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RINGS WITH A SETWISE POLYNOMIAL-LIKE CONDITION

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Abstract. Let R be an infinite ring. Here, we prove that if 0_R belongs to $\{x_1x_2\cdots x_n \mid x_1,x_2,\ldots,x_n \in X\}$ for every infinite subset X of R, then R satisfies the polynomial identity $x^n=0$. Also, we prove that if 0_R belongs to $\{x_1x_2\cdots x_n - x_{n+1} \mid x_1,x_2,\ldots,x_n,x_{n+1} \in X\}$ for every infinite subset X of R, then $x^n=x$, for all $x\in R$.

1. Introduction

If $X_1,\ldots,X_m$ are non-empty subsets of a ring, we define as usual $X_1\cdots X_m := \{a_1\cdots a_m \mid a_i \in X_i,\ i=1,\ldots,m\}$,

$$\sum_{i=1}^{m} X_i := \Big\{\sum_{i=1}^{m} a_i \mid a_i \in X_i,\ i=1,\ldots,m\Big\};$$

and if $X_1=\cdots=X_m$, then we denote $\sum_{i=1}^m X_i$ and $X_1\cdots X_m$ by mX_1 and X_1^m, respectively. We denote the set $\{-x \mid x\in X_1\}$ by $-X_1$ and define $(-n)X_1$ as the set $-(nX_1)$ for all positive integers n. Suppose that $h(x_1,\ldots,x_n)$ is a nonzero polynomial in non-commuting indeterminates

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$x_1, \ldots, x_n$ with coefficients from the integers $\mathbb{Z}$ and zero constant. Then, we define $h^*(X_1, \ldots, X_n)$ as follows: if

$$h(x_1, \ldots, x_n) = \sum_{i=1}^{t} h_i(x_1, \ldots, x_n),$$

where $h_i(x_1, \ldots, x_n) = x_{i_1}^{\alpha_{i_1}} \cdots x_{i_n}^{\alpha_{i_n}}$ are monomials of h, we define

$$h_i^*(X_1, \ldots, X_n) = X_{i_1}^{\alpha_{i_1}} \cdots X_{i_n}^{\alpha_{i_n}},$$

and

$$h^*(X_1, \ldots, X_n) := \sum_{i=1}^{t} h_i^*(X_1, \ldots, X_n).$$

A ring R is called an h-ring if $h(r_1, \ldots, r_n) = 0$, for all $r_1, \ldots, r_n \in R$. We say that a ring R is an h^*-ring if for every n infinite subsets $X_1, \ldots, X_n$ (not necessarily distinct) of R, we have $0 \in h^*(X_1, \ldots, X_n)$.

In Theorem 1 of [2], it is proved that if $XY \cap YX \neq \varnothing$, for all infinite subsets X and Y of an infinite ring R, then R is commutative. In fact, if $c(x_1, x_2)$ is the polynomial $x_1x_2 - x_2x_1$ in non-commuting indeterminates x_1, x_2, then this result means that every infinite c^*-ring is a c-ring. In [1], a ring is called a virtually h-ring, if in every n infinite subsets $X_1, \ldots, X_n$ of R, there exist elements $a_i \in X_i$ $(i = 1, \ldots, n)$ such that $h(a_1, \ldots, a_n) = 0$. It is clear that every virtually h-ring is an h^*-ring. It is asked in [1] if every infinite virtually h-ring is an h-ring. We ask the same question for h^*-rings.

Suppose that $\alpha_1, \ldots, \alpha_t$ are positive integers and $t > 0$ is an integer such that either $t > 1$ or ($t = 1$ and $\alpha_1 > 1$). Suppose that $\mathcal{M}(x_1, \ldots, x_n)$ is the polynomial $x_{i_1}^{\alpha_1} \cdots x_{i_t}^{\alpha_t}$ in non-commuting indeterminates $x_1, \ldots, x_n$; and let $J_n(x) = x^n - x$, where $n > 1$ is a positive integer. Our main results are the followings

Theorem 1. *Every infinite $\mathcal{M}^*$-ring is an $\mathcal{M}$-ring.*

Corollary 1. *Let $n > 1$ be an integer and $\mathcal{N}(x) = x^n$. Then, every infinite $\mathcal{N}^*$-ring is an $\mathcal{N}$-ring.*

Theorem 2. *Every infinite J_n^*-ring is a J_n-ring.*

Theorem 1 generalizes Theorem 1.3 of [1], which says that every infinite virtually $\mathcal{M}$-ring is an $\mathcal{M}$-ring. In [3], it is proved that every infinite virtually J_n-ring is a J_n-ring, and so Theorem 2 generalizes the latter result.

Let R be a ring and Y be a non-empty subset or an element of R. We denote the ring of all $m \times m$ matrices over R by $\mathrm{Mat}_m(R)$; $J(R)$ and $\mathrm{A}(Y)$ denote the Jacobson radical of R and the annihilator of Y in R, respectively; and for a left R-module V, $\mathrm{End}_R(V)$ denotes the ring of all left R-module endomorphisms of V. Following [7]; a ring R is called an FZS-ring if every zero subring (i.e., every subring with trivial multiplication) of R is finite.

2. **Some general results on h^*-rings**

Throughout we assume that $h(x_1, \ldots, x_n)$ is a nonzero polynomial in non-commuting indeterminates $x_1, \ldots, x_n$ with coefficients from $\mathbb{Z}$ and zero constant. We need the following famous theorem due to Kaplansky [6].

Kaplansky's Theorem. [6] If R is a left primitive ring satisfying a polynomial identity of degree d, then R is a finite dimensional simple algebra over its center, of dimension at most $[d/2]^2$.

The proof of the following lemma is similar to that of Lemma 2.4 of [1].

Lemma 2.1. *Every left primitive h^*-ring is Artinian.*

Lemma 2.2. *Let R be an infinite h^*-ring. If I is an infinite ideal of R, then R/I is an h-ring.*

Proof. Let $r_1, \ldots, r_n \in R$ and consider the infinite subsets $X_i = r_i + I$, for $i \in \{1, \ldots, n\}$. Since R is an h^*-ring, $0 \in h^*(r_1 + I, \ldots, r_n + I)$. Since I is an ideal of R, it follows that $h(r_1, \ldots, r_n) \in I$. Thus, R/I is an h-ring. □

3. **Proofs**

Suppose that $\alpha_1, \ldots, \alpha_t$ are positive integers and $t > 0$ is an integer such that either $t > 1$ or ($t = 1$ and $\alpha_1 > 1$); and $\mathcal{M}(x_1, \ldots, x_n) = x_{i_1}^{\alpha_1} \cdots x_{i_t}^{\alpha_t}$. Also suppose that $J_n(x) = x^n - x$, where $n > 1$ is an integer.

Proof of Theorem 1. Let R be an infinite $\mathcal{M}^*$-ring. We have that $R/J(R)$ is the subdirect product of R/P_i, where P_i is a left primitive ideal of R, for all $i \in I$. By Lemma 2.1, R/P_i is Artinian, for all $i \in I$. Since for every infinite division ring D, the full matrix ring $\mathrm{Mat}_k(D)$ is not an $\mathcal{M}^*$-ring (consider the set DI_k which contains no non-zero zero divisor, where I_k is the identity $k \times k$ matrix), it follows that R/P_i cannot be infinite. Thus, P_i is infinite, for all $i \in I$.
Now, it follows from Lemma 2.2 that R/P_i is an $\mathcal{M}$-ring, for all $i \in I$ and so $R/J(R)$ is an $\mathcal{M}$-ring. Thus, $x^{\alpha_1+\cdots+\alpha_t} \in J(R)$, for all $x \in R$. Now, we prove that R is periodic, that is, for every element $x \in R$, there exist two distinct positive integers s and t such that $x^s = x^t$. Consider the set $X = \{x^k \mid k \in \mathbb{N}\}$ for an arbitrary element $x \in R$. If X is infinite, then by the hypothesis, there are positive integers $k_1, \ldots, k_n$ such that $x^{k_1+\cdots+k_t} = 0$. Thus, $x^{k_1+\cdots+k_t+1} = x^{k_1+\cdots+k_t} = 0$. If X is finite, then there exist two distinct integers t and s such that $x^t = x^s$. Hence, R is periodic.
Now, we prove that R is a nil ring. Let $a \in R$. Since R is periodic, there exist two distinct positive integers s and t such that $s > t$ and $(a^d)^s = (a^d)^t$, where $d = \alpha_1 + \cdots + \alpha_t$. Thus, $a^{dt(s-t)}$ is idempotent. Since every idempotent element in $J(R)$ is zero, $a^{dt(s-t)} = 0$. Let $x \in R$. Then, by Lemma 2.6 of [4], $A(x)$ is infinite and Theorem 1.3 of [1] implies that $A(x)$ contains an infinite zero subring T. Now, consider the infinite set $x + T$. Thus, $0 \in (x+T)^{\alpha_1} \cdots (x+T)^{\alpha_t}$, and so $x^d = 0$. Let $S = \mathrm{Lev}(R)$, the Levitski radical of R, i.e., the unique maximal locally nilpotent ideal of R. It follows that $\overline{R} = R/S$ is a nil ring of degree at most d. If $\overline{R}$ is non-zero, then it follows from Lemma 1.6.24 of [9] that $\overline{R}$ contains a non-zero nilpotent ideal, which is not possible, since $\mathrm{Lev}(\overline{R}) = 0$. Thus, $\overline{R} = 0$ and R is locally nilpotent.
Now, let $x_1, \ldots, x_n \in R$. Then, $A(x_1, \ldots, x_n)$ is infinite, by Lemma 2.6 of [4] and it contains an infinite zero subring T, by Theorem 1.3 of [1]. By the hypothesis, $0 \in (x_{i_1} + T)^{\alpha_1} \cdots (x_{i_t} + T)^{\alpha_t}$, which implies that $\mathcal{M}(x_1, \ldots, x_n) = x_{i_1}^{\alpha_1} \cdots x_{i_t}^{\alpha_t} = 0$, as required. □

Proof of the Corollary 1. By considering $t = n$ and $\alpha_1 = \cdots = \alpha_t = 1$ in Theorem 1, the proof follows from Theorem 1. □

We need the following easy lemma in the proof of Lemma 3.2.

Lemma 3.1. *Let $n > 1$ be a positive integer, and let p be a prime larger than n. Then, for all positive integers $t_1, \ldots, t_n, t_{n+1}$, we have $p^{t_1} + \cdots + p^{t_n} \neq p^{t_{n+1}}$.*

Proof. It is straightforward. □

Lemma 3.2. *Let R be an infinite J_n^*-ring. Then, R is a periodic FZS-ring.*

Proof. We first prove that R is a periodic ring. Let x be an element of R. Let $p > n$ be a prime number and suppose $X = \{x^{p^t} \mid t \in \mathbb{N}\}$. If X is finite, then $x^{p^t} = x^{p^s}$, for some distinct integers t and s. Now, suppose that X is infinite. Then, by the hypothesis, there exist positive integers $t_1, \ldots, t_n, t_{n+1}$ such that $x^{p^{t_1}+\cdots+p^{t_n}} = x^{p^{t_{n+1}}}$. Since $p > n$, it follows from Lemma 3.1 that $p^{t_1} + \cdots + p^{t_n} \neq p^{t_{n+1}}$. Thus, in any case there are distinct positive integers r_1 and r_2 such that $x^{r_1} = x^{r_2}$, and so R is periodic.
Now, we show that every zero subring of R is finite (i.e., R is an FZS-ring). Suppose, for a contradiction, that S is an infinite zero subring. By the hypothesis, there are n elements $s_1, \ldots, s_n, s_{n+1} \in S\backslash\{0\}$ such that $s_1 \cdots s_n = s_{n+1}$, and so $s_{n+1} = 0$, a contradiction. Thus, R is an FZS-ring. □

Lemma 3.3. *If F is a field which is a J_n^*-ring, then F is finite.*

Proof. Suppose, for a contradiction, that F is infinite. It follows from Lemma 3.2 that F is a periodic ring, and so every nonzero element of F satisfies a polynomial of the form $x^m - 1$, for some positive integer m. Thus, F has prime characteristic p, since the rational number $\frac{1}{2}$ does not satisfy a polynomial of the form $x^m - 1$. Thus, for each element a of $F^* = F\backslash\{0\}$, there exists a positive integer k such that $a^{p^k-1} = 1$, and so F^* is an infinite torsion locally cyclic group. It follows that there exists an infinite sequence of positive integers $n_1 < n_2 < \cdots$, such that for each $i \in \mathbb{N}$ there is an element $a_i \in F^*$ of order $p^{n_i} - 1$. Now, by Theorems I and V of [5], for $i > 6$, each $p^{n_i}-1$ has a prime divisor q_i such that $q_i \equiv 1 \mod n_i$. Therefore, the set $\{q_i \mid i > 6\}$ of primes is infinite, and so the abelian group F^* has infinitely many primary components. It follows that F^* contains two infinite subgroups N and M such that $N \cap M = 1$. Now, let $a \in F^*$ and consider the infinite sets aN and aM. Since F is a J_n^*-ring, there exist elements $a_1, \ldots, a_{n+1} \in N$ and $b_1, \ldots, b_{n+1} \in M$ such that

$$(aa_1) \cdots (aa_n) = aa_{n+1} \text{ and } (ab_1) \cdots (ab_n) = ab_{n+1}.$$

It follows that $a^{n-1} \in N \cap M$, and so $a^{n-1} = 1$. Since a is an arbitrary element of F^*, it follows that each q_i divides $n-1$, a contradiction. This completes the proof. □

We will use the following result due to Herstein in the proof of Theorem 2.

Theorem 3.4 (Herstein [8])**.** *Let R be a periodic ring in which every nilpotent element is central. Then, R is commutative.*

Proof of Theorem 2. For any ring S, we will denote by $N = N(S)$ and $Z = Z(S)$ the set of nilpotent elements and the center, respectively. Let R be an infinite J_n^*-ring. By Theorem 6 of [7], N is finite; and since R is infinite, Z is infinite by Theorem 7 of [7]. For $u \in N$, consider the additive homomorphism $\phi : Z \to uZ$, given by $z \mapsto uz$. Since $\phi(Z)$ is finite, $W = \ker\phi = Z \cap A(u)$ is of finite index in $(Z, +)$, and hence is infinite. Since $u + W$ is infinite, there exist $z_1, z_2, \ldots, z_{n+1} \in W$ such that

$$(u + z_1)(u + z_2)\cdots(u + z_n) = u + z_{n+1},$$

that is,

$$(*) \quad u^n + z_1 z_2 \cdots z_n = u + z_{n+1}.$$

Multiplying $(*)$ by u gives $u^2 = u^{n+1}$, and it follows easily that $u^2 = 0$, and therefore $u^n = 0$. We now conclude from $(*)$ that $u \in Z$. Thus, by Theorem 3.4, R is commutative.

Now, write R as a subdirect product of subdirectly irreducible rings R_α, and note that each finite R_α is a J_n-ring by Lemma 2.2. Suppose that there exists $\alpha = \alpha_0$ such that R_{α_0} is infinite. Since R_{α_0} is subdirectly irreducible and commutative, the only nonzero idempotent is 1; and since R_{α_0} is periodic, each nonnilpotent element has a power which is idempotent, i.e., it is invertible. Consequently, $\frac{R_{\alpha_0}}{N(R_{\alpha_0})}$ is a field, which by Lemma 3.3 must be finite. Thus, R_{α_0} is finite, and so we have a contradiction. Therefore, all R_α are finite and R is a J_n-ring. □

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References

[1] A. Abdollahi and S. Akbari, Rings virtually satisfying a polynomial identity, *J. Pure Appl. Algebra* **198** (2005), no. 1-3, 9–19.
[2] A. Abdollahi, H. E. Bell and A. A. Klein, On commutativity and centrality in infinite rings, *Comm. Algebra* **35** (2007), no. 4, 1323–1332.
[3] A. Abdollahi and B. Taeri, A combinatorial condition on infinite rings, *Proceedings of the 31st Iranian Mathematics Conference (Tehran, 2000)*, pp. 23–27, Univ. Tehran, Tehran, 2000.
[4] A. Abdollahi and B. Taeri, On a class of infinite rings, *Algebra Colloq.* **8** (2001), no. 2, 153–157.
[5] G. D. Birkhoff and H. S. Vandiver, On the integral divisors of $a^n - b^n$, *Ann. of Math.* (2) **5** (1904), no. 4, 173–180.
[6] I. Kaplansky, Rings with a polynomial identity, *Bull. Amer. Math. Soc.* **54** (1948), no. 6, 575–580.
[7] A. A. Klein and H. E. Bell, Rings with finitely many nilpotent elements, *Comm. Algebra* **22** (1994), no. 1, 349–354.
[8] I. N. Herstein, A note on rings with central nilpotent elements, *Proc. Amer. Math. Soc.* **5** (1954), 620.
[9] L. H. Rowen, Polynomial Identities in Ring Theory, Academic Press Inc., New York-London, 1980.

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