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CONVERGENCE THEOREMS OF AN IMPLICIT ITERATION PROCESS FOR ASYMPTOTICALLY PSEUDO-CONTRACTIVE MAPPINGS

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Abstract. The purpose of this paper is to study the strong convergence of an implicit iteration process with errors to a common fixed point for a finite family of asymptotically pseudocontractive mappings and nonexpansive mappings in normed linear spaces. The results in this paper improve and extend the corresponding results of Xu and Ori, Zhou and Chang, Sun, Yang and Yu in some aspects.

1. **Introduction and Preliminaries**

Throughout this paper we assume that E is an arbitrary real Banach space and E^* denotes the dual space of E. The normalized duality map $J : E \to 2^{E^*}$ is defined by

$$Jx := \{u^* \in E^* : \langle x, u^* \rangle = \|x\|^2; \|u^*\| = \|x\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between elements of E and E^*. If E^* is strictly convex, then J is single-valued.

We first recall some definitions and conclusions.

Definition 1.1. *Let $T : D(T) \subset E \to E$ be a mapping.*

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(1) *T is said to be asymptotically nonexpansive* (*see, e.g.,* [5]) *if there exists a sequence $\{k_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ such that*

$$\|T^n x - T^n y\| \le k_n \|x-y\|, \quad \forall x,\ y \in D(T), \quad n \ge 1;$$

(2) *T is said to be asymptotically pseudo-contractive* (*see, e.g.,* [10]) *with sequence $\{k_n\} \subset [0,\infty)$, if and only if $\lim_{n\to\infty} k_n = 1$, for all $n \ge 1$, x, $y \in D(T)$ and there exists $j(x-y) \in J(x-y)$ such that*

$$\langle T^n x - T^n y, j(x-y)\rangle \le k_n \|x-y\|^2;$$

(3) *T is said to be strictly asymptotically pseudo-contractive with sequence $\{k_n\} \subset [0,\infty)$, if and only if $\lim_{n\to\infty} k_n = k \in (0,1)$, for all $n \ge 1$, x, $y \in D(T)$ and there exists $j(x-y) \in J(x-y)$ such that*

$$\langle T^n x - T^n y, j(x-y)\rangle \le k_n \|x-y\|^2;$$

(4) *T is said to be asymptotically nonexpansive in the intermediat sense* (*see, e.g.,* [1]) *if*

$$\limsup_{n\to\infty}\{\sup_{x,y\in D(T)} (\|T^n x - T^n y\| - \|x-y\|)\} \le 0.$$

(5) *T is called uniformly L-Lipschitzian* (*see, e.g.,* [4]) *if there exists a constant $L > 0$ such that*

$$\|T^n x - T^n y\| \le L\|x-y\|, \quad \text{for all } x, y \in D(T), \quad n \ge 1.$$

It is easy to see that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian and every asymptotically nonexpansive mapping is asymptotically pseudo-contractive. Rhoades [9] constructed an example to show that the class of asymptotically pseudo-contractive mappings properly contains the class of asymptotically nonexpansive mappings. It is clear that an asymptotically nonexpansive mapping is asymptotically nonexpansive in the intermediate sense if the domain of T is bounded. But the converse is not true.

Example 1.2. *Let $E = R = (-\infty,\infty)$ with the usual norm. Take $K = [0,1]$ and define $T : K \to K$ by*

$$Tx = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{9} & \text{if } x = 1, \\ x - \frac{1}{3^{n+1}} & \text{if } \frac{1}{3^{n+1}} \le x < \frac{1}{3}(\frac{1}{3^{n+1}} + \frac{1}{3^n}), \\ \frac{1}{3^n} - x & \text{if } \frac{1}{3}(\frac{1}{3^{n+1}} + \frac{1}{3^n}) \le x < \frac{1}{3^n} \end{cases}$$

for all $n \geq 0$*. Then* $F(T) = \{0\}$ *and* T *is not continuous at* $x = 1$*. We can verify that*

$$Tx \leq \frac{1}{3}x, \quad x \in K.$$

Thus T^2 *is continuous in* K *and* $T^2 K \subset [0, 3^{-n}]$ *for all* $n \geq 1$*. Then for any* $x \in K$*, there exists* $j(x-0) \in J(x-0)$ *satisfying*

$$\langle T^n x - T^n 0, j(x-0) \rangle = T^n x \cdot x \leq \frac{1}{3}\|x\|^2 < \|x\|^2$$

for all $n \geq 1$*. That is,* T *is asymptotically pseudo-contractive mapping with sequence* $\{k_n\} = 1$*. It follows that*

$$\limsup_{n\to\infty} \sup\{\|T^x - T^n y\| - \|x - y\| : x, y \in K\} \leq \limsup_{n\to\infty} 3^{-n} = 0.$$

That is, T *is also asymptotically nonexpansive in the intermediate sense.*

Let K be a nonempty closed convex subset of E and $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings from K into itself (i.e., $\|T_i x - T_i y\| \leq \|x - y\|$ for $x, y \in K$ and $i = 1, 2, \ldots, N$). In 2001, Xu and Ori [14] introduced the following implicit iteration process. For an arbitrary $x_0 \in K$ and $\{\alpha_n\}_{n=1}^\infty \subset [0, 1]$, the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows:

$$\begin{cases} x_1 = (1-\alpha_1)x_0 + \alpha_1 T_1 x_1, \\ x_2 = (1-\alpha_2)x_1 + \alpha_2 T_2 x_2, \\ \quad \vdots \\ x_N = (1-\alpha_N)x_{N-1} + \alpha_N T_N x_N, \\ x_{N+1} = (1-\alpha_{N+1} x_N + \alpha_{N+1} T_1 x_{N+1}, \\ \quad \vdots \end{cases}$$

The scheme is expressed in its compact form by

$$x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_{n(\bmod N)} x_n, \ n \geq 1. \tag{1.1}$$

Using this iteration, they proved that the sequence $\{x_n\}$ converges weakly to a common fixed point of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ in a Hilbert space under certain conditions. Since then, construction of fixed points for nonexpansive mappings and strictly pseudo-contractive mappings and some other mappings via the implicit iterative algorithm has been extensively investigated by many authors (see, e.g., [2, 6, 7, 8, 12, 14, 15, 16, 17, 18] and the references cited therein). An implicit process is generally desirable when no explicit scheme is available.

Such a process is generally used as a "tool" to establish the convergence of an explicit scheme.

In 2006, Chang et al. [2] introduced another implicit iteration process with error. In the sense of [2], the implicit iteration process with errors for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^N$ is generated from an arbitrary $x_0 \in K$ by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} x_n + u_n, \quad \forall n \geq 1, \tag{1.2}$$

where $n = (k-1)N + i, i = i(n) \in \{1, 2, \cdots, N\}, k = k(n) \geq 1$ is a positive integer and $k(n) \to \infty$, as $n \to \infty$. $\{\alpha_n\}$ is a suitable sequence in $[0,1]$ and $\{u_n\} \subset K$ is such that $\sum_{n=1}^{\infty} \|u_n\| < \infty$. They extended the results of [14] from Hilbert spaces to more general uniformly convex Banach spaces and from nonexpansive mappings to asymptotically nonexpansive mappings.

It is clear that if K is a nonempty convex subset of E and $\{u_n\} \subset K$ such that $\sum_{n=1}^{\infty} \|u_n\| < \infty$, then the implicit iterative sequence with errors in the sense of [2] need not be well defined, i.e., $\{x_n\}_{n=1}^{\infty}$ may fail to be in K. More precisely, the conditions imposed on the error terms are not compatible with the randomness of the occurrence of errors.

As a generalization of [2], Yang and Hu [15] proposed another implicit iteration process which appears to be more satisfactory as follows:

$$x_n = \alpha_n x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_n + \gamma_n u_n, \quad \forall n \geq 1, \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0,1]$, and $\alpha_n + \beta_n + \gamma_n = 1$. $\{u_n\}$ is bounded in K.

Recently, Yang [16] introduced composite implicit iteration process with errors for a finite family of asymptotically demicontractive mappings defined as follows:

$$\begin{cases} x_n = \alpha_n x_{n-1} + \beta_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n, \\ y_n = \delta_n x_n + \lambda_n T_{i(n)}^{k(n)} x_n + \tau_n v_n, \quad \forall n \geq 1, \end{cases} \tag{1.4}$$

where $n = (k-1)N + i$, $i = i(n) \in \{1, 2, \ldots, N\}$, $k = k(n) \geq 1$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$. $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\delta_n\}$, $\{\tau_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \delta_n + \lambda_n + \tau_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K. The results of [16] improve and extend the corresponding results of [15] and others.

Since for each $n \geq 1$, it can be written as $n = (k-1)N + i$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $k = k(n) \geq 1$ is a positive integer and $k(n) \to$

∞ as $n \to \infty$. Hence, (1.3) can be expressed in the following form:

$$x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T^n_{i(n)} x_n + \gamma_n u_n, \quad \forall n \geq 1, \tag{1.5}$$

where $\{\alpha_n\}$, $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \gamma_n \leq 1$, and $\{u_n\}$ is a bounded sequence in K.

It is much more interesting and important for one to establish strong convergence theorems without compactness assumptions on the mapping considered or on its domain. The aim of this paper is to prove the strong convergence theorems using the modified implicit iteration process with errors for a finite family of asymptotically pseudo-contractive mappings and nonexpansive mappings in a real normed linear space. The results in this paper improve and extend the corresponding results of Xu and Ori, Zhou and Chang, Sun, Yang and Yu in some aspects.

In what follows we shall use the following results:

Lemma 1.3. *Let E be a normed linear space, then for all $x, y \in E$ and for all $j(x+y) \in J(x+y)$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle.$$

Lemma 1.4. *(see* [13, Lemma 1]*) Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + c_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then

(i) $\lim_{n\to\infty} a_n$ *exists;*

(ii) *In particular, if $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ converging to 0, then* $\lim_{n\to\infty} a_n = 0$.

The following lemma appears useful for the proof of the main result.

Lemma 1.5. *Let $\phi : [0, \infty) \to [0, \infty)$ be a strictly increasing function with $\phi(0) = 0$ and let $\{\rho_n\}, \{\lambda_n\}, \{\mu_n\}, \{c_n\}$ be nonnegative real sequences such that $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n=1}^{\infty} c_n < \infty$. Suppose that*

$$\rho^2_{n+1} \leq \rho^2_n - \lambda_n \phi(\rho_{n+1}) + \lambda_n \mu_n + c_n, \quad n \geq 1. \tag{1.6}$$

Then $\rho_n \to 0$ as $n \to \infty$.

Proof. The proof follows from the following two claims.

Claim 1. $\liminf_{n\to\infty} \rho_n = 0$.

Suppose on the contrary that $\liminf_{n\to\infty} \rho_n = \delta > 0$. Then there exists some positive integer N_0 such that $\rho_n \geq \delta > 0$ for all $n \geq N_0$.

Since $\phi(t)$ is strictly increasing and $\delta > 0$, $\phi(\rho_{n+1}) \geq \phi(\delta) > 0$ for all $n \geq N_0$. Also, since $\lim_{n\to\infty} \mu_n = 0$, there exists a positive integer $N_1 \geq N_0$ such that $\mu_n \leq \phi(\delta)/2$ for all $n \geq N_1$. Then, it follows from (1.6) that for all $n \geq N_1$

$$\begin{aligned} \rho_{n+1}^2 &\leq \rho_n^2 - \lambda_n \phi(\delta) + \frac{1}{2}\lambda_n \phi(\delta) + c_n \\ &= \rho_n^2 - \frac{1}{2}\lambda_n \phi(\delta) + c_n. \end{aligned}$$

Hence, for any $n \geq N_1$,

$$\frac{1}{2}\phi(\delta) \sum_{j=N_1}^{n} \lambda_n \leq \rho_{N_1}^2 - \rho_{n+1}^2 + \sum_{j=N_1}^{n} c_n \leq \rho_{N_1}^2 + \sum_{j=N_1}^{n} c_n.$$

This implies that $\sum_{n=1}^{\infty} \lambda_n < \infty$, which is contradiction. Thus, $\delta = 0$, i.e., $\liminf_{n\to\infty} \rho_n = 0$.

Claim 2. $\lim_{n\to\infty} \rho_n = 0$.

Suppose, on the contrary that $\limsup_{n\to\infty} \rho_n = b > 0$. Then there exists a subsequence $\{\rho_{n_j}\}$ of $\{\rho_n\}$ such that $\lim_{j\to\infty} \rho_{n_j} = b$. Thus there exists a positive integer k such that $\rho_{n_j} \geq b/2$ for all $n_j \geq n_k$. Since $\phi(t)$ is strictly increasing, $\phi(\rho_{n_j}) \geq \phi(b/2)$ for all $n_j \geq n_k$. Note that $\lim_{n\to\infty} \mu_n = 0$, without loss of generality, we have $\mu_n \leq \phi(b/2)$ for all $n \geq n_k$. It follows from (1.6) that

$$\begin{aligned} \rho_{n_j+1}^2 &\leq \rho_{n_j}^2 - \lambda_{n_j} \phi(b/2) + \lambda_{n_j} \phi(b/2) + c_{n_j} \\ &= \rho_{n_j}^2 + c_{n_j} \end{aligned}$$

for all $n_j \geq n_k$. This with Lemma 1.4(i) implies that $\lim_{j\to\infty} \rho_{n_j}$ exists. Since $\liminf_{n\to\infty} \rho_n = 0$, it follows from Lemma 1.4(ii) that we have $\lim_{j\to\infty} \rho_{n_j} = 0$, which is a contradiction with $b > 0$. Thus $\limsup_{n\to\infty} \rho_n = 0$. Therefore $\lim_{n\to\infty} \rho_n = 0$. This completes the proof. $\square$

Lemma 1.6. *Let E be a real normed linear space, and K be a nonempty convex subset of E. Let $T_i : K \to K$ $(i = 1, 2, \ldots, N)$ be a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically pseudo-contractive mapping with $\{k_{in}\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, where $k_n = \max_{1\leq i\leq N}\{k_{in}\}$. Assume that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ denotes the set of common fixed points of $\{T_i\}_{i=1}^{N}$. Let $\{x_n\}$ be the sequence defined by (1.5). Suppose that $\{u_n\}$ is bounded in K and that $\{\alpha_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \gamma_n < \infty, \quad \sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty;$
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0.$

Then $\{x_n\}$, $\{T_i^n x_n\}_{i=1}^N$ *are bounded in* K.

Proof. Since $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. If we denote $M = \sup\{\|u_n - p\| : n \geq 1\}$, for any given $p \in F$, then $M < \infty$. Since for each $i = 1, 2, \dots, N$, $T_i : K \to K$ is an asymptotically pseudo-contractive mappings with $\{k_{in}\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, then we have

$$\langle T_i^n x - T_i^n y, j(x - y)\rangle \leq k_{in}\|x - y\|^2 \leq k_n\|x - y\|^2, \tag{1.7}$$

for all $n \geq 1$, $x, y \in K$. It follows from Lemma 1.4, (1.5) and (1.7) that

$$\begin{aligned} \|x_n - p\|^2 &= \langle (1 - \alpha_n - \gamma_n)(x_{n-1} - p) + \alpha_n(T_i^n x_n - p) \\ &\quad + \gamma_n(u_n - p), j(x_n - p)\rangle \\ &= (1 - \alpha_n - \gamma_n)\langle x_{n-1} - p, j(x_n - p)\rangle \\ &\quad + \alpha_n\langle T_i^n x_n - p, j(x_n - p)\rangle \\ &\quad + \gamma_n\langle u_n - p, j(x_n - p)\rangle \\ &\leq (1 - \alpha_n)\|x_{n-1} - p\|\|x_n - p\| + \alpha_n k_n\|x_n - p\|^2 \\ &\quad + M\gamma_n\|x_n - p\|. \end{aligned} \tag{1.8}$$

If $\|x_n - p\| = 0$ for infinitely many n, then $\{x_n\}$ is bounded. If $\|x_n - p\| > 0$, it follows from (1.8) that we have

$$(1 - \alpha_n k_n)\|x_n - p\| \leq (1 - \alpha_n)\|x_{n-1} - p\| + M\gamma_n. \tag{1.9}$$

Noticing $\alpha_n \to 0$, $k_n \to 1$ as $n \to \infty$, we have $\lim_{n\to\infty}(1 - \alpha_n k_n) = 1 > 0$. Without loss of generality, we assume that $(1 - \alpha_n k_n) > 0$ for all $n \geq 1$. Therefore, from (1.9), we obtain that

$$\begin{aligned} \|x_n - p\| &\leq \frac{1 - \alpha_n}{1 - \alpha_n k_n}\|x_{n-1} - p\| + \frac{M}{1 - \alpha_n k_n}\gamma_n \\ &= \left(1 + \frac{\alpha_n(k_n - 1)}{1 - \alpha_n k_n}\right)\|x_{n-1} - p\| + \frac{M}{1 - \alpha_n k_n}\gamma_n, \quad \forall n \geq 1. \end{aligned} \tag{1.10}$$

By virtue of $\lim_{n\to\infty}(1 - \alpha_n k_n) = 1 > 1/2$, there exists a positive integer n_0 such that $(1 - \alpha_n k_n) \geq 1/2$ for all $n \geq n_0$. It follows from (1.10) that

$$\|x_n - p\| \leq [1 + 2\alpha_n(k_n - 1)]\,\|x_{n-1} - p\| + 2M\gamma_n, \tag{1.11}$$

for all $n \geq n_0$. Taking $a_n = \|x_{n-1} - p\|$, $b_n = 2M\gamma_n$, $c_n = 2\alpha_n(k_n - 1)$ in Lemma 1.4, we know that $\{\|x_n - p\|\}$ is bounded. Therefore, $\{x_n\}$ is bounded in K.

On the other hand, we have

$$\begin{aligned} \|T_i^n x_n - p\| &= (\|T_i^n x_n - p\| - \|x_n - p\|) + \|x_n - p\| \\ &\le \sup_{x,y\in K} (\|T_i^n x - T_i^n y\| - \|x - y\|) + D_1, \end{aligned} \tag{1.12}$$

where $\|x_n - p\| \le D_1$. By the asymptotically nonexpansiveness of $T_i (i = 1, 2, \ldots, N)$ in the intermediate sense, i.e.,

$$\limsup_{n\to\infty} \{ \sup_{x,y\in K} (\|T_i^n x - T_i^n y\| - \|x - y\|)\} \le 0.$$

There exists a positive integer n_0 such that $\sup_{x,y\in K}(\|T_i^n x - T_i^n y\| - \|x - y\|) \le 1$ for all $n > n_0$ and $i = 1, 2, \ldots, N$, so from (1.12), we get

$$\|T_i^n x_n - p\| \le 1 + D_1$$

for all $n > n_0$ and $i = 1, 2, \ldots, N$. Setting

$$M_1 = \max_{1\le i\le N} \left\{ \|T_i x_1 - p\|, \|T_i^2 x_2 - p\|, \ldots, \|T_i^{n_0} x_{n_0} - p\|, 1 + D_1 \right\},$$

we have

$$\|T_i^n x_n - p\| \le M_1, \quad n \ge 1, \text{ and } i = 1, 2, \ldots, N.$$

Noticing $\|T_i^n x_n\| \le \|T_i^n x_n - p\| + \|p\|$, we get $\{T_i^n x_n\}$ is bounded in K for $i = 1, 2, \ldots, N$. This completes the proof. □

2. **Main Results**

Theorem 2.1. *Let E be a real normed linear space, and K be a nonempty convex subset of E. Let $T_i : K \to K$ $(i = 1, 2, \ldots, N)$ be a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically pseudo-contractive mappings with $\{k_{in}\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, where $k_n = \max_{1\le i\le N}\{k_{in}\}$. Assume that $F = \bigcap_{i=1}^N F(T_i) \ne \emptyset$ denotes the set of common fixed points of $\{T_i\}_{i=1}^N$. Let $\{x_n\}$ be the sequence defined by* (1.5)*. Suppose that $\{u_n\}$ is bounded in K and that $\{\alpha_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\sum_{n=1}^\infty \gamma_n < \infty, \quad \sum_{n=1}^\infty \alpha_n(k_n - 1) < \infty$;
(ii) $\sum_{n=1}^\infty \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0.$

Assume that there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ such that

$$\begin{aligned} &\limsup_{n\to\infty}\{\langle T_i^n x_n - p, j(x_n - p)\rangle - k_n\|x_n - p\|^2 \\ &\quad + \phi(\|x_n - p\|)\} \le 0 \end{aligned} \tag{2.1}$$

for $p \in F$ and $i = 1, 2, \dots, N$. Then $\{x_n\}$ converges strongly to a common fixed point p of $\{T_i\}_{i=1}^N$.

Proof. It follows from Lemma 1.4 and (1.5) that

$$\begin{aligned} \|x_n - p\|^2 &= \|(x_{n-1} - p) - \alpha_n(x_{n-1} - T_i^n x_n) + \gamma_n(u_n - x_{n-1})\|^2 \\ &\le \|x_{n-1} - p\|^2 - 2\alpha_n\langle x_{n-1} - T_i^n x_n, j(x_n - p)\rangle \\ &\quad + 2\gamma_n\langle u_n - x_{n-1}, j(x_n - p)\rangle \\ &= \|x_{n-1} - p\|^2 + 2\alpha_n\langle T_i^n x_n - p, j(x_n - p)\rangle \\ &\quad - 2\alpha_n\langle x_n - p, j(x_n - p)\rangle + 2\langle x_n - x_{n-1}, j(x_n - p)\rangle \\ &\quad + 2\gamma_n\langle u_n - x_{n-1}, j(x_n - p)\rangle \\ &= \|x_{n-1} - p\|^2 + 2\alpha_n\langle T_i^n x_n - p, j(x_n - p)\rangle \\ &\quad - 2\alpha_n\|x_n - p\|^2 + 2\alpha_n^2\langle T_i^n x_n - x_{n-1}, j(x_n - p)\rangle \\ &\quad + 2(\alpha_n + 1)\gamma_n\langle u_n - x_{n-1}, j(x_n - p)\rangle \\ &\le \|x_{n-1} - p\|^2 + 2\alpha_n\langle T_i^n x_n - p, j(x_n - p)\rangle \\ &\quad - 2\alpha_n\|x_n - p\|^2 + 2\alpha_n^2\|T_i^n x_n - x_{n-1}\|\|x_n - p\| \\ &\quad + 4\gamma_n\|u_n - x_{n-1}\|\|x_n - p\|. \end{aligned} \tag{2.2}$$

Notice that

$$\begin{aligned} 2\alpha_n\langle T_i^n x_n - p, j(x_n - p)\rangle &= 2\alpha_n d_n + 2\alpha_n[k_n\|x_n - p\|^2 \\ &\quad -\phi(\|x_n - p\|)], \end{aligned} \tag{2.3}$$

where $d_n = \langle T_i^n x_n - p, j(x_n - p)\rangle - k_n\|x_n - p\|^2 + \phi(\|x_n - p\|)$. It follows from (2.2) and (2.3) that we have

$$\begin{aligned} \|x_n - p\|^2 &\le \|x_{n-1} - p\|^2 + 2\alpha_n(d_n + D^2\alpha_n) - 2\alpha_n\phi(\|x_n - p\|) + e_n \\ &\le \|x_{n-1} - p\|^2 - \alpha_n\phi(\|x_n - p\|) + 2\alpha_n(d_n + D^2\alpha_n) + e_n, \end{aligned} \tag{2.4}$$

where $e_n = D^2[2\alpha_n(k_n - 1) + 4\gamma_n]$ and $D = \max\{\sup_{n\ge1}\{\|x_n - p\|\}, \sup_{n\ge1}\{\|u_n - x_{n-1}\|\}, \max\{\sup_{n\ge1}\{\|x_{n-1} - T_i^n x_n\|\}_{i=1}^N\}$. From condition (i), we have that $\sum_{n=1}^\infty e_n < \infty$. It follows from (2.4) and Lemma 1.5 that $\|x_n - p\| \to 0$ as $n \to \infty$. This completes the proof. □

It follows from Theorem 2.1 that we have the following result.

Corollary 2.2. *Let E be a real normed linear space, and K be a nonempty bounded convex subset of E. Let $T_i : K \to K$ $(i = 1, 2, \dots, N)$ be a finite family of asymptotically nonexpansive mappings with $\{k_{in}\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, where $k_n = \max_{1\le i\le N}\{k_{in}\}$. Assume that $F = \bigcap_{i=1}^N F(T_i) \ne \emptyset$ denotes the set of common fixed points of $\{T_i\}_{i=1}^N$.*

Let $\{x_n\}$ *be the sequence defined by* (1.5)*. Suppose that* $\{\alpha_n\}$*,* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty}\gamma_n<\infty, \quad \sum_{n=1}^{\infty}\alpha_n(k_n-1)<\infty;$
(ii) $\sum_{n=1}^{\infty}\alpha_n=\infty, \quad \lim_{n\to\infty}\alpha_n=0.$

Assume that there exists a strictly increasing function $\phi:[0,\infty)\to[0,\infty)$*,* $\phi(0)=0$ *such that*

$$\limsup_{n\to\infty}\left\{\langle T_i^n x_n-p, j(x_n-p)\rangle-k_n\|x_n-p\|^2+\phi(\|x_n-p\|)\right\}\le 0$$

for $p\in F$ *and* $i=1,2,\ldots,N$*. Then* $\{x_n\}$ *converges strongly to a common fixed point* p *of* $\{T_i\}_{i=1}^N$*.*

Proof. Since T_i is an asymptotically nonexpansive mapping with $\{k_n\}\subset[1,\infty)$ such that $\lim_{n\to\infty}k_n=1$, we have

$$\begin{aligned}\limsup_{n\to\infty}\{&\sup_{x,\ y\in K}(\|T_i^n x-T_i^n y\|-\|x-y\|)\}\\ &\le\limsup_{n\to\infty}[(k_n-1)diam(K)]=0,\end{aligned}$$

where $diam(K)=\sup_{x,y\in K}\|x-y\|<\infty$. This implies that every asymptotically nonexpansive mapping is asymptotically nonexpansive in the intermediate sense. Since every asymptotically nonexpansive mapping is asymptotically pseudo-contractive mapping. The conclusion now follows easily from Theorem 2.1. □

If $\gamma_n=0$ in (1.5) for all $n\ge1$, similar to prove Theorem 2.1, we have the following result.

Theorem 2.3. *Let* E *be a real normed linear space, and* K *be a nonempty convex subset of* E*. Let* $T_i:K\to K$ $(i=1,2,\ldots,N)$ *be a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically pseudo-contractive mappings with* $\{k_{in}\}\subset[1,\infty)$ *such that* $\lim_{n\to\infty}k_n=1$*, where* $k_n=\max_{1\le i\le N}\{k_{in}\}$*. Assume that* $F=\bigcap_{i=1}^N F(T_i)\neq\emptyset$ *denotes the set of common fixed points of* $\{T_i\}_{i=1}^N$*. Let* $\{x_n\}$ *be the sequence defined by*

$$x_n=(1-\alpha_n)x_{n-1}+\alpha_n T_i^n x_n,\quad n\ge1. \tag{2.5}$$

Suppose that $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *satisfying the following conditions:*
(i) $\sum_{n=1}^{\infty}\alpha_n(k_n-1)<\infty;$ *(ii)* $\sum_{n=1}^{\infty}\alpha_n=\infty, \quad \lim_{n\to\infty}\alpha_n=0.$

Assume that there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$, $\phi(0) = 0$ *such that*

$$\limsup_{n\to\infty} \left\{ \langle T_i^n x_n - p, j(x_n - p)\rangle - k_n\|x_n - p\|^2 + \phi(\|x_n - p\|) \right\} \le 0$$

for $p \in F$ *and* $i = 1, 2, \ldots, N$. *Then* $\{x_n\}$ *converges strongly to a common fixed point* p *of* $\{T_i\}_{i=1}^N$.

It follows from Theorem 2.3 that we have the following result.

Corollary 2.4. *Let* E *be a real normed linear space, and* K *be a nonempty bounded convex subset of* E. *Let* $T_i : K \to K$ $(i = 1, 2, \ldots, N)$ *be a finite family of asymptotically nonexpansive mappings with* $\{k_{in}\} \subset [1,\infty)$ *such that* $\lim_{n\to\infty} k_n = 1$, *where* $k_n = \max_{1\le i\le N}\{k_{in}\}$. *Assume that* $F = \bigcap_{i=1}^N F(T_i) \ne \emptyset$ *denotes the set of common fixed points of* $\{T_i\}_{i=1}^N$. *Let* $\{x_n\}$ *be the sequence defined by* (2.5). *Suppose that* $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *satisfying the following conditions:*
(i) $\sum_{n=1}^\infty \alpha_n(k_n - 1) < \infty$; *(ii)* $\sum_{n=1}^\infty \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.
Assume that there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$, $\phi(0) = 0$ *such that*

$$\limsup_{n\to\infty} \left\{ \langle T_i^n x_n - p, j(x_n - p)\rangle - k_n\|x_n - p\|^2 + \phi(\|x_n - p\|) \right\} \le 0$$

for $p \in F$ *and* $i = 1, 2, \ldots, N$. *Then* $\{x_n\}$ *converges strongly to a common fixed point* p *of* $\{T_i\}_{i=1}^N$.

Since each nonexpansive mapping from K into K is an asymptotically nonexpansive mapping from $K \to K$ with $k_n = 1, \forall n \ge 1$ from Corollary 2.2, we have the following result.

Theorem 2.5. *Let* E *be a real normed linear space, and* K *be a nonempty convex subset of* E. *Let* $T_i : K \to K$ $(i = 1, 2, \ldots, N)$ *be a finite family of nonexpansive mappings. Assume that* $F = \bigcap_{i=1}^N F(T_i) \ne \emptyset$ *denotes the set of common fixed points of* $\{T_i\}_{i=1}^N$. *Let* $\{\alpha_n\}$, $\{\gamma_n\} \subset [0,1]$ *be such that* $\sum_{n=1}^\infty \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \gamma_n < \infty$. *For arbitrary* $x_0 \in K$, *let* $\{x_n\}$ *be iteratively defined by*

$$x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T_i x_n + \gamma_n u_n, \quad \forall n \ge 1. \tag{2.6}$$

Suppose that there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$, $\phi(0) = 0$ *such that*

$$\limsup_{n\to\infty} \left\{ \langle T_i x_n - p, j(x_n - p)\rangle - \|x_n - p\|^2 + \phi(\|x_n - p\|) \right\} \le 0$$

for $p \in F$. *Then* $\{x_n\}$ *converges strongly to* $p \in F$.

If $\gamma_n = 0,\ \forall n \geq 1$, from Theorem 2.5, we have the following result.

Corollary 2.6. *Let E be a real normed linear space, and K be a nonempty convex subset of E. Let $T_i : K \to K$ $(i = 1, 2, \ldots, N)$ be a finite family of nonexpansive mappings. Assume that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ denotes the set of common fixed points of $\{T_i\}_{i=1}^{N}$. Let $\{\alpha_n\} \subset [0,1]$ be such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$. For arbitrary $x_0 \in K$, let $\{x_n\}$ be iteratively defined by*

$$x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_i x_n, \quad \forall n \geq 1. \tag{2.7}$$

Suppose that there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$, $\phi(0) = 0$ such that

$$\limsup_{n\to\infty} \big\{ \langle T_i x_n - p, j(x_n - p)\rangle - \|x_n - p\|^2 + \phi(\|x_n - p\|) \big\} \leq 0$$

for $p \in F$. Then $\{x_n\}$ converges strongly to $p \in F$.

Remark 2.7. *(1) Corollary 2.6 gives an affirmative answer to the following open question raised by Xu and Ori* [14]*: "It is unclear what assumptions on the mappings $\{T_1, T_2, \ldots, T_N\}$ and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$ defined by* (2.7)*."*
(2) Lemma 1.6 shows that the sequence of iterates $\{x_n\}$ defined by (1.5) *is bounded so that the boundedness assumption imposed on K in Theorem 3.3 of Sun* [12] *is not necessary. Theorem 2.1 improves and generalizes Theorem 3.3 of Sun* [12] *to the case of implicit iteration process with errors for a finite family of asymptotically nonexpansive mappings. Theorem 2.1 extends the main results of* [2, 8, 12, 17, 18] *from real uniformly convex Banach space to arbitrary real normed linear space.*
(3) We delete the key condition in [18, Theorem 1] *: there exists a constant $L > 0$ such that for any $i,\ j \in \{T_1, T_2, \ldots, T_N\},\ i \neq j$,*

$$\|T_i^n x - T_j^n y\| \leq L\|x - y\|, \quad \forall n \geq 1, \quad \forall x, y \in K.$$

We now give a nontrivial example which illustrates Theorem 2.1 for $\gamma_n = 0$ $(n \geq 1)$.

Example 2.8. *Let $E = R = (-\infty, +\infty)$ with the usual norm. Take $K = [0,1]$ and $T_i : K \to K$ $(i = 1, 2, 3, 4)$ defined by*

$$T_1 x = \begin{cases} 0 & \text{if } x = 1 \\ x/2 & \text{if } x \in [0,1), \end{cases} \qquad T_2 x = \begin{cases} 0 & \text{if } x = 1 \\ x/3 & \text{if } x \in [0,1), \end{cases}$$

$$T_3x = \begin{cases} 0 & \textit{if } x = 1 \\ x/4 & \textit{if } x \in [0,1), \end{cases} \qquad T_4x = \begin{cases} 0 & \textit{if } x = 1 \\ x/5 & \textit{if } x \in [0,1), \end{cases}$$

Then, it is obvious that each $\{T_i\}_{i=1}^4$ *is a discontinuous mapping with unique common fixed point* $x = 0$. *And* $T_1^n x = x/2^n$, $T_2^n x = x/3^n$, $T_3^n x = x/4^n$, $T_4^n x = x/5^n$. *Thus* $\{T_i\}_{i=1}^4$ *is asymptotically nonexpansive mapping in the intermediate sense and asymptotically pseudocontractive mapping with* $k_n = 1$ *for all* $n \geq 1$. *Additionally,* $\{T_i\}_{i=1}^4$ *also satisfies the condition (2.1) with sequence* $k_n = 1$ *and* $\phi(t) = t^2/2$ *for* $t \geq 0$.

Setting $\alpha_n = 1/(n+1)$. *It is easy to see that the control conditions (i) and (ii) in Theorem 2.1 are satisfied.*

If $x_0 = 1/2$. *It follows from* (1.5) *that* $x_1 = (1-1/2)/2 + (1/2)T_1x_1 = 1/4 + x_1/4$. *Thus,* $x_1 = 1/3$. *And* $x_2 = (1-1/3)x_1 + (1/3)T_2^2 x_2 = 2/9 + x_2/3^3$, *then we get* $x_2 = 3/13$. *It follows from*

$$x_3 = (1-1/4)x_2 + (1/4)T_3^3 x_3 = (3/4)\cdot(3/13) + (1/4)\cdot(x_3/4^3)$$

that we obtain $x_3 = 192/1105$. *And from*

$$x_4 = (1-1/5)x_3 + (1/5)T_4^4 x_4 = (4/5)\cdot(192/1105) + (1/4)\cdot(x_4/5^4),$$

we get $x_4 \simeq 0.139049$. *From*

$$x_5 = (1-1/6)x_4 + (1/6)T_1^5 x_5 = (5/6)\cdot 0.139049 + (1/6)\cdot(x_5/2^5),$$

we have $x_5 \simeq 0.116481$. *From*

$$x_6 = (1-1/7)x_5 + (1/7)T_2^6 x_6 = (6/7)\cdot 0.116481 + (1/7)\cdot(x_6/3^6),$$

we have $x_6 \simeq 0.098700$. *And so on. Therefore, we have that* $x_1 < x_2 < \cdots < x_n < \cdots$, *and* $\lim_{n\to\infty} x_n = 0$.

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