

$$\psi(d(fx, gy)) \le \psi(\max N_{f,g}^4(x, y)) - \varphi(u),$$

where $u \in N_{f,g}^4(x, y)$, instead of (4.1)*, then it may happen that common fixed point of f and g is not unique. Indeed, take $f = g = i_X$. Then $N_{f,g}^4(x, y) = \{0, d(x, y)\}$, so $\max N_{f,g}^4(x, y) = d(x, y)$ and when $u = 0$ the condition is fulfilled.*

5. Weak conditions of Hardy-Rogers type

If (X, d) is a metric space and $f : X \to X$ a selfmap, then the following distances are usually used in forming several contractive conditions:

$$d(x, y),\ d(x, fx),\ d(y, fy),\ d(x, fy),\ d(y, fx),\ d(fx, fy),$$

for distinct points $x, y \in X$. In Sections 1, 3 and 4 we have constructed several M- and N-sets and used them to form weak contractive conditions. In this section we consider a kind of convex combinations of these distances (as is done in the so-called Hardy-Rogers contractive conditions) and form the respective weak conditions.

For example, we can consider the expressions

(5.1)
$$\Theta_f^5(x, y) = Ad(x, y) + Bd(x, fx) + Cd(y, fy) + Dd(x, fy) + Ed(y, fx),$$

(5.2)
$$\Theta_f^4(x, y) = ad(x, y) + bd(x, fx) + cd(y, fy) + e[d(x, fy) + d(y, fx)],$$

where 1° $A > 0$, $B, C, D, E \geq 0$, $A + B + C + D + E \leq 1$ (for expression (5.1)), and 2° $a > 0$, $b, c, e \geq 0$, $a + b + c + 2e \leq 1$ (for expression (5.2)).

In the recent paper [15], H.K. Nashine and I. Altun considered (in the frame of ordered metric spaces) the weak contractive condition in the form

$$\psi(d(fx, fy)) \leq \psi(\Theta_f^4(x, y)) - \varphi(\Theta_f^4(x, y)),$$

where $\psi \in \Psi$ and $\varphi \in \Phi$. In this section we use the respective (more general) condition with the expression $\Theta_f^5(x, y)$. For the sake of simplicity, we stay in the frame of metric spaces without order.

Theorem 5.1. *Let (X, d) be a complete metric space and $f : X \to X$. If there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that for all $x, y \in X$,*

$$\psi(d(fx, fy)) \leq \psi(\Theta_f^5(x, y)) - \varphi(\Theta_f^5(x, y)), \tag{5.3}$$

holds, then f has a unique fixed point.

Proof. The given condition (5.3) and properties of functions ψ and φ imply that

$$d(fx, fy) \leq \Theta_f^5(x, y) \tag{5.4}$$

for each $x, y \in X$. Starting with an arbitrary $x_0 \in X$, construct the Picard sequence by $x_{n+1} = fx_n$. The condition (5.4) implies that

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(fx_n, fx_{n+1}) \\ &\leq Ad(x_n, x_{n+1}) + Bd(x_n, x_{n+1}) + Cd(x_{n+1}, x_{n+2}) \\ &\quad + Dd(x_n, x_{n+2}) + Ed(x_{n+1}, x_{n+1}) \\ &\leq (A + B + D)d(x_n, x_{n+1}) + (C + D)d(x_{n+1}, x_{n+2}), \end{aligned}$$

implying that

$$(1 - C - D)d(x_{n+1}, x_{n+2}) \leq (A + B + D)d(x_n, x_{n+1}),$$

and, similarly,

$$(1 - B - E)d(x_{n+2}, x_{n+1}) \leq (A + C + E)d(x_{n+1}, x_n).$$

Adding up, one obtains that

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}),$$

where $\lambda = \frac{2A+B+C+D+E}{2-B-C-D-E} \leq 1$. It follows that $\{d(x_{n+1}, x_n)\}$ is a nonincreasing sequence of nonnegative numbers, which tends to some $r \geq 0$. In order to prove that $r = 0$, put $x = x_{n+1}$ and $y = x_n$ in (5.3) to obtain

$$\psi(d(x_{n+2}, x_{n+1})) \leq \psi(\Theta_f^5(x_{n+1}, x_n)) - \varphi(\Theta_f^5(x_{n+1}, x_n)), \tag{5.5}$$

where

$$\begin{aligned}\Theta_f^5(x_{n+1},x_n) &= Ad(x_{n+1},x_n)+Bd(x_{n+1},x_{n+2})+Cd(x_n,x_{n+1})\\ &\quad+Dd(x_{n+1},x_{n+1})+Ed(x_n,x_{n+2})\\ &\le (A+C+E)d(x_n,x_{n+1})+(B+E)d(x_{n+1},x_{n+2}).\end{aligned}$$

Similarly,

$$\Theta_f^5(x_n,x_{n+1})\le (A+C+D)d(x_n,x_{n+1})+(B+D)d(x_{n+1},x_{n+2}).$$

On the other hand, (5.4) implies that

$$\Theta_f^5(x_{n+1},x_n)\ge d(x_{n+1},x_{n+2}).$$

In the case when $D=E$, passing to the limit when $n\to\infty$, we obtain that $\lim_{n\to\infty}\Theta_f^5(x_{n+1},x_n)=r$; the same conclusion is obtained if $D<E$ (or $D>E$). Hence, passing to the (upper) limit in (5.5), we get that $\psi(r)\le\psi(r)-\varphi(r)$, implying that $r=0$.

As in previous proofs, in order to obtain that $\{x_n\}$ is a Cauchy sequence, suppose that it is not the case and using Lemma 2.1 deduce that there exist $\varepsilon>0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k>m_k>k$ and the sequences

$$d(x_{2m_k},x_{2n_k}),\quad d(x_{2m_k},x_{2n_k+1}),\quad d(x_{2m_k-1},x_{2n_k}),\quad d(x_{2m_k-1},x_{2n_k+1})$$

all tend to ε. Putting $x=x_{2n_k}$ and $y=2_{2m_k-1}$ in (5.3) gives

$$\begin{aligned}\psi(d(x_{2n_k+1},x_{2m_k}) &= \psi(d(fx_{2n_k},fx_{2m_k-1}))\\ &\le \psi(\Theta_f^5(x_{2n_k},x_{2m_k-1}))-\varphi(\Theta_f^5(x_{2n_k},x_{2m_k-1})).\end{aligned}$$

Here

$$\begin{aligned}\Theta_f^5(x_{2n_k},x_{2m_k-1}) &= Ad(x_{2n_k},x_{2m_k-1})+Bd(x_{2n_k},x_{2n_k+1})\\ &\quad+Cd(x_{2m_k-1},x_{2n_k})+Dd(x_{2n_k},x_{2m_k})+Ed(x_{2m_k-1},x_{2n_k+1})\\ &\to A\varepsilon+B\cdot 0+c\cdot 0+D\varepsilon+E\varepsilon=(A+D+E)\varepsilon,\end{aligned}$$

when $k\to\infty$. Since also $d(x_{2n_k+1},x_{2m_k})\to\varepsilon$ when $k\to\infty$, we obtain that

$$\psi(\varepsilon)\le\psi((A+D+E)\varepsilon)-\varphi((A+D+E)\varepsilon)\le\psi(\varepsilon)-\varphi((A+D+E)\varepsilon),$$

implying that $\varepsilon=0$ (because $A>0$).

Thus, the sequence $\{x_n\}$ converges to some z in the complete metric space X. In order to prove that $fz=z$, suppose the contrary and put

$x = x_n$ and $y = z$ in (5.4). It follows that

$$d(fx_n, fz) \le Ad(x_n, z) + Bd(x_n, x_{n+1}) + Cd(z, fz) + Dd(x_n, fz) + Ed(z, x_{n+1}).$$

Passing to the limit when $n \to \infty$ gives that

$$d(z, fz) \le (C + D)d(z, fz) < (A + B + C + D + E)d(z, fz) \le d(z, fz).$$

A contradiction, since $A > 0$.

The proof that the fixed point of f is unique is standard. $\square$

When two functions $f, g : X \to X$ are given, expressions

$$\Theta^5_{f,g}(x, y) = Ad(x, y) + Bd(x, fx) + Cd(y, gy) + Dd(x, gy) + Ed(y, fx),$$

$$\Theta^4_{f,g}(x, y) = ad(x, y) + bd(x, fx) + cd(y, gy) + e[d(x, gy) + d(y, fx)],$$

can be used, with the following assumptions on coefficients: 1° $A > 0$, $B, C, D, E \ge 0$, $A+B+C+D+E \le 1$, and ($B = C$ or $D = E$); 2° $a > 0$, $b, c, e \ge 0$, $a+b+c+2e \le 1$. Again, the condition with $\Theta^4_{f,g}(x, y)$ was used by Nashine and Altun [15]. The proof of the following generalization of their result is similar to that of Theorem 5.1, by using the procedure as in [13].

Theorem 5.2. *Let (X, d) be a complete metric space and $f, g : X \to X$ be two maps. If there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that for all $x, y \in X$,*

$$\psi(d(fx, gy)) \le \psi(\Theta^5_{f,g}(x, y)) - \varphi(\Theta^5_{f,g}(x, y)), \tag{5.6}$$

holds, then f and g have a unique common fixed point.

Acknowledgments

This work was supported by the Ministry of Science and Technological Development of Serbia.

References

[1] M. Abbas and M. Ali Khan, Common fixed point theorem of two mappings satisfying a generalized weak contractive condition, *Int. J. Math. Math. Sci.* (2009), Article ID 131068, 9 pp.

[2] M. Abbas and D. Djorić, Common fixed point theorem for four mappings satisfying generalized weak contractive conditions, *Filomat* **24** (2010), no. 2, 1–10.

[3] Ya. I. Alber and S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, New results in operator theory, Advances and Appl. **98** (ed. by I. Gohberg and Yu Lyubich), Birkhäuser Verlag, Basel, 1997.

[4] I. Beg and M. Abbas, Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition, *Fixed Point Theory Appl.* (2006), Article ID 74503, 7 pp.
[5] V. Berinde, Approximating fixed points of weak φ-contractions using the Picard iteration, *Fixed Point Theory* **4** (2003), no. 2, 131–142.
[6] B. S. Choudhury, P. Konor, B. E. Rhoades and N. Metiya, Fixed point theorems for generalized weakly contractive mapping, *Nonlinear Anal.* **74** (2011), no. 6, 2116–2126.
[7] Lj. B. Ćirić, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.* **45** (1974) 267–273.
[8] K. M. Das and K. V. Naik, Common fixed-point theorems for commuting maps on a metric space, *Proc. Amer. Math. Soc.* **77** (1979), no. 3, 369–373.
[9] B. Djafari Rouhani and S. Moradi, Common fixed point of multivalued generalized φ-weak contractive mappings, *Fixed Point Theory Appl.* (2010), Article ID 708984, 13 pp.
[10] D. Djorić, Common fixed point for generalized (ψ, ϕ)-weak contractions, *Appl. Math. Lett.* **22** (2009), no. 12, 1896–1900.
[11] P. N. Dutta and B. S. Choudhury, A generalisation of contraction principle in metric spaces, *Fixed Point Theory Appl.* (2008), Article ID 406368, 8 pp.
[12] J. Jachymski, Equivalent conditions for generalized contractions on (ordered) metric spaces, *Nonlinear Anal.* **74** (2011), no. 3, 768–774.
[13] Z. Kadelburg, M. Pavlović and S. Radenović, Common fixed point theorems for ordered contractions and quasicontractions in ordered cone metric spaces, *Comput. Math. Appl.* **59** (2010), no. 9, 3148–3159.
[14] S. Moradi, Z. Fahti and E. Analouee, Common fixed point of single valued generalized φ_f-weak contractive mappings, *Appl. Math. Lett.* **24** (2011), no. 5, 771–776.
[15] H. K. Nashine and I. Altun, Fixed point theorems for generalized weakly contractive condition in ordered metric spaces, *Fixed Point Theory Appl.* (2011), Article ID 132367, 20 pp.
[16] O. Popescu, Fixed points for (ψ, ϕ)-weak contractions, *Appl. Math. Lett.* **24** (2011), no. 1, 1–4.
[17] B. E. Rhoades, Some theorems on weakly contractive maps, *Nonlinear Anal.* **47** (2001), no. 4, 2683–2693.
[18] Q. Zhang and Y. Song, Fixed point theory for generalized φ-weak contractions, *Appl. Math. Lett.* **22** (2009), no. 1, 75–78.

Stojan Radenović University of Belgrade, Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Beograd, Serbia
Email: `radens@beotel.net`

Zoran Kadelburg University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Beograd, Serbia
Email: `kadelbur@matf.bg.ac.rs`

Davorka Jandrlić University of Belgrade, Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Beograd, Serbia
Email: djandrlic@mas.bg.ac.rs

Andrija Jandrlić University of Belgrade, Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Beograd, Serbia
Email: ajandrlic@mas.bg.ac.rs