φ-AMENABILITY OF BANACH ALGEBRAS

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ABSTRACT. Let A be an arbitrary Banach algebra and φ a homomorphism from A onto $\mathbb{C}$. Our first purpose in this paper is to give some equivalent conditions under which guarantees a φ-mean of norm one. Then we find some conditions under which there exists a φ-mean in the weak* cluster of $\{a \in A;\ \|a\| = \varphi(a) = 1\}$ in A^{**}.

1. Introduction

In this paper, the second dual A^{**} of a Banach algebra A will always be equipped with the first Arens product. The first Arens product of A^{**} is constructed in three steps as follows: Let a, b be in A; f in A^*; and m, n be in A^{**}. We define the elements $f.a$, nf of A^* and mn of A^{**} by the identities

$$\langle f.a, b \rangle = \langle f, ab \rangle, \quad \langle nf, a \rangle = \langle n, f.a \rangle, \text{ and } \langle mn, f \rangle = \langle m, nf \rangle.$$

Ample information about Arens multiplications and Arens regularity notion can be found in the books [6] and [7].

Let A be a Banach algebra, and let φ be a nonzero multiplicative linear functional on A. Then A is called φ-amenable if there exists a

bounded linear functional m on A^* satisfying $\langle m, \varphi \rangle = 1$ and

$$\langle m, f.a \rangle = \varphi(a) \langle m, f \rangle$$

for all $f \in A^*$ and $a \in A$. The functional m is called a φ-mean. This extends the notion of left amenability of a Lau algebra (or F-algebra) which was first studied by Lau [14]. The concept of φ-amenable Banach algebras was introduced recently by Kaniuth, Lau and Pym [13] (see also [12]). Kaniuth, Lau and Pym [13] characterized φ- amenability of A in terms of vanishing cohomology groups $H^1(A, X^*)$ for a particular class of Banach A-bimodules X, and in terms of the existence of a bounded net $(a_\alpha)_\alpha$ in A satisfying $\varphi(a_\alpha) = 1$ for each α and $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$ for each $a \in A$. In some cases, these extend known results concerning topologically invariant means on Fourier algebras, and properties of group algebras of amenable locally compact groups. They established several characterizations of φ-amenability as well as some hereditary properties. In particular, these involve the projective tensor product $A \widehat{\otimes} B$ for Banach algebras A and B.

Character amenability has also been studied recently by Monfared in [16] and by Hu, Monfared and Traynor [11]. In [16], Monfared studied the character amenability of Lau product algebras. He also proved that all bounded cohomologies $H^n(A, E)$ vanish for a commutative character amenable A when E is a finite-dimensional Banach A-bimodule, and thus deduces that all finite-dimensional extensions of A split strongly. Both of theses concepts generalize the earlier concept of left amenability for F-algebras introduced by Lau in [14]. Recently the notion of α-amenable hypergroups was introduced and studied in [3, 4] and [8].

Throughout the paper, $\Delta(A)$ will denote the set of all non-zero homomorphisms from A into $\mathbb{C}$. For notations and terminologies not explained here, see [13] and also [12].

In this paper, we continue the study of φ-amenable Banach algebras. We present a few results in the theory of φ-amenable Banach algebras, and we obtain necessary and sufficient conditions for A^* to have a φ-mean. The relationship between φ-amenability of a Banach algebra A and amenability of a certain space of functions on $S_\varphi := \{a \in A; \|a\| = \varphi(a) = 1\}$ are investigated. As a sample result, the following is proved: The Banach algebra A^* has a φ-mean in $\overline{S_\varphi}^{w^*}$ if a special space of functions on S_φ has a left invariant mean or, equivalently, if and only if S_φ has the fixed point property for certain affine actions of S_φ on compact convex sets.

2. **Main results**

For a locally compact group G, $L^1(G)$ is its group algebra and $L^\infty(G)$ is the dual of $L^1(G)$. Note that $f.\phi = \widetilde{\phi} * f$ where $\widetilde{\phi}(x) = \Delta(x^{-1})\phi(x^{-1})$, $f \in L^\infty(G)$, and Δ is the Haar modulus function on G [10]. By Proposition 14.5 in [18], $G = SL(2, \mathbb{R})$ is non-amenable. If M is a mean on $L^\infty(G)$, then there exist $f \in L^\infty(G)$ and $\phi \in P^1(G) = \{\phi \in L^1(G); \ \phi \geq 0, \ \|\phi\|_1 = 1\}$ such that $\langle M, f.\widetilde{\phi} \rangle = \langle M, \phi * f \rangle \neq \langle M, f \rangle$. So $L^\infty(G)^*$ contains no 1-mean of norm 1.

Let G be a locally compact group and $A(G)$ the Fourier algebra of G. Then $\Delta(A(G))$ consists of all point evaluations φ_x, $x \in G$. It is known that $A(G)$ is φ_x-amenable for every $x \in G$ [13].

Let A be a Banach algebra and $\varphi \in \Delta(A)$. In this section, we establish several criteria for A to possess a φ-mean of norm 1.

Theorem 2.1. *Let A be any Banach algebra and $\varphi \in \Delta(A)$. Let $\mathcal{H}$ denote the real-linear span of the set $\{\varphi(a)f.b - \varphi(b)f.a; \ a, b \in A, \ f \in A^*\}$. The following assertions are equivalent:*

(i) *There exists a φ-mean m with $\|m\| = 1$.*

(ii) *For every $\epsilon > 0$ and $h \in \mathcal{H}$,*

$$\sup \big\{ Re \ \langle h, c \rangle; \ \varphi(c) = 1, \ \|c\| \leq 1 + \epsilon \big\} \geq 0.$$

(iii) *There exists $m \in A^{**}$ such that $\|m\| = \langle m, \varphi \rangle = 1$ and*

$$\varphi(b)\langle m, f.a \rangle = \varphi(a)\langle m, f.b \rangle$$

for every $f \in A^*$ and $a, b \in A$.

(iv) *For every $f \in A^*$, there exists $m_f \in A^{**}$ such that $\|m_f\| = \langle m_f, \varphi \rangle = 1$ and $\varphi(b)\langle m_f, f.a \rangle = \varphi(a)\langle m_f, f.b \rangle$ for every $a, b \in A$.*

Proof. (i) implies (ii). Let m be as (i), and let $(a_\alpha)_\alpha$ be a net in A such that $a_\alpha \to m$ in the weak*-topology of A^{**} and $\|a_\alpha\| \leq \|m\| = 1$ for all α [19]. Let us assume on the contrary that there exist $\epsilon > 0$, $f_1, ..., f_n \in A^*$, $a_1, ..., a_n \in A$ and $b_1, ..., b_n \in A$ such that

$$\sup \big\{ Re \ \langle h, c \rangle; \ \varphi(c) = 1, \ \|c\| \leq 1 + \epsilon \big\} < 0 \qquad (*)$$

for $h = \sum_{i=1}^n \varphi(a_i)f_i.b_i - \varphi(b_i)f_i.a_i$. Since $\varphi(a_\alpha) \to \langle m, \varphi \rangle = 1$, after passing to a subnet and replacing a_α by $\frac{1}{\varphi(a_\alpha)}a_\alpha$, we can assume that

$\varphi(a_\alpha) = 1$ and $\|a_\alpha\| \leq 1 + \epsilon$ for all α. We have

$$
\begin{aligned}
\lim_\alpha \langle h, a_\alpha \rangle &= \sum_{i=1}^n \lim_\alpha \langle \varphi(a_i) f_i.b_i - \varphi(b_i) f_i.a_i, a_\alpha \rangle \\
&= \sum_{i=1}^n \langle m, \varphi(a_i) f_i.b_i - \varphi(b_i) f_i.a_i \rangle \\
&= \sum_{i=1}^n \varphi(a_i) \langle m, f_i.b_i \rangle - \varphi(b_i) \langle m, f_i.a_i \rangle \\
&= \sum_{i=1}^n \varphi(a_i) \varphi(b_i) \big(\langle m, f_i \rangle - \langle m, f_i \rangle \big) = 0.
\end{aligned}
$$

This shows that $\sup \big\{ Re\langle h, c \rangle; \ \varphi(c) = 1, \ \|c\| \leq 1 + \epsilon \big\} \geq 0$, which contradicts $(*)$. So that (i) implies (ii).

(ii) implies (iii). Before proving (iii), note that if X is a vector space over $\mathbb{C}$, it is also a vector space over $\mathbb{R}$. Also, if $f : X \to \mathbb{C}$ is complex-linear functional, then $Re f : X \to \mathbb{R}$ is real-linear functional. Assume that the scalar field is $\mathbb{R}$.

Let $\epsilon > 0$ be given. For each $f \in A^*$ define

$$
p(f) = \sup \big\{ Re \ \langle f, c \rangle; \ \varphi(c) = 1, \ \|c\| \leq 1 + \epsilon \big\}.
$$

Then p is a subadditive positively homogeneous function on the real-linear space A^*. By hypothesis, $p(h) \geq 0$ for all $h \in \mathcal{H}$. By the Hahn-Banach theorem, there exists a real-linear functional n_ϵ on A^* such that $\langle n_\epsilon, h \rangle = 0$ for all $h \in \mathcal{H}$ and $\langle n_\epsilon, f \rangle \leq p(f)$ for all $f \in A^*$. In particular,

$$
\langle n_\epsilon, \varphi \rangle \leq p(\varphi) = \sup \big\{ Re \ \varphi(c); \ \varphi(c) = 1, \ \|c\| \leq 1 + \epsilon \big\} = 1.
$$

This together with linearity of n_ϵ, imply that $\langle n_\epsilon, \varphi \rangle = 1$, and also $\|n_\epsilon\| \leq 1 + \epsilon$. Similarly,

$$
\langle n_\epsilon, i\varphi \rangle \leq p(i\varphi) = \sup \big\{ Re \ i\varphi(c); \ \varphi(c) = 1, \ \|c\| \leq 1 + \epsilon \big\} = Re \ i = 0,
$$

and also $-\langle n_\epsilon, i\varphi \rangle = \langle n_\epsilon, -i\varphi \rangle \leq p(-i\varphi) \leq 0$. This shows that $\langle n_\epsilon, i\varphi \rangle = 0$. If $a, b \in A$ and $f \in A^*$, then $\varphi(a) f.b - \varphi(b) f.a \in \mathcal{H}$. Hence

$$
\varphi(a) \langle n_\epsilon, f.b \rangle = \varphi(b) \langle n_\epsilon, f.a \rangle.
$$

Let n be a weak*-cluster point of the net $(n_\epsilon)_\epsilon$. Then $\|n\| = \langle n, \varphi \rangle = 1$, $\langle n, i\varphi \rangle = 0$ and $\varphi(a) \langle n, f.b \rangle = \varphi(b) \langle n, f.a \rangle$ for all $f \in A^*$ and $a, b \in A$.

Assume that the scalar field is $\mathbb{C}$. Let m be the complex-linear functional on A^* whose real part is n. Obviously $\langle m, g \rangle = \langle n, g \rangle - i \langle n, ig \rangle$

for any $g \in A^*$. Clearly $\langle m, \varphi \rangle = 1$, $\|m\| = 1$ and

$$
\begin{aligned}
\varphi(a)\langle m, f.b \rangle &= \varphi(a)\langle n, f.b \rangle - i\varphi(a)\langle n, if.b \rangle \\
&= \varphi(b)\langle n, f.a \rangle - i\varphi(b)\langle n, if.a \rangle \\
&= \varphi(b)\langle m, f.a \rangle,
\end{aligned}
$$

for all $f \in A^*$ and $a, b \in A$.

(iii) implies (iv). Trivial.

It remains to show that (iv) implies (i). Assume that (iv) holds. Define a subset $\mathcal{M}$ of A^{**} by

$$
\mathcal{M} = \big\{ m \in A^{**}; \ \|m\| = \langle m, \varphi \rangle = 1 \big\} = \big\{ m \in A^{**}; \ \|m\| \le 1, \ \langle m, \varphi \rangle = 1 \big\}.
$$

For every $f \in A^*$, let

$$
\mathcal{M}_f = \big\{ m \in \mathcal{M}; \ \varphi(a)\langle m, f.b \rangle = \varphi(b)\langle m, f.a \rangle \text{ for all } a, b \in A \big\}.
$$

By Theorem 3.15 in [19], $\mathcal{M}_f$ is a weak*-compact subset in A^{**}. We establish this part by showing that the family $\big\{ \mathcal{M}_f; \ f \in A^* \big\}$ has the finite intersection property. By assumption, $\mathcal{M}_f \ne \emptyset$ for all $f \in A^*$. Let $n \in \mathbb{N}$, $f_1, ..., f_n \in A^*$ and assume that $\bigcap \big\{ \mathcal{M}_{f_i}; \ 1 \le i \le n-1 \big\} \ne \emptyset$. Let m_1 be a member of this intersection, and let $m_2 \in \mathcal{M}_{m_1 f_n}$. Let $(a_\alpha)_\alpha$ be a net in A such that $\|a_\alpha\| = 1$, $\varphi(a_\alpha) \to 1$ and $a_\alpha \to m_2$ in the weak*-topology. If $a, b \in A$ and $i \in \{1, ..., n-1\}$ are fixed, then

$$
\varphi(ba_\alpha)\langle m_1, f_i.aa_\alpha \rangle = \varphi(aa_\alpha)\langle m_1, f_i.ba_\alpha \rangle
$$

for all α. Taking the limit on α, we find that $\varphi(b)\langle m_2 m_1, f_i.a \rangle = \varphi(a)\langle m_2 m_1, f_i.b \rangle$. This shows that $m_2 m_1 \in \bigcap \big\{ \mathcal{M}_{f_i}; \ 1 \le i \le n-1 \big\}$. On the other hand,

$$
\begin{aligned}
\varphi(b)\langle m_2 m_1, f_n.a \rangle &= \varphi(b)\langle m_2, m_1 f_n.a \rangle = \varphi(a)\langle m_2, m_1 f_n.b \rangle \\
&= \varphi(a)\langle m_2 m_1, f_n.b \rangle.
\end{aligned}
$$

Consequently $m_2 m_1 \in \bigcap \big\{ \mathcal{M}_{f_i}; \ 1 \le i \le n \big\}$. Thus $\big\{ \mathcal{M}_f; \ f \in A^* \big\}$ has the finite intersection property, as required.

Let $m \in \bigcap \big\{ \mathcal{M}_f; \ f \in A^* \big\}$, and take any net $(a_\alpha)_\alpha$ in A such that $\|a_\alpha\| \le 1$ and $a_\alpha \to m$ in the weak*-topology. Then $\varphi(a_\alpha) \to 1$. For every $f \in A^*$ and $a \in A$, we have

$$
\begin{aligned}
\varphi(a)\langle mm, f \rangle &= \varphi(a) \lim_\alpha \langle m, f.a_\alpha \rangle = \lim_\alpha \varphi(a)\varphi(a_\alpha)\langle m, f.a_\alpha \rangle \\
&= \lim_\alpha \varphi(aa_\alpha)\langle m, f.a_\alpha \rangle = \lim_\alpha \varphi(a_\alpha)\langle m, f.aa_\alpha \rangle \\
&= \lim_\alpha \langle m, f.aa_\alpha \rangle = \langle mm, f.a \rangle.
\end{aligned}
$$

Since $\langle mm, \varphi \rangle = \langle m, \varphi \rangle^2 = 1$ and $\|m\| = 1$, so m satisfies all the requirements in (i). The proof is finished. $\qquad\square$

As an immediate application of the preceding result, we have the following

Corollary 2.2. *Let A be any Banach algebra and $\varphi \in \Delta(A)$. Then the following conditions are equivalent:*

- (i) *A admits a φ-mean of norm 1.*
- (ii) *There exists a net $(a_\alpha)_\alpha$ in $\{a \in A; \|a\| = 1\}$ such that $\varphi(a_\alpha) \to 1$ and also for every $a, b \in A$, $\big(\varphi(a)ba_\alpha - \varphi(b)aa_\alpha\big)_\alpha$ converges to 0 in the weak*-topology.*
- (iii) *There exists a net $(a_\alpha)_\alpha$ in $\{a \in A; \|a\| = 1\}$ such that $\varphi(a_\alpha) \to 1$ and also for every $a, b \in A$, $\big(\varphi(a)ba_\alpha - \varphi(b)aa_\alpha\big)_\alpha$ converges to 0 in the norm-topology.*

Proof. Suppose that (i) holds. Then by Theorem 2.1, let m be such that $\|m\| = \langle m, \varphi \rangle = 1$ and $\varphi(a)\langle m, f.b \rangle = \varphi(b)\langle m, f.a \rangle$ for all $a, b \in A$. By Goldstein's theorem [6], there exists a net $(a_\alpha)_\alpha$ in A such that $a_\alpha \to m$ in the weak*-topology and $\|a_\alpha\| \le 1$ for all α. Passing to a subnet if necessary, we replacing a_α by $\frac{a_\alpha}{\|a_\alpha\|}$, we can assume that $\varphi(a_\alpha) \to 1$ and $\|a_\alpha\| = 1$ for all α. So (i) implies (ii).

Suppose that (ii) holds. An argument similar to the proof of Lemma 1.4 in [13], shows that there exists a net $(a_\alpha)_\alpha$ in $\{a \in A; \|a\| = 1\}$ such that $\varphi(a_\alpha) \to 1$ and also for every $a, b \in A$, $\big(\varphi(a)ba_\alpha - \varphi(b)aa_\alpha\big)_\alpha$ converges to 0 in the norm-topology. (iii) implies (i). By Banach-Alaoglu's theorem [19], the net $(a_\alpha)_\alpha$ admits a subnet (a_β) converging to a φ-mean m in the weak*-topology of A^*. Hence for every $f \in A^*$ and every $a, b \in A$, we have

$$
\begin{aligned}
\varphi(b)\langle m, f.a \rangle &= \varphi(b) \lim_\beta \langle f.a, a_\beta \rangle = \varphi(b) \lim_\beta \langle f, aa_\beta \rangle \\
&= \varphi(a) \lim_\beta \langle f, ba_\beta \rangle = \varphi(a)\langle m, f.b \rangle.
\end{aligned}
$$

By Theorem 2.1, A is φ-amenable. This completes the proof. $\qquad\square$

Following Kaniuth, Lau and Pym [12], we say that an element a of A is φ-*maximal* if it satisfies $\|a\| = \varphi(a) = 1$. We define $S_\varphi = \{a \in A; \|a\| = \varphi(a) = 1\}$. Let $X(A, \varphi)$ be the closed vector space spanned by S_φ. It is shown that if A is a commutative Banach algebra and $X(A, \varphi) = A$, then A is φ-amenable, see Proposition 2.10 in [12].

Definition 2.3. *A* right action *of A on A^* is an anti-homomorphism of A into the algebra of linear operators in A^*, denoted by $T : A \times A^* \to A^*$ where $(a, f) \to T_a(f)$ (this means that $(a, f) \to T_a(f)$ is bilinear and $T_{ab} = T_b T_a$ for any $a, b \in A$) such that $T_a{}^*(S_\varphi) \subseteq S_\varphi$ and $T_a(\varphi) = \varphi$ for all $a \in S_\varphi$.*

Let X be a subspace of A^*, X is called S_φ-invariant under the right action T if $T_a(X) \subseteq X$ for all $a \in S_\varphi$.

Theorem 2.4. *Let A be any Banach algebra and $\varphi \in \Delta(A)$. Let $X(A, \varphi) = A$. Then the following conditions are equivalent:*

  (i) *There exists $m \in \overline{S_\varphi}^{w^*}$ such that $\langle m, f.a \rangle = \varphi(a)\langle m, f \rangle$ for all $f \in A^*$ and $a \in A$, where the closure is taken in weak*-topology.*
  (ii) *For any weakly-weakly separately continuous right action $T : A \times A^* \to A^*$ of A on A^*, and any S_φ-invariant subspace X of A^* containing φ, any φ-invariant $n \in \overline{S_\varphi}^{w^*}$ on X ($\langle n, T_a(f) \rangle = \langle n, f \rangle$ for any $f \in X$ and $a \in S_\varphi$) can be extended to a φ-invariant $m \in \overline{S_\varphi}^{w^*}$ on A^*.*

Proof. (i) implies (ii). Assume that A has a φ-mean, say $m \in \overline{S_\varphi}^{w^*}$, and let $T : A \times A^* \to A^*$ be any separately continuous right action of A on A^*. Assume that X is a S_φ-invariant subspace of A^* containing φ. Let $n \in \overline{S_\varphi}^{w^*}$ be a φ-invariant on X, and let

$$M = \left\{ m \in \overline{S_\varphi}^{w^*}; \ m|_X = n \right\}.$$

In fact M is a weak*-closed convex subset of the unit ball in A^{**} and is therefore weak*-compact. For $a \in A$, the mapping T_a from A^* to itself is weakly-weakly continuous, and so T_a is norm-norm continuous [1]. Define $T_a{}^* : A^{**} \to A^{**}$ by $\langle T_a{}^*(F), f \rangle = \langle F, T_a(f) \rangle$ where $F \in A^{**}$ and $f \in A^*$. We claim that $T_a{}^*(M) \subseteq M$ for any $a \in S_\varphi$. Since $T_a{}^*$ is weak*-weak* continuous, by assumption $T_a{}^*(\overline{S_\varphi}^{w^*}) \subseteq \overline{S_\varphi}^{w^*}$. Now let $m_1 \in M$, $f \in X$ and $a \in S_\varphi$. Then

$$\langle T_a{}^*(m_1), f \rangle = \langle m_1, T_a(f) \rangle = \langle n, T_a(f) \rangle = \langle n, f \rangle.$$

Consequently $T_a{}^*(M) \subseteq M$. By assumption, there is a net $(a_\alpha)_\alpha$ in S_φ such that $\|a a_\alpha - a_\alpha\| \to 0$ (for all $a \in S_\varphi$) and $a_\alpha \to m$ in the weak*-topology, see Lemma 1.4 in [13] and its proof. Now choose $m_1 \in M$. Since M is weak*-compact, there is a subnet of $(T_{a_\alpha}{}^*(m_1))_\alpha$ which converges to an element m_2 of M in the weak*-topology. So without loss

generality we assume that $T_{a_\alpha}{}^*(m_1) \to m_2$. For each $a \in S_\varphi$, $f \in A^*$ and α,

$$
\begin{aligned}
\langle T_a{}^* T_{a_\alpha}{}^*(m_1), f \rangle &= \langle m_1, T_{a_\alpha}(T_a(f)) \rangle = \langle m_1, T_{aa_\alpha}(f) \rangle \\
&= \langle m_1, T_{aa_\alpha - a_\alpha}(f) \rangle + \langle m_1, T_{a_\alpha}(f) \rangle \\
&= \langle m_1, T_{aa_\alpha - a_\alpha}(f) \rangle + \langle T_\alpha{}^*(m_1), f \rangle.
\end{aligned}
$$

By weakly-weakly separate continuity of T, linearity of $a \to T_a(f)$ and the fact that $\|aa_\alpha - a_\alpha\| \to 0$, we have $T_a{}^*(m_2) = m_2$. This shows that $\langle m_2, T_a(f) \rangle = \langle m_2, f \rangle$ for every $f \in A^*$ and $a \in S_\varphi$. On the other hand $X(A, \varphi) = A$, and so $\langle m_2, T_a(f) \rangle = \varphi(a) \langle m_2, f \rangle$ for every $f \in A^*$ and $a \in A$.

(ii) implies (i): Define $T : A \times A^* \to A^*$ by $T_a(f) = f.a$. Clearly $T_a(\varphi) = \varphi.a = \varphi$ for every $a \in S_\varphi$. Since $T_a T_b = T_{ba}$, it is clear that T defines a right action of A on A^*. Suppose $a_\alpha \to a$ in the weak-topology, then for fixed $f \in A^*$ and $F \in A^{**}$, we have

$$
\langle F, T_{a_\alpha}(f) \rangle - \langle F, T_a(f) \rangle = \langle F, f.a_\alpha \rangle - \langle F, f.a \rangle = \langle Ff, a_\alpha \rangle - \langle Ff, a \rangle \to 0.
$$

To show continuity in the other variable, let $f_\alpha \to f$ in the weak-topology. Then for each $a \in A$ and α, $\langle F, T_a(f_\alpha) \rangle = \langle F, f_\alpha.a \rangle = \langle aF, f_\alpha \rangle$. Thus $T_a(f_\alpha) \to T_a(f)$ in the weak-topology, so the action of A on A^* is separately continuous.

Now, let $X = \{c\varphi; \ c \in \mathbb{C}\}$. Then X is obviously S_φ-invariant under the separately continuous right action $T : A \times A^* \to A^*$. Choose $b \in S_\varphi$, and define $\langle n, c\varphi \rangle = c\varphi(b) = c$. It is easy to see that $\langle n, T_a(c\varphi) \rangle = \langle n, c\varphi.a \rangle = \varphi(a) \langle n, c\varphi \rangle$ for all $a \in A$ and $c\varphi \in X$. Any invariant extension $m \in \overline{S_\varphi}^{w^*}$ of n to A^* is necessarily a φ-mean. This completes our proof.   □

Theorem 2.5. *Let A be any Banach algebra and $\varphi \in \Delta(A)$. Then the following conditions are equivalent:*

(i) *A admits a φ-mean of norm 1.*
(ii) *For every $f \in A^*$ and $a \in A$, there exists a mean $m_{f,a}$ on A^* such that $\langle m_{f,a}, \varphi \rangle = 1$, $\|m_{f,a}\| = 1$ and $\varphi(b)\langle m_{f,a}, f.ab \rangle = \varphi(ab)\langle m_{f,a}, f.b \rangle$ whenever $b \in A$.*
(iii) *For every two finite subset $F \subseteq A$ and $F^* \subseteq A^*$, and $\epsilon > 0$ there exists a $m_{F^*,F,\epsilon} \in A^{**}$ such that $\|m_{F^*,F,\epsilon}\| = 1$, $\langle m_{F^*,F,\epsilon}, \varphi \rangle = 1$ and $|\langle m_{F^*,F,\epsilon}, f.a \rangle - \varphi(a)\langle m_{F^*,F,\epsilon}, f \rangle| < \epsilon$ whenever $a \in F$ and $f \in F^*$.*

Proof. Note first that (i) obviously implies (iii). Next suppose that (i) holds. If m is a φ-mean of norm 1, we can choose $m_{f,a} = m$ for all $f \in A^*$ and $a \in A$. Thus for every $b \in A$,

$$\langle m, f.ab \rangle = \varphi(ab)\langle m, f \rangle \text{ and } \varphi(ab)\langle m, f \rangle = \varphi(a)\langle m, f.b \rangle.$$

Clearly $\varphi(b)\langle m, f.ab \rangle = \varphi(ab)\langle m, f.b \rangle$. Thus (i) implies (ii).

Now assume that (ii) holds. For every $f \in A^*$ and $a \in A$, let

$$\mathcal{M}_{f,a} = \big\{ m \in \mathcal{M}; \ \varphi(b)\langle m, f.ab \rangle = \varphi(ab)\langle m, f.b \rangle \text{ for all } b \in A \big\},$$

where $\mathcal{M} = \{ m \in A^{**}; \ \|m\| = \langle m, \phi \rangle = 1 \}$. These sets are nonempty weak*-closed subsets of A^{**} and we want to show that $\bigcap \{ \mathcal{M}_{f,a}; \ f \in A^*, \ a \in A \}$ is nonempty. Since $\{ m \in A^{**}; \ \|m\| \leq 1 \}$ is weak*-compact, it suffices to show that $\bigcap \{ \mathcal{M}_{f,a}; \ f \in F^*, \ a \in F \} \neq \emptyset$ for any two finite subset $F^* \subseteq A^*$ and $F \subseteq A$. So assume that $m_1 \in \bigcap \{ \mathcal{M}_{f,a}; \ f \in F^*, \ a \in F \}$ for finite subsets $F^* \subseteq A^*$ and $F \subseteq A$, and let $h \in A^*$ and $c \in A$. Let $m_2 \in \mathcal{M}_{m_1 h, c}$. Let $(a_\alpha)_\alpha$ be a net in A such that $\|a_\alpha\| = 1$ for all α, $\varphi(a_\alpha) \to 1$ and $a_\alpha \to m_2$ in the weak*-topology. Then

$$\begin{aligned} \varphi(b)\langle m_2 m_1, f.ab \rangle &= \lim_\alpha \varphi(ba_\alpha)\langle m_1, f.aba_\alpha \rangle = \lim_\alpha \varphi(aba_\alpha)\langle m_1, f.ba_\alpha \rangle \\ &= \varphi(ab)\langle m_2 m_1, f.b \rangle. \end{aligned}$$

for every $f \in F^*$, $a \in F$ and $b \in A$. Thus $m_2 m_1 \in \bigcap \{ \mathcal{M}_{f,a}; \ f \in F^*, \ a \in F \}$. On the other hand, $\varphi(b)\langle m_2 m_1, h.cb \rangle = \varphi(cb)\langle m_2 m_1, h.b \rangle$ for all $b \in A$. This shows that $\{ \mathcal{M}_{f,a}; \ f \in A^*, \ a \in A \}$ has the finite intersection property. Therefore there is some m in $\mathcal{M}$ such that $\varphi(b)\langle m, f.ab \rangle = \varphi(ab)\langle m, f.b \rangle$ for all $f \in A^*$ and $a, b \in A$. It is easy to see that $mm \in A^{**}$ is a φ-mean of norm 1. Thus (i) and (ii) are equivalent.

(iii) implies (i). We denote $\mathcal{F}^*$ the family of finite subsets of A^* and by $\mathcal{F}$ the family of finite subsets of A. For every $F^* \in \mathcal{F}^*$, $F \in \mathcal{F}$ and $\epsilon > 0$,

$$\mathcal{C}_{F^*,F,\epsilon} = \big\{ m \in \mathcal{M}; \ |\langle m, f.a \rangle - \varphi(a)\langle m, f \rangle| < \epsilon \text{ for all } f \in F^*, \ a \in F \big\} \neq \emptyset.$$

It is easy to see that $\bigcap \{ \overline{\mathcal{C}_{F^*,F,\epsilon}}; \ F^* \in \mathcal{F}^*, \ F \in \mathcal{F}, \ \epsilon > 0 \} \neq \emptyset$ where closure is taken in the weak*-topology. Now, let m be any member of this intersection, and let $f \in A^*$, $a \in A$ and $\epsilon > 0$ be given. The set of all $n \in A^{**}$ such that $|m - n| < \frac{\epsilon}{3(|\varphi(a)|+1)}$ at $f.a$ and at f is a weak*-neighborhood of m. Therefore $\mathcal{C}_{\{f\},\{a\},\frac{\epsilon}{3}}$ contains such an n. We

have

$$|\langle m, f.a \rangle - \varphi(a)\langle m, f \rangle| \leq |\langle m, f.a \rangle - \langle n, f.a \rangle| + |\langle n, f.a \rangle - \varphi(a)\langle n, f \rangle|$$
$$+ |\varphi(a)||\langle m, f \rangle - \langle n, f \rangle| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $\langle m, fa \rangle = \varphi(a)\langle m, f \rangle$. This completes our proof. $\qquad\qquad\square$

Remark 2.6. *Let A be a Banach algebra and let $\varphi \in \Delta(A)$. Let $m \in \overline{S_\varphi}^{w^*}$ be a φ-mean on A^*. Let $a_1, ..., a_n \in A$ and let $\epsilon > 0$ be given. Since A has a φ-mean m, there exists a net $(a_\alpha)_\alpha$ in S_φ such that $\|a_i a_\alpha - \varphi(a_i)a_\alpha\| \to 0$ for all i (see Theorem 1.4 in [13] and its proof).*

By the first paragraph, there exists $a \in S_\varphi$ such that $\|a_i a - \varphi(a_i)a\| < \epsilon$ whenever $i \in \{1, ..., n\}$. Hence

$$\sup\{\|a_i a\|; \ 1 \leq i \leq n\} < \epsilon + \sup\{|\varphi(a_i)|; \ 1 \leq i \leq n\}.$$

Since $\epsilon > 0$ may be chosen arbitrarily, we have

$$\inf \big\{ \sup\{\|a_i a\|; \ 1 \leq i \leq n\}; \ a \in S_\varphi \big\} \leq \sup\{|\varphi(a_i)|; \ 1 \leq i \leq n\}.$$

The converse is also true. Indeed, fix $b \in S_\varphi$. For every $\epsilon > 0$ and every finite subset $F = \{a_1, ..., a_n\}$ in A, there exists $a \in S_\varphi$ such that $\|a_i a - \varphi(a_i)ba\| < \epsilon$, whenever $1 \leq i \leq n$ (since $\varphi\big(a_i - \varphi(a_i)b\big) = 0$ for all $i \in \{1, ..., n\}$). An argument similar to the proof of Theorem 2.5 $((iii) \to (i))$, shows that there exists $n \in \overline{S_\varphi}^{w^*}$ such that $\langle n, f.a \rangle = \varphi(a)\langle n, f.b \rangle$ for every $a \in A$. Finally, let $m = bn$. Then $bn \in \overline{S_\varphi}^{w^*}$. Moreover,

$$\langle m, f.a \rangle = \langle bn, f.a \rangle = \langle n, f.ab \rangle = \varphi(ab)\langle n, f.b \rangle = \varphi(a)\langle m, f \rangle$$

for all $f \in A^*$ and $a \in A$. So m is a φ-mean on A^*. Note that this is in fact Proposition 15.5 of [18] which was proved for group algebras.

By the same argument as above, one can prove that A has a φ-mean of norm 1 if and only if for all $n \in \mathbb{N}$, $a_1, ..., a_n \in A$ and $\gamma > 1$,

$$\inf \big\{ \sup\{\|a_i a\|; \ 1 \leq i \leq n\}; \ \|a\| < \gamma, \ \varphi(a) = 1 \big\}$$
$$\leq \sup\{|\varphi(a_i)|; \ 1 \leq i \leq n\}.$$

Let K be a convex set in a locally convex linear topological space E, and let $A(K)$ denote the Banach space of bounded continuous affine functions on K under the supremum norm.

Let A be a Banach algebra, and let S_φ be the set of all φ-maximal elements of A. Under the norm topology, S_φ is a semi-topological semi-group. Let $C_b(S_\varphi)$ be the space of bounded continuous functions on S_φ with usual sup norm. For $a \in S_\varphi$, the translation L_a of $C_b(S_\varphi)$ by a

is given by $L_a f(b) = f(ab)$, where $f \in C_b(S_\varphi)$ and $b \in S_\varphi$. Denote by $U(S_\varphi)$ the space of all functions $f \in C_b(S_\varphi)$ such that the map $a \mapsto L_a f$ of S_φ into $C_b(S_\varphi)$ is uniformly continuous with respect to the uniformity induced on S_φ and uniformity induced on $C_b(S_\varphi)$. An element $M \in (A(S_\varphi) \cap U(S_\varphi))^*$ is a *mean* on $A(S_\varphi) \cap U(S_\varphi)$ if $\|M\| = \langle M, 1 \rangle = 1$. A mean M on $A(S_\varphi) \cap U(S_\varphi)$ is left invariant if $\langle M, L_a f \rangle = \langle M, f \rangle$ for all $f \in A(S_\varphi) \cap U(S_\varphi)$ and $a \in S_\varphi$. More information on this matter can be found in [5, 9, 17].

Theorem 2.7. *Suppose A is a Banach algebra and $\varphi \in \Delta(A)$. Among the following two properties, the implication $(i) \rightarrow (ii)$ hold. If $X(A, \varphi) = A$, then also $(ii) \rightarrow (i)$.*

(i) *There exists a φ-mean m in $\overline{S_\varphi}^{w^*}$.*
(ii) *$A(S_\varphi) \cap U(S_\varphi)$ has a left invariant mean.*

Proof. We can prove Theorem 2.7 by using an argument similar to the proof of Lemma 2.1 in [15].

(i) implies (ii). Let $m \in \overline{S_\varphi}^{w^*}$ be a mean on A^*. By Theorem 1.4 in [13] and its proof, there exists a net $(a_\alpha)_\alpha$ in S_φ such that $\|aa_\alpha - \varphi(a)a_\alpha\| \rightarrow 0$ for every $a \in A$. Let δ_{a_α} denote evaluation at a_α. By possibility passing to subnets we can assume that $f(a_\alpha) = \delta_{a_\alpha}(f) \rightarrow M(f)$ for all $f \in A(S_\varphi) \cap U(S_\varphi)$ to some mean M on $A(S_\varphi) \cap U(S_\varphi)$ (see Theorem 1.8 in [5]). Fix $f \in A(S_\varphi) \cap U(S_\varphi)$ and define $M_l f(a) = \langle M, L_a f \rangle$ where $a \in S_\varphi$. Then $a \mapsto M_l f(a)$ is obviously in $A(S_\varphi) \cap U(S_\varphi)$. Define $MM : A(S_\varphi) \cap U(S_\varphi) \rightarrow \mathbb{C}$ by $\langle MM, f \rangle = \langle M, M_l f \rangle$. It is easy to see that MM is a mean on $A(S_\varphi) \cap U(S_\varphi)$. Let $f \in A(S_\varphi) \cap U(S_\varphi)$, $a \in S_\varphi$ and let $\epsilon > 0$ be given. There is a $\delta > 0$ such that $b, c \in S_\varphi$, and $\|b - c\| < \delta$ implies

$$|\langle M, L_b f \rangle - \langle M, L_c f \rangle| \leq \|L_b f - L_c f\| < \epsilon.$$

Also, there is some α_0 such that $\alpha \geq \alpha_0$ implies $\|aa_\alpha - a_\alpha\| < \delta$. For every $\alpha \geq \alpha_0$,

$$\begin{aligned} |M_l(L_a f)(a_\alpha) - M_l(f)(a_\alpha)| &= |\langle M, L_{a_\alpha} L_a f \rangle - \langle M, L_{a_\alpha} f \rangle| \\ &= |\langle M, L_{aa_\alpha} f \rangle - \langle M, L_{a_\alpha} f \rangle| < \epsilon, \end{aligned}$$

which implies that $\langle MM, L_a f \rangle = \langle MM, f \rangle$, i.e., MM is a left invariant mean on $A(S_\varphi) \cap U(S_\varphi)$.

(ii) implies (i). Let M be a left invariant mean on $A(S_\varphi) \cap U(S_\varphi)$. Any element in $co\{\delta_a;\ a \in S_\varphi\}$ is said to be a finite mean. The finite means on $A(S_\varphi) \cap U(S_\varphi)$ are weak*-dense in the weak*-compact convex

set of means on $A(S_\varphi) \cap U(S_\varphi)$ (see Theorem 1.8 in [5]). If $\sum_{i=1}^{n} \alpha_i \delta_{a_i}$ is a finite mean on $A(S_\varphi) \cap U(S_\varphi)$, then

$$\sum_{i=1}^{n} \alpha_i \delta_{a_i} = \delta_{\sum_{i=1}^{n} \alpha_i a_i} \in \{\delta_a; \ a \in S_\varphi\}.$$

This shows that $\{\delta_a; \ a \in S_\varphi\} = co\{\delta_a; \ a \in S_\varphi\}$. Hence, there is a net $a_\alpha \in S_\varphi$ such that $\delta_{a_\alpha} \to M$ in the weak*-topology. We define $\Lambda(A^*)$ to be the set of all functions $\Lambda(f)$ with domain S_φ, that arise from an $f \in A^*$ by formula $\Lambda(f)(a) = \langle f, a \rangle$. The mapping $f \mapsto \Lambda(f)$ is a linear map of A^* into $A(S_\varphi) \cap U(S_\varphi)$ which is continuous.

Let $\Lambda^* : (A(S_\varphi) \cap U(S_\varphi))^* \to A^{**}$ denote the adjoint map of Λ. Since Λ^* is weak*-weak* continuous, $a_\alpha = \Lambda^*(\delta_{a_\alpha}) \to \Lambda^*(M)$ in the weak*-topology. Thus $\Lambda^*(M) \in \overline{S_\varphi}^{w^*}$. Let $f \in A^*$ and $a \in S$. Left invariance of M gives

$$\begin{aligned} \langle \Lambda^*(M), f.a \rangle &= \langle M, \Lambda(f.a) \rangle = \langle M, L_a \Lambda(f) \rangle \\ &= \langle M, \Lambda(f) \rangle = \langle \Lambda^*(M), f \rangle. \end{aligned}$$

This shows that $\Lambda^*(M)$ is a φ-mean on A^*. This completes the proof.   □

Let K be a compact convex set in a locally convex linear topological space E. Associate to each $T_0 \in \mathcal{B}(A(K), A(K))$, $h \in A(K)$, $k \in K$ and to each positive real ϵ the set

$$V(T_0, h, k, \epsilon) = \{T \in \mathcal{B}(A(K), A(K)); \ |T(h)(k) - T_0(h)(k)| < \epsilon\}.$$

Let $\mathcal{B}$ be the collection of all finite intersections of the sets $V(T_0, h, x, \epsilon)$. Then $\mathcal{B}$ is a convex balanced local base for a topology τ on $\mathcal{B}(A(K), A(K))$.

Some additional definitions are needed in order to state our results of the paper. An affine action of S_φ on K is a map $T : S_\varphi \times K \to K$ (denoted by $(a, k) \mapsto T_a(k)$, $a \in S_\varphi$, $k \in K$) such that $T_{ab} = T_a o T_b$ for all $a, b \in S_\varphi$ and $a \mapsto T_a(k)$ is an affine continuous map for each $k \in K$. If T is an affine action, then T induces an anti-action $\tau : S_\varphi \times A(K) \to A(K)$ of S_φ on the Banach space $A(K)$, defined by $\tau_a(h) = h o T_a$. τ is said to be uniformly continuous if to each neighborhood W of 0 in $\mathcal{B}(A(K), A(K))$ corresponds a $\delta > 0$ such that $\|a - b\| < \epsilon$ implies $\tau_a - \tau_b \in W$.

The relationship between an affine action T and its induced anti-action τ with respect to measurability, continuity, and compactness conditions on them have been studied by Wong in [20].

Theorem 2.8. *Let A be a Banach algebra and $\varphi \in \Delta(A)$. Among the following two properties, the implication $(i) \to (ii)$ hold. If $X(A, \varphi) = A$, then also $(ii) \to (i)$.*

(i) *There exists a φ-mean m in $\overline{S_\varphi}^{w^*}$.*

(ii) *For any affine action $T : S_\varphi \times K \to K$ such that the induced anti-action $\tau : S_\varphi \times A(K) \to A(K)$ is uniformly continuous, there is some $k \in K$ such that $T_a(k) = k$ for all $a \in S_\varphi$.*

Proof. Assume that A has a φ- mean, say $m \in \overline{S_\varphi}^{w^*}$. By Theorem 2.7, $A(S_\varphi) \cap U(S_\varphi)$ has a left invariant mean. Let $T : S_\varphi \times K \to K$ be any affine action such that the induced anti-action $\tau : S_\varphi \times A(K) \to A(K)$ is uniformly continuous. Select a specific $k \in K$. Define a map $Tk : A(K) \to C_b(S_\varphi)$ by $Tk(h)(a) = h(T_a(k))$ for $h \in A(K)$ and $a \in S_\varphi$. Designate $Tk(h)$ by f. Then for $a, b \in S_\varphi$, we have

$$L_a f(b) = f(ab) = Tk(h)(ab) = h(T_{ab}(k)) = \tau_{ab}(h)(k).$$

Since τ is uniformly continuous, it follows that $Tk(h) \in A(S_\varphi) \cap U(S_\varphi)$. Hence by Theorem 1 in [2], there is some $k \in K$ such that $T_a(k) = k$ for all $a \in S_\varphi$.

(ii) implies (i). Let $E = A^{**}$ with weak*-topology and put $K = \overline{S_\varphi}^{w^*}$. Define an affine action of S_φ on K by $T_a(m) = am$ for $a \in S_\varphi$ and $m \in K$. Clearly the induced anti-action $\tau : S_\varphi \times A(K) \to A(K)$ of S_φ on $A(K)$ is uniformly continuous. By assumption, there exists $m \in K$ such that $am = m$ for all $a \in S_\varphi$. m is a φ-mean on A^*. $\square$

Acknowledgments

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