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HYBRID STEEPEST-DESCENT METHOD WITH SEQUENTIAL AND FUNCTIONAL ERRORS IN BANACH SPACE

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ABSTRACT. Let X be a reflexive Banach space, $T : X \to X$ a nonexpansive mapping with $C = Fix(T) \neq \emptyset$ and $F : X \to X$ δ-strongly accretive and λ- strictly pseudocontractive with $\delta + \lambda > 1$. In this paper, we present modified hybrid steepest-descent methods involving sequential and functional errors with functions admitting a center which generate convergent sequences to the unique solution of the variational inequality $VI^*(F, C)$. We also present similar results for a strongly monotone and Lipschitzian operator in the context of a Hilbert space and apply the results for solving a minimization problem.

1. Introduction

Let C be a nonempty closed convex subset of a Hilbert space H and $F : C \to H$ be a nonlinear map. The classical variational inequality which is denoted by $VI(F, C)$ is used to find $x^* \in C$ so that

$$\langle Fx^*, v - x^* \rangle \geq 0,$$

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for all $v \in C$. We recall that F is called r-strongly monotone, if for each $x, y \in C$ we have

$$\langle Fx - Fy, x - y\rangle \geq r\|x - y\|^2$$

for a constant $r > 0$. Existence and uniqueness of solutions are important problems of the $VI(F, C)$. It is known that, if F is a strongly monotone and Lipschitzian mapping on C, then $VI(F, C)$ has a unique solution. An important problem is how to find a solution of $VI(F, C)$. It is known that

$$x^* \in VI(F, C) \Longleftrightarrow x^* = P_C(x^* - \lambda F x^*), \tag{1.1}$$

where $\lambda > 0$ is an arbitrarily fixed constant and P_C is the projection of H onto C. This alternative equivalence has been used to study the existence theory of the solution and to develop several iterative type algorithms for solving variational inequalities. But, the fixed point formulation in (1.1) involves the projection P_C, which may not be easy to compute, due to the complexity of the convex set C. So, projection methods and their variant forms can be implemented for solving variational inequalities.

In order to reduce the probable complexity as the results of the projection P_C, Yamada [19] (see also [7]) introduced a hybrid steepest-descent method for solving $VI(F, C)$. His idea is as follow. Assume that C is the fixed point set of a nonexpansive mapping $T : H \to H$. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \ \forall x, y \in H.$$

Assume that F is r-strongly monotone and μ-Lipschitzian on C. Take a fixed number $\eta \in (0, 2r/\mu^2)$ and a sequence $\{\lambda_n\}$ in $(0, 1)$ satisfying the following conditions:

$(C1)$ $\lim_n \lambda_n = 0$;
$(C2)$ $\sum_{n=1}^{\infty} \lambda_n = \infty$;
$(C3)$ $\lim_n (\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$.

Starting with an arbitrary initial guess $u_0 \in H$, generate a sequence $\{u_n\}$ by the following algorithm:

$$u_{n+1} := Tu_n - \lambda_{n+1}\eta F(Tu_n), \quad n \geq 0. \tag{1.2}$$

Yamada [19] proved that the sequence $\{u_n\}$ converges strongly to a unique solution of $VI(F,C)$. Xu and Kim [18] studied the hybrid steepest-descent algorithm (1.2). Their major contribution is that the strong convergence of (1.2) holds with condition $(C3)$ being replaced by the following condition:

$(C3)'$ $\lim_n(\lambda_n - \lambda_{n+1})/\lambda_{n+1} = 0$.

It is clear that condition $(C3)'$ is strictly weaker than condition $(C3)$, coupled with conditions $(C1)$ and $(C2)$. Moreover, $(C3)'$ includes the important and natural choice $\{1/n\}$ for $\{\lambda_n\}$, whereas $(C3)$ does not.

Let X be a real Banach space and let $J : X \to 2^{X^*}$ denote the normalized duality mapping defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\},$$

for all $x \in X$. Now, let $F : X \to X$ be a mapping and $T : X \to X$ be a nonexpansive mapping with $C = Fix(T) \neq \emptyset$. For the rest of the paper, we denote by J the single-valued duality mapping. The variational inequality problem in Banach space is to find a point $x* \in X$ such that

$$\langle F(x^*), J(x - x^*)\rangle \geq 0, \forall x \in C.$$

The problem $VI^*(F,C)$ for an inverse strongly accretive operator F over a nonempty closed convex subset C of a smooth Banach space X has already been presented in Aoyama et al., [3, 4]. For the study of the variational inequality problem for monotone operators in a Banach space, see, e.g., [1–4, 6, 10, 11]. If F is strongly accretive and strictly pseudocontractive, then F is inverse strongly accretive. Recently, L. C. Ceng et al. [6] have proved, in case that F is strongly accretive and strictly pseudocotractive on a reflexive Banach space that admits a weak sequentially continuous duality mapping, $VI^*(F,C)$ has a unique solution and extended steepest-descent method for solving $VI^*(F,C)$. They have shown that if $\{\lambda_n\}$ and $\{\mu_n\}$ are two sequences in $(0,1)$ satisfying the following conditions:

$(C1)$ $\lim_{n\to\infty} \lambda_n/\mu_n = 0$;
$(C2)$ $\sum_{n=0}^{\infty} \lambda_n\mu_n = \infty$;
$(C3)$ $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;

$(C4)$ $\sum_{n=0}^{\infty} |\mu_{n+1} - \mu_n| < \infty$ or $\lim_{n\to\infty} \mu_n/\mu_{n+1} = 1$;
then, the sequence $\{x_n\}$ defined by $x_0 \in X$ and

$$\begin{cases} y_n = \lambda_n x_n + (1-\lambda_n)T(x_n) \\ x_{n+1} = y_n - \lambda_n \mu_n F(x_n), \quad n \geq 0 \end{cases}$$

converges strongly to the unique solution of $VI^*(F, C)$.

In this paper, we introduce modified hybrid steepest-descent methods involving sequential and functional errors. Among many other results, we mention the following: Let X be a reflexive Banach space that admits a weak sequentially continuous duality mapping J from X to X^*, $T : X \to X$ be a nonexpansive mapping with $C = Fix(T) \neq \emptyset$ and $F : X \to X$ be a δ-strongly accretive and λ- strictly pseudocotractive mapping with $\delta + \lambda > 1$. Suppose that $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, 1]$ satisfy the following conditions:

$(C1)$ $\lambda_n \to 0$;
$(C2)$ $\sum_{n=0}^{\infty} \lambda_n = \infty$;
$(C3)$ $(1-\alpha_n)/\lambda_n \to 0$;
$(C4)$ $0 < \liminf_{n\to\infty} \beta_n$ and $\limsup_{n\to\infty} \beta_n < 1$;

and either $\{y_n\}$ is an arbitrary bounded sequence in X or $y_n \in \overline{co}\{T^i x_n : i = 0, 1, 2, \dots\}$, for all n. Let $x_0 \in X$ and $\{x_n\}$ be generated by

$$\begin{cases} z_n = \alpha_n T x_n + (1-\alpha_n) y_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n \{z_n - \lambda_n F z_n\}. \end{cases}$$

We show that $\{x_n\}$ converges in norm to the unique solution of $VI^*(F, C)$. The results presented in this paper are new even for Hilbert spaces. Our results extend some results of e.g., [6, 7, 14, 18, 19].

2. **Preliminaries**

Let X be a real Banach space and J the duality mapping from X to X^*. If X admits sequentially continuous duality mapping from weak topology to weak-star topology, then by [[9], Lemma 1] we know that the duality mapping J is single-valued, hence hence X is also smooth (see [1, 16]). In this case, duality mapping J is also called weak sequentially continuous, that is, for each $\{x_n\} \subset X$ with $x_n \rightharpoonup x$ we have $J(x_n) \rightharpoonup^* J(x)$.

A Banach space X is said to satisfy Opial's condition if whenever $\{x_n\}$ is a sequence in X which converges weakly to x, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \text{for all } \ y \in X, \ y \neq x.$$

By [[9], Theorem 1], we know that if X admits a weak sequentially continuous duality mapping, then X satisfies Opial's condition. The following lemma will be needed in the sequel.

Lemma 2.1. *([1, 12, 16]) Let C be a nonempty closed convex subset of a reflexive Banach space X that satisfies Opial's condition and suppose that $T : C \to C$ is nonexpansive then the mapping $I - T$ is demiclosed at zero, that is, if $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \to 0$ then $Tx = x$.*

Recall that a mapping F with domain $D(F)$ and range $R(F)$ in X is called δ-strongly accretive if for each $x, y \in D(F)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Fx - Fy, j(x-y)\rangle \geq \delta\|x-y\|^2 \quad \text{for some} \quad \delta \in (0, 1).$$

F is called λ-strictly pseudocontractive [5] if for each $x, y \in D(F)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Fx - Fy, j(x-y) \leq \|x-y\|^2 - \lambda\|x-y-(Fx-Fy)\|^2,$$

for some $\lambda \in (0, 1)$.

The following Lemmas will be used throughout the paper:

Lemma 2.2. *([6]) Let X be a smooth Banach space and $F : X \to X$ be a mapping. If F is δ-strongly accretive and λ-strictly pseudocontractive with $\delta+\lambda > 1$, then for any fixed number $\alpha \in (0, 1)$, $I-\alpha F$ is contractive with constant $1 - \alpha(1 - \sqrt{(1-\delta)/\lambda})$.*

Lemma 2.3. *([6]) Let X be a reflexive Banach space that admits a weak sequentially continuous duality mapping J from X to X^*. Suppose that $T : X \to X$ is a nonexpansive mapping and $C = F(T) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta+\lambda > 1$. Then the variationally inequality $VI^*(F, C)$ has a unique solution.*

Lemma 2.4. *([16]) Let X be a real smooth Banach space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, J(x+y) > \quad ; \quad \forall x, y \in X$$

Lemma 2.5. *([15]) Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n$*

and $\limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$ *for all integers* $n \geq 0$ *and*

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.6. *([13, 17]) Let* $\{s_n\}, \{c_n\} \subset \mathbb{R}_+$, $\{a_n\} \subset (0,1)$ *and* $\{b_n\} \subset \mathbb{R}$ *be sequences such that* $s_{n+1} \leq (1-a_n)s_n + b_n + c_n$ *for all* $n \geq 0$. *Assume* $\sum_{n=0}^{\infty} c_n < \infty$ *then the following results hold:*
1) If $b_n \leq \beta a_n$ *(for some* $\beta > 0$*), then* $\{s_n\}$ *is a bounded sequence.*
2) If we have $\sum_{n=0}^{\infty} a_n = \infty$ *and* $\limsup_{n\to\infty} b_n/a_n \leq 0$, *then* $s_n \to 0$.

3. Iterative methods with sequential errors

Theorem 3.1. *Let* X *be a reflexive Banach space that admits a weak sequentially continuous duality mapping* J *from* X *to* X^*, $T : X \to X$ *be a nonexpansive mapping with* $C = Fix(T) \neq \emptyset$ *and* $F : X \to X$ *be a* δ*-strongly accretive and* λ*- strictly pseudocotractive mapping with* $\delta + \lambda > 1$. *Suppose that* $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ *and* $\{\lambda_n\} \subset (0,1]$ *satisfying the following conditions:*

$(C1)$ $\lambda_n \to 0$;
$(C2)$ $\sum_{n=0}^{\infty} \lambda_n = \infty$;
$(C3)$ $(1-\alpha_n)/\lambda_n \to 0$;
$(C4)$ $0 < \liminf_{n\to\infty} \beta_n$ *and* $\limsup_{n\to\infty} \beta_n < 1$;

and $\{y_n\}$ *is an arbitrary bounded sequence in* X. *Let* $x_0 \in X$ *and* $\{x_n\}$ *be generated by*

$$\begin{cases} z_n = \alpha_n T x_n + (1-\alpha_n) y_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n\{z_n - \lambda_n F z_n\}. \end{cases}$$

Then $\{x_n\}$ *converges strongly to the unique solution* x^* *of the variational inequality* $VI^*(F, C)$.

Proof. Note that, taking $F_n := (I - \lambda_n F)$, we can write

$$x_{n+1} = (1-\beta_n)x_n + \beta_n F_n z_n$$

and by Lemma 2.2, we have for all $x, y \in C$,

$$\|F_n x - F_n y\| \leq (1 - \lambda_n \tau)\|x - y\|,$$

where $\tau := 1 - \sqrt{(1-\delta)/\lambda}$.

Now we shall divide the proof into several steps.

Step 1. $\{x_n\}$ is bounded:

From $(1-\alpha_n)/\lambda_n \to 0$ we can choose N such that $(1-\alpha_n) \le \lambda_n$, $(\forall n > N)$. Let $x^* \in C$. Since $\{y_n\}$ is bounded, we can choose some big enough constant $K > 0$ such that

$$\sup_n\{\|y_n - x^*\| + \|Fx^*\|\} < K.$$

So, for $n > N$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &\le (1-\beta_n)\|x_n - x^*\| + \beta_n\|F_n z_n - x^*\| \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n\{\|F_n z_n - F_n x^*\| + \|F_n x^* - x^*\|\} \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n\{(1-\lambda_n\tau)\|z_n - x^*\| + \lambda_n\|Fx^*\|\} \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n\{(1-\lambda_n\tau)\|x_n - x^*\| \\
&\quad +(1-\alpha_n)\|y_n - x^*\| + \lambda_n\|Fx^*\|\} \\
&= (1-\beta_n\lambda_n\tau)\|x_n - x^*\| + \beta_n(1-\alpha_n)\|y_n - x^*\| + \beta_n\lambda_n\|Fx^*\| \\
&\le (1-\beta_n\lambda_n\tau)\|x_n - x^*\| + \beta_n\lambda_n\|y_n - x^*\| + \beta_n\lambda_n\|Fx^*\| \\
&\le (1-\beta_n\lambda_n\tau)\|x_n - x^*\| + \beta_n\lambda_n K.
\end{aligned}$$

From this inequality and the first part of Lemma 2.6, it follows that $\{x_n\}$ is bounded.

Step 2. Let $\{\omega_n\}$ be a bounded sequence in C. Then $\|F_{n+1}w_n - F_n\omega_n\| \to 0$.

In fact, since $\{\omega_n\}$ is bounded and F is a Lipschitzian mapping, $\sup_n\{\|F\omega_n\|\} < \infty$. Now, from $\lambda_n \to 0$, we obtain

$$\|F_{n+1}w_n - F_n\omega_n\| = \|\lambda_{n+1}Fw_n - \lambda_n F\omega_n\| \to 0, \text{ as } n \to \infty.$$

Step 3. $\|x_{n+1} - x_n\| \to 0$.

To prove it, define a sequence $\{u_n\}$ by $u_n = (x_{n+1} - (1-\beta_n)x_n)/\beta_n$ so that $x_{n+1} = (1-\beta_n)x_n + \beta_n u_n$.

Now we compute

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|(x_{n+2} - (1-\beta_{n+1})x_{n+1})/\beta_{n+1} - (x_{n+1} - (1-\beta_n)x_n)/\beta_n\| \\
&= \|F_{n+1}z_{n+1} - F_n z_n\| \le \|F_{n+1}z_{n+1} - F_{n+1}z_n\| + \|F_{n+1}z_n - F_n z_n\| \\
&\le \|z_{n+1} - z_n\| + \|F_{n+1}z_n - F_n z_n\| \\
&\le \alpha_{n+1}\|Tx_{n+1} - Tx_n\| + |\alpha_{n+1} - \alpha_n|\|Tx_n\| \\
&\quad +\|(1-\alpha_{n+1})y_{n+1} + (1-\alpha_n)y_n\| + \|F_{n+1}z_n - F_n z_n\|
\end{aligned}$$

$$\leq \|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n| \|Tx_n\|$$
$$+\|(1 - \alpha_{n+1})y_{n+1} + (1 - \alpha_n)y_n\| + \|F_{n+1}z_n - F_n z_n\|.$$

Since $(1-\alpha_n) \to 0$ and the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded, applying Step 2, we obtain

$$\limsup_n (\|u_{n+1} - u_n\| - \|x_{n+1} - x_n\|)$$
$$\leq \limsup_n \{|\alpha_{n+1} - \alpha_n| \|Tx_n\| + \|(1 - \alpha_{n+1})y_{n+1} + (1 - \alpha_n)y_n\|$$
$$+\|F_{n+1}z_n - F_n z_n\|\} \leq 0.$$

Apply Lemma 2.5 to get $\lim_n \|x_{n+1} - x_n\| = \lim_n \beta_n \|x_n - z_n\| = 0$.

Step 4. $\|TF_n z_n - F_n z_n\| \to 0$.

Indeed, by Step 3 and the conditions (C1) and (C3) we have

$$\|z_n - Tx_n\| = (1 - \alpha_n)\|Tx_n - y_n\| \to 0$$

and

$$\beta_n \|x_n - z_n\| \leq \|x_{n+1} - x_n\| + \lambda_n \|Fz_n\| \to 0.$$

So, considering (C4), we obtain

$$\|Tz_n - z_n\| \leq \|Tz_n - Tx_n\| + \|z_n - Tx_n\|$$
$$\leq \|z_n - x_n\| + \|z_n - Tx_n\| \to 0.$$

Consequently,

$$\|TF_n z_n - F_n z_n\| \leq \|TF_n z_n - Tz_n\| + \|Tz_n - F_n z_n\|$$
$$\leq \|F_n z_n - z_n\| + \|Tz_n - z_n\| + \|z_n - F_n z_n\|$$
$$= \{2\lambda_n \|Fz_n\| + \|Tz_n - z_n\|\} \to 0.$$

Step 5. $\limsup_{n\to\infty} \langle -Fx^*, J(F_n z_n - x^*)\rangle \leq 0$, where, x^* is the unique solution of the variational inequality $VI^*(F, C)$.

To prove it, we pick a subsequence $\{F_{n_i}(z_{n_i})\}$ of $\{F_n z_n\}$ so that

$$\limsup_{n\to\infty} \langle -Fx^*, J(F_n z_n - x^*)\rangle = \lim_{i\to\infty} \langle -Fx^*, J(F_{n_i}(z_{n_i}) - x^*)\rangle$$

Because X is reflexive and $\{F_n z_n\}$ is bounded, we may assume that $F_{n_i}(z_{n_i}) \rightharpoonup y^*$ for some $y^* \in X$. But by lemma 2.1 and Step 4, we have $y^* \in Fix(T) = C$. Now, since x^* solves $VI^*(F, C)$, we obtain

$$\limsup_{n\to\infty} \langle -Fx^*, J(F_n z_n - x^*)\rangle = \langle -Fx^*, J(y^* - x^*)\rangle \leq 0.$$

Step 6. $x_n \to x^*$.

Observe that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le (1-\beta_n)\|x_n-x^*\|^2+\beta_n\|F_nz_n-x^*\|^2\\
&=(1-\beta_n)\|x_n-x^*\|^2+\beta_n\|F_nz_n-F_nx^*-\lambda_nFx^*\|^2\\
&\le (1-\beta_n)\|x_n-x^*\|^2+\beta_n\{(1-\lambda_n\tau)\|z_n-x^*\|^2\\
&\quad +2\langle -\lambda_nFx^*, J(F_nz_n-x^*)\rangle\}\\
&\le (1-\beta_n)\|x_n-x^*\|^2+\beta_n(1-\lambda_n\tau)\{\alpha_n\|Tx_n-x^*\|^2+(1-\alpha_n)\|y_n-x^*\|^2\}\\
&\quad +2\beta_n\lambda_n\langle -Fx^*, J(F_nz_n-x^*)\rangle\\
&\le (1-\beta_n)\|x_n-x^*\|^2+\beta_n(1-\lambda_n\tau)\{\|x_n-x^*\|^2+(1-\alpha_n)\|y_n-x^*\|^2\}\\
&\quad +2\beta_n\lambda_n\langle -Fx^*, J(F_nz_n-x^*)\rangle\\
&\le (1-\beta_n\lambda_n\tau)\|x_n-x^*\|^2+\beta_n(1-\alpha_n)\|y_n-x^*\|^2\\
&\quad +2\beta_n\lambda_n\langle -Fx^*, J(F_nz_n-x^*)\rangle
\end{aligned}
$$

Now, from conditions (C2) and (C3), Step 5 and the second part of Lemma 2.6, we get $\|x_n-x^*\|\to 0$. □

Theorem 3.2. *Let X, T, C and F be as in Theorem 3.1, and suppose that $\{\alpha_n\},\{\beta_n\}\subset[0,1]$ and $\{\lambda_n\}\subset(0,1]$ satisfy the following conditions:*

$(C1)$ $\lambda_n\to 0$;
$(C2)$ $\sum_{n=0}^{\infty}\beta_n\lambda_n=\infty$;
$(C3)$ $\lambda_n/\lambda_{n+1}\to 1$;
$(C4)$ $(1-\alpha_n)/\lambda_n\to 0$;
$(C5)$ $\sum_{n=0}^{\infty}|\beta_n-\beta_{n-1}|<\infty$;

and $\{y_n\}$ is an arbitrary bounded sequence in X. Let $x_0\in X$ and $\{x_n\}$ be generated by

$$
\begin{cases} z_n=\alpha_nTx_n+(1-\alpha_n)y_n,\\ x_{n+1}=(1-\beta_n)Tx_n+\beta_n\{z_n-\lambda_nFz_n\}.\end{cases}
$$

Then $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality $VI^*(F,C)$.

Proof. Following the proof of Theorem 3.1, we may write

$$x_{n+1}=(1-\beta_n)Tx_n+\beta_nF_nz_n,$$

and, using a similar argument, prove the theorem according to the following steps:
Step 1. $\{x_n\}$ ia bounded;

Step 2. $\|x_{n+1} - x_n\| \to 0$;
Step 3. $\|TF_n z_n - F_n z_n\| \to 0$;
Step 4. $\limsup_{n\to\infty}\langle -Fx^*, J(F_n z_n - x^*)\rangle \le 0$, where, x^* is the unique solution of the variational inequality $VI^*(F,C)$;
Step 5. $x_n \to x^*$.

To prove Step 1, it suffices to repeat the proof of Step 1 in Theorem 3.1. To prove Step 2, we estimate

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\le (1-\beta_n)\|Tx_n - Tx_{n-1}\| + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\| \\
&+\beta_n\|F_n z_n - F_n z_{n-1}\| + |\beta_n - \beta_{n-1}|\|F_n z_{n-1}\| \\
&+\beta_{n-1}\|F_n z_{n-1} - F_{n-1} z_{n-1}\| \\
&\le (1-\beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\| \\
&+\beta_n(1-\lambda_n\tau)\|z_n - z_{n-1}\| \\
&+|\beta_n - \beta_{n-1}|\|F_n z_{n-1}\| + \beta_{n-1}|\lambda_n - \lambda_{n-1}|\|Fz_{n-1}\|.
\end{aligned} \tag{3.1}$$

Moreover, we have

$$\begin{aligned}
&\beta_n(1-\lambda_n\tau)\|z_n - z_{n-1}\| \\
&\le \beta_n(1-\lambda_n\tau)\{\alpha_n\|Tx_n - Tx_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Tx_{n-1}\| \\
&+(1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|y_{n-1}\|\} \\
&\le \beta_n(1-\lambda_n\tau)\|x_n - x_{n-1}\| + \beta_n|\alpha_n - \alpha_{n-1}|\|Tx_{n-1}\| \\
&+\beta_n(1-\alpha_n)\|y_n - y_{n-1}\| + \beta_n|\alpha_n - \alpha_{n-1}|\|y_{n-1}\|,
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\beta_{n-1}|\lambda_n - \lambda_{n-1}|\|Fz_{n-1}\| \\
&\le \beta_n\lambda_n|1 - \frac{\lambda_{n-1}}{\lambda_n}|\|Fz_{n-1}\| \\
&+|\beta_n - \beta_{n-1}||\lambda_n - \lambda_{n-1}|\|Fz_{n-1}\|.
\end{aligned} \tag{3.3}$$

Note that being a convex combination of two bounded sequences, $\{z_n\}$ is bounded. Set

$$L = \sup_n\{\|Fz_n\|, \|F_n z_{n-1}\|, \|Tx_n\|, \|y_n\|\},$$

by (3.1), (3.2) and (3.3), we obtain

$$
\begin{aligned}
&\|x_{n+1}-x_n\| \\
&\le (1-\beta_n)\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\|Tx_{n-1}\| \\
&+\{\beta_n(1-\lambda_n\tau)\|x_n-x_{n-1}\| + \beta_n|\alpha_n-\alpha_{n-1}|\|Tx_{n-1}\| \\
&+\beta_n(1-\alpha_n)\|y_n-y_{n-1}\| + \beta_n|\alpha_n-\alpha_{n-1}|\|y_{n-1}\|\} \\
&+|\beta_n-\beta_{n-1}|\|F_nz_{n-1}\| + \{\beta_n\lambda_n|1-\frac{\lambda_{n-1}}{\lambda_n}|\|Fz_{n-1}\| \\
&+|\beta_n-\beta_{n-1}||\lambda_n-\lambda_{n-1}|\|Fz_{n-1}\|\} \\
&\le (1-\beta_n\lambda_n\tau)\|x_n-x_{n-1}\| + \{|\beta_n-\beta_{n-1}| + \beta_n|\alpha_n-\alpha_{n-1}| \\
&+\beta_n(1-\alpha_n) + \beta_n|\alpha_n-\alpha_{n-1}| + |\beta_n-\beta_{n-1}| \\
&+\beta_n\lambda_n|1-\frac{\lambda_{n-1}}{\lambda_n}| + |\beta_n-\beta_{n-1}||\lambda_n-\lambda_{n-1}|\}L \\
&\le (1-\beta_n\lambda_n\tau)\|x_n-x_{n-1}\| + \{2\beta_n|\alpha_n-\alpha_{n-1}| \\
&+\beta_n(1-\alpha_n) + \beta_n\lambda_n|1-\frac{\lambda_{n-1}}{\lambda_n}|\}L + 3|\beta_n-\beta_{n-1}|L. \qquad (3.4)
\end{aligned}
$$

Moreover, by (C5),

$$\sum_{n=0}^{\infty} 3|\beta_n-\beta_{n-1}|L < \infty \qquad (3.5)$$

and, by (C3) and (C4),

$$
\begin{aligned}
&\{2\beta_n|\alpha_n-\alpha_{n-1}| + \beta_n(1-\alpha_n) + \beta_n\lambda_n|1-\frac{\lambda_{n-1}}{\lambda_n}|\}/\beta_n\lambda_n \\
&= 2\frac{|\alpha_n-\alpha_{n-1}|}{\lambda_n} + \frac{1-\alpha_n}{\lambda_n} + |1-\frac{\lambda_{n-1}}{\lambda_n}| \\
&\le 2\{\frac{1-\alpha_n}{\lambda_n} + \frac{1-\alpha_{n-1}}{\lambda_n}\} + \frac{1-\alpha_n}{\lambda_n} + |1-\frac{\lambda_{n-1}}{\lambda_n}| \\
&= 3\frac{1-\alpha_n}{\lambda_n} + 2\frac{1-\alpha_{n-1}}{\lambda_{n-1}} \times \frac{\lambda_{n-1}}{\lambda_n} + |1-\frac{\lambda_{n-1}}{\lambda_n}| \to 0. \qquad (3.6)
\end{aligned}
$$

So, combining (3.4), (3.5), (3.6) and Lemma 2.6, we conclude that $\|x_{n+1}-x_n\| \to 0$.

Now we prove Step 3: First, note that

$$
\begin{aligned}
\|TF_nz_n - F_nz_n\| &\le \|TF_nz_n - Tz_n\| + \|Tz_n - F_nz_n\| \\
&\le 2\|F_nz_n - z_n\| + \|Tz_n - z_n\|,
\end{aligned}
$$

and

$$\begin{aligned}\|Tz_n - z_n\| &\le \|Tz_n - T^2x_n\| + \|T^2x_n - Tx_n\| + \|Tx_n - z_n\| \\ &\le 2\|z_n - Tx_n\| + \|Tx_n - x_n\|.\end{aligned}$$

So,

$$\begin{aligned}&\|TF_nz_n - F_nz_n\| \\ &\le 2\|F_nz_n - z_n\| + 2\|z_n - Tx_n\| + \|Tx_n - x_n\|. \qquad (3.7)\end{aligned}$$

On the other hand,

$$\|F_nz_n - z_n\| = \lambda_n\|Fz_n\| \to 0, \qquad (3.8)$$

and, combining (C1) and (C4),

$$\|z_n - Tx_n\| = (1 - \alpha_n)\|Tx_n - y_n\| \to 0. \qquad (3.9)$$

Moreover, by (3.8) and (C1), we obtain

$$\|x_{n+1} - Tx_n\| \le \{\|z_n - Tx_n\| + \lambda_n\|Fz_n\|\} \to 0.$$

From this inequality and Step 2, we get

$$\|Tx_n - x_n\| \le \|x_{n+1} - x_n\| + \|x_{n+1} - Tx_n\| \to 0. \qquad (3.10)$$

Now, considering (3.7)-(3.10), the desired result follows.

To prove Steps 4 and 5, it is enough to repeat the proofs of Steps 5 and 6 of Theorem 3.1. □

4. **Iterative methods with functional errors admitting a center**

The aim of this section is to obtain the results of Section 3 by replacing the sequential error $\{y_n\}$ with functional errors for functions admitting a center [8].

Theorem 4.1. *Let X be a reflexive Banach space that admits a weak sequentially continuous duality mapping J from X to X^*, $T : X \to X$ be a nonexpansive mapping with $C = Fix(T) \neq \emptyset$ and $\{S_n : n = 0, 1, 2, \dots\}$ be a family of mappings from X to X such that for some $z_0 \in C$ and $l \in (0, \infty)$ we have $\|S_nx - z_0\| \le l\|x - z_0\|$ for all $x \in X$ and $n = 0, 1, 2, \dots$. Suppose that $F : X \to X$ is a δ-strongly accretive and λ-strictly pseudocontractive mapping with $\delta + \lambda > 1$ and the sequences $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, 1]$ satisfy the following conditions:*

$(C1)$ $\lambda_n \to 0$;
$(C2)$ $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(*C*3) $(1-\alpha_n)/\lambda_n \to 0$;
(*C*4) $0 < \liminf_{n\to\infty} \beta_n$ *and* $\limsup_{n\to\infty} \beta_n < 1$.

Let $x_0 \in X$ *and* $\{x_n\}$ *be generated by*

$$\begin{cases} z_n = \alpha_n T x_n + (1-\alpha_n) S_n(x_n), \\ x_{n+1} = (1-\beta_n) x_n + \beta_n \{z_n - \lambda_n F z_n\}. \end{cases}$$

Then $\{x_n\}$ *converges strongly to the unique solution* x^* *of the variational inequality* $VI^*(F,C)$.

Proof. By an argument similar to the one used in Step 1 of Theorem 3.1, we obtain that

$$\begin{aligned} &\|x_{n+1} - z_0\| \\ &\le (1-\beta_n\lambda_n\tau)\|x_n - z_0\| + \beta_n(1-\alpha_n)\|S_n x_n - z_0\| + \beta_n\lambda_n\|F z_0\|, \end{aligned} \tag{4.1}$$

for $n = 0, 1, 2, \ldots$ and $z_0 \in C$. From the our assumptions, we may consider a fixed z_0 in C such that

$$\|S_n x - z_0\| \le l\|x - z_0\|, \tag{4.2}$$

for all $x \in X$ and $n = 0, 1, 2, \ldots$. Now, by (C3), we can choose N such that

$$(1-\alpha_n) \le \lambda_n(\tau/2l), \tag{4.3}$$

for all $n > N$. So, combining (4.1), (4.2) and (4.3), we obtain

$$\begin{aligned} &\|x_{n+1} - z_0\| \\ &\le (1-\beta_n\lambda_n\tau)\|x_n - z_0\| + \beta_n\lambda_n(\tau/2l)\|S_n x_n - z_0\| + \beta_n\lambda_n\|F z_0\| \\ &\le (1-\beta_n\lambda_n\tau)\|x_n - z_0\| + \beta_n\lambda_n(\tau/2l)l\|x_n - z_0\| + \beta_n\lambda_n\|F z_0\| \\ &= (1-\beta_n\lambda_n\tau/2)\|x_n - z_0\| + \beta_n\lambda_n\|F z_0\|, \end{aligned} \tag{4.4}$$

for every $n > N$. So, combining (C2), (C4), (4.4) and Lemma 2.6, it follows that $\{x_n\}$ is bounded. Consequently, (4.2) implies that $\{S_n x_n\}$ is bounded. Now, by taking $\{y_n\} = \{S_n x_n\}$ in Theorem 3.1, we get the desired result. □

Applying Theorem 3.2 and using an argument similar to the one used in Theorem 4.1, we get the following result:

Theorem 4.2. *Let* X, T, C, F *and* $\{S_n\}$ *be as in Theorem 4.1, and suppose that* $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ *and* $\{\lambda_n\} \subset (0,1]$ *satisfy the following conditions:*

$(C1)$ $\lambda_n \to 0$;
$(C2)$ $\sum_{n=0}^{\infty} \beta_n \lambda_n = \infty$;
$(C3)$ $\lambda_n/\lambda_{n+1} \to 1$;
$(C4)$ $(1-\alpha_n)/\lambda_n \to 0$;
$(C5)$ $\sum_{n=0}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Let $x_0 \in X$ and $\{x_n\}$ be generated by

$$\begin{cases} z_n = \alpha_n T x_n + (1-\alpha_n) S_n(x_n), \\ x_{n+1} = (1-\beta_n) T x_n + \beta_n \{z_n - \lambda_n F z_n\}. \end{cases}$$

Then $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality $VI^*(F, C)$.

Remark 4.3. *If S_n is defined for any $x \in X$, as any arbitrary element of $\overline{co}\{T^i x : i = 0, 1, 2, \dots\}$, then the assertion*

$$\|S_n x - z_0\| \leq \|x - z_0\|$$

holds for all $x \in X$ and $z_0 \in C = Fix(T)$. So, in Theorems 4.1 and 4.2, we may replace $S_n x_n$ with some arbitrary element of $\overline{co}\{T^i x_n : i = 0, 1, 2, \dots\}$.

5. Some results in Hilbert spaces

In this section, we deduce some results for a strongly monotone and Lipschitzian mapping in the context of a Hilbert space H. Let $F : H \to H$ be r-strongly monotone and μ-Lipschitzian. Let λ be a number in $[0, 1]$ and let $0 < \eta < 2r/\mu^2$. Then

$$\|(I - \lambda\eta F)x - (I - \lambda\eta F)y\| \leq (1 - \lambda\tau)\|x - y\|, \quad x, y \in H, \tag{5.1}$$

where $\tau = 1 - \sqrt{1 - \eta(2r - \eta\mu^2)} \in (0, 1)$. (See [18]). In addition, if $\lambda \leq \eta$ then by (5.1) we have

$$\begin{aligned} &\|(I - \lambda F)x - (I - \lambda F)y\| \\ &= \|(I - (\lambda/\eta)\eta F)x - (I - (\lambda/\eta)\eta F)y\| \\ &\leq (1 - (\lambda/\eta)\tau)\|x - y\| \\ &= (1 - \lambda\tau')\|x - y\|, \quad \forall x, y \in H, \end{aligned}$$

where $\tau' = \tau/\eta$. Furthermore we may choose the above η so that $2r/\mu^2 - \eta < 1/\mu^2$. Then, we have

$$\tau' = \tau/\eta = \{1 - \sqrt{1 - \eta(2r - \eta\mu^2)}\}/\eta$$

$$= (2r - \eta\mu^2)/\{1 + \sqrt{1 - \eta(2r - \eta\mu^2)}\} < 1.$$

According to the discussion above, we get the following:

Lemma 5.1. *Let $F : H \to H$ be r-strongly monotone and μ-Lipschitzian. Then, there exist constants $b \in (0,1]$ and $\tau' \in (0,1)$ such that*

$$\|(I - \lambda F)x - (I - \lambda F)y\| \leq (1 - \lambda\tau')\|x - y\|,$$

for all $\lambda \in [0,b]$ and $x, y \in H$.

Theorem 5.2. *Let $T : H \to H$ be a nonexpansive mapping with $C = Fix(T) \neq \emptyset$ and $F : H \to H$ be r-strongly monotone and μ-Lipschitzian. Suppose that $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,1]$ satisfy the following conditions:*

- $(C1)$ $\lambda_n \to 0$;
- $(C2)$ $\sum_{n=0}^{\infty} \lambda_n = \infty$;
- $(C3)$ $(1 - \alpha_n)/\lambda_n \to 0$;
- $(C4)$ $0 < \liminf_{n\to\infty} \beta_n$ *and* $\limsup_{n\to\infty} \beta_n < 1$;

and either $\{y_n\}$ is an arbitrary bounded sequence in H or $y_n \in \overline{co}\{T^i x_n : i = 0, 1, 2, \dots\}$, for all n. Let $x_0 \in H$ and $\{x_n\}$ be generated by

$$\begin{cases} z_n = \alpha_n T x_n + (1 - \alpha_n) y_n, \\ x_{n+1} = (1 - \beta_n) x_n + \beta_n \{z_n - \lambda_n F z_n\}. \end{cases}$$

Then $\{x_n\}$ converges strongly to the unique solution of the variational inequality $VI(F, C)$.

Proof. From Lemma 5.1, there are constants $b \in (0,1]$ and $\tau' \in (0,1)$ such that

$$\|(I - \lambda F)x - (I - \lambda F)y\| \leq (1 - \lambda\tau')\|x - y\|,$$

for all $\lambda \in [0,b]$ and $x, y \in H$. Since $\lambda_n \to 0$, without lose of generality, we may assume that $\lambda_n \leq b$ for all n. Now, taking $F_n := (I - \lambda_n F)$, we have

$$\|F_n x - F_n y\| \leq (1 - \lambda_n \tau')\|x - y\|,$$

for all $x, y \in C$. For the rest of the proof, it suffices to repeat the proof of Theorem 3.1 and consider Remark 4.3 and Theorem 4.1. □

If we take $\alpha_n = 1$ for all n, then we obtain in some sense the result of Theorem 3.1 in [18], without the assumption $\lambda_n/\lambda_{n+1} \to 1$.

Corollary 5.3. *Let T, C and F be as in Theorem 5.2, and suppose that $\{\beta_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,1]$ satisfy the following conditions:*

$(C1)$ $\lambda_n \to 0$;
$(C2)$ $\sum_{n=0}^{\infty} \lambda_n = \infty$;
$(C3)$ $0 < \liminf_{n\to\infty} \beta_n$ *and* $\limsup_{n\to\infty} \beta_n < 1$.

Let $x_0 \in H$ and $\{x_n\}$ be generated by

$$x_{n+1} = (1-\beta_n)x_n + \beta_n\{Tx_n - \lambda_n F(Tx_n)\}.$$

Then $\{x_n\}$ converges strongly to the unique solution of the variational inequality $VI(F,C)$.

Considering Theorems 3.2 and 4.2, using an argument similar to the one used in Theorem 5.2, we get the following result which is a generalization of [[18], Theorem 3.1] in the sense that, with the same assumptions on coefficients, an arbitrary sequence will be inserted in the algorithm.

Theorem 5.4. *Let T, C and F be as in Theorem 5.2, and suppose that $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,1]$ satisfy the following conditions:*

$(C1)$ $\lambda_n \to 0$;
$(C2)$ $\sum_{n=0}^{\infty} \lambda_n = \infty$;
$(C3)$ $\lambda_n/\lambda_{n+1} \to 1$;
$(C4)$ $(1-\alpha_n)/\lambda_n \to 0$;

and either $\{y_n\}$ is an arbitrary bounded sequence in H or $y_n \in \overline{co}\{T^i x_n : i = 0,1,2,\dots\}$, for all n. Let $x_0 \in H$ and $\{x_n\}$ be generated by

$$\begin{cases} z_n = \alpha_n T x_n + (1-\alpha_n) y_n, \\ x_{n+1} = z_n - \lambda_n F z_n. \end{cases}$$

Then $\{x_n\}$ converges strongly to the unique solution of the variational inequality $VI(F,C)$.

6. **Application in minimization**

Let K be a nonempty closed and convex subset of a real Hilbert space H. Let A be a bounded linear operator on H. Given an element $b \in H$, consider the minimization problem

$$\min_{x\in K}\|Ax-b\|^2. \tag{6.1}$$

Let S_b denote the solution set of (6.1). Then, S_b is closed convex. In the sequel, it is assumed that $S_b\neq\varnothing$. (It is known that $S_b\neq\varnothing$ if and only if $P_{\overline{A(K)}}(b)\in A(K)$).

For each $\lambda>0$, define a mapping $T:H\to H$ by

$$Tx=P_K(A^*b+(I-\lambda A^*A)x),\forall x\in H, \tag{6.2}$$

where A^* is the adjoint of A. It is shown that $Fix(T)=S_b$ and for $\lambda\in(0,2\|A\|^{-2})$ the mapping T is nonexpansive (see [18]).

Let $\theta:H\to\mathbb{R}$ be a differentiable convex function such that θ' is a μ-Lipschitzian and r-strongly monotone operator for some $\mu>0$ and $r>0$. Under these assumptions, there exists a unique point $\tilde{x}\in S_b$ such that

$$\theta(\tilde{x})=\min\{\theta(x):x\in S_b\}. \tag{6.3}$$

The minimization problem (6.3) is equivalent to the following variational inequality problem:

$$\langle\theta'(\tilde{x}),x-\tilde{x}\rangle\geq 0,\ \ x\in S_b.$$

We now apply the results in Section 5 for finding the unique solution of the minimization problem (6.3).

Theorem 6.1. *Suppose that* $\{\alpha_n\},\{\beta_n\}\subset[0,1]$ *and* $\{\lambda_n\}\subset(0,1]$ *satisfy the following conditions:*

$(C1)$ $\lambda_n\to 0$;
$(C2)$ $\sum_{n=0}^{\infty}\lambda_n=\infty$;
$(C3)$ $(1-\alpha_n)/\lambda_n\to 0$;
$(C4)$ $0<\liminf_{n\to\infty}\beta_n$ *and* $\limsup_{n\to\infty}\beta_n<1$;

and either $\{y_n\}$ *is an arbitrary bounded sequence in* H *or* $y_n\in\overline{co}\{T^ix_n: i=0,1,2,\dots\}$, *for all* n, *where* T *is given in* (6.2). *Let* $x_0\in H$ *and* $\{x_n\}$ *be generated by*

$$\begin{cases} z_n=\alpha_nTx_n+(1-\alpha_n)y_n,\\ x_{n+1}=(1-\beta_n)x_n+\beta_n\{z_n-\lambda_n\theta'(z_n)\}.\end{cases}$$

Then, $\{x_n\}$ *strongly converges to the unique solution* $\tilde{x}$ *of the minimization problem* (6.3).

Theorem 6.2. *Suppose that* $\{\alpha_n\} \subset [0,1]$ *and* $\{\lambda_n\} \subset (0,1]$ *satisfy the following conditions:*

$(C1)$ $\lambda_n \to 0$;
$(C2)$ $\sum_{n=0}^{\infty} \lambda_n = \infty$;
$(C3)$ $\lambda_n/\lambda_{n+1} \to 1$;
$(C4)$ $(1-\alpha_n)/\lambda_n \to 0$;

and either $\{y_n\}$ *is an arbitrary bounded sequence in* H *or* $y_n \in \overline{co}\{T^i x_n : i = 0,1,2,\dots\}$, *for all* n, *where* T *is given in* (6.2). *Let* $x_0 \in H$ *and* $\{x_n\}$ *be generated by*

$$\begin{cases} z_n = \alpha_n T x_n + (1-\alpha_n) y_n, \\ x_{n+1} = z_n - \lambda_n \theta' z_n. \end{cases}$$

Then, $\{x_n\}$ *strongly converges to the unique solution* $\tilde{x}$ *of the minimization problem* (6.3).

References

[1] R. P. Agarwal, D. O'Regan and D. R. Sahu, Fixed Point Theory for Lipschitzian-Type Mappings with Applications, Topological Fixed Point Theory and Its Applications, 6, Springer, New York, 2009.

[2] Y. I. Alber, Metric and Generalized Projection Operators in Banach Spaces: Properties and Applications, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, 15–50, Lecture Notes in Pure and Applied Mathematics, 178, Marcel Dekker, New York, 2006.

[3] K. Aoyama, H. Iiduka and W. Takahashi, Strong convergence of Halpern's sequence for accretive operators in a Banach space, *Panamer. Math. J.* **17** (2007), no. 3, 75–89.

[4] K. Aoyama, H. Iiduka and W. Takahashi, Weak convergence of an iterative sequence for accretive operators in Banach spaces, *Fixed Point Theory Appl.* (2006), Article ID 35390, 13 pages.

[5] F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.* **20** (1967) 197–228.

[6] L. C. Ceng, Q. H. Ansari and J. C. Yao, Mann-type steepest-descent and modified hybrid steepest-descent methods for variational inequalities in Banach spaces, *Numer. Funct. Anal. Optim.* **29** (2008), no. 9-10, 987–1033.

[7] F. Deutsch and I. Yamada, Minimizing certain convex functions over the intersection of the fixed points of nonexpansive mapping, *Numer. Funct. Anal. Optim.* **19** (1998), no. 9-10, 33–56.

[8] J. Garcia-Falset, E. Llorens-Fuster and S. Prus, The fixed point property for mappings admitting a center, *Nonlinear Anal.* **66** (2007), no. 6, 1257–1274.

[9] J. P. Gossez and E. Lami Dozo, Some geometric properties related to the fixed point theory for nonexpansive mappings, *Pacific J. Math.* **40** (1972) 565–573.

[10] H. Iiduka and W. Takahashi, Strong convergence studied by a hybrid type method for monotone operators in a Banach space, *Nonlinear Anal.* **68** (2008), no. 12, 3679–3688.
[11] H. Iiduka and W. Takahashi, Weak convergence of a projection algorithm for variational inequalities in a Banach space, *J. Math. Anal. Appl.* **339** (2008), no. 1, 668–679.
[12] J. S. Jung, Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* **302** (2005), no. 2, 509–520.
[13] P. E. Maingé, Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.* **325** (2007), no. 1, 469–479.
[14] S. Saeidi, Modified hybrid steepest-descent methods for variational inequalities and fixed points, *Math. Comput. Modelling* **52** (2010), no. 1-2, 134–142.
[15] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals, *J. Math. Anal. Appl.* **305** (2005), no. 1, 227–239.
[16] W. Takahashi, Nonlinear Functional Analysis, Fixed Point Theory and its Applications, Yokohama Publishers, Yokohama, 2000.
[17] H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc. (2)* **66** (2002), no. 1, 240–256.
[18] H. K. Xu and T. H. Kim, Convergence of hybrid steepest-descent methods for variational inequalities, *J. Optim. Theory Appl.* **119** (2003), no. 1, 185–201.
[19] I. Yamada, The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive Feasibility and Optimization and their Applications, 473–504, Stud. Comput. Math., 8, North-Holland, Amsterdam, 2001.

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