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# ON THE RELATIONS BETWEEN THE POINT SPECTRUM OF $A$ AND INVERTIBILITY OF $I + f(A)B$

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Abstract. Let $A$ be a bounded linear operator on a Banach space $X$. We investigate the conditions of existing rank-one operator $B$ such that $I + f(A)B$ is invertible for every analytic function $f$ on $\sigma(A)$. Also, we compare the invariant subspaces of $f(A)B$ and $B$. This work is motivated by an operator method on the Banach space $\ell^2$ for solving some PDEs, extended to general operator space under some conditions here.

## 1. Introduction

Let $X$ be a Banach space and $T \in \mathcal{L}(X)$ (i.e., $T : X \to X$ is a bounded linear operator). A (closed) subspace $\mathcal{M}$ of $X$ is called an invariant subspace of $T$, if $T\mathcal{M} \subseteq \mathcal{M}$. $\mathcal{M}$ is called a nontrivial invariant subspace, if it is different from (0) and $X$. If $\lambda$ is an eigenvalue for $T$, then the subspace of all eigenvectors of $T$ corresponding to $\lambda$ is an invariant subspace of $T$. So, if the point spectrum of $T$ is nonempty, then $T$ has a nontrivial invariant subspace. The collection of all invariant subspaces of $T$ is a lattice and it is denoted by $\mathrm{Lat}(T)$. The adjoint of $T$, which is denoted by $T^*$, is a linear operator defined on the dual space $X'$ by $(T^*\phi)x = \phi(Tx)$, for each $x \in X$ and $\phi \in X'$. The spectrum, the point spectrum and the spectral radius of $T$ are denoted by $\sigma(T)$, $\sigma_p(T)$ and

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$r(T)$, respectively. For $T \in \mathcal{L}(X)$ and $S \subseteq X$, let $\overline{S}$, $N(T)$, $R(T)$, $\{T\}'$, and $\mathrm{Hol}(T)$ denote respectively the norm closure of $S$ in $X$, the null space of $T$, the range of $T$, the commutant of $T$, and the collection of all analytic functions on $\sigma(T)$. A bounded linear operator $T$ on $X$ is said to be of rank one if the dimension of the range of $T$ is one. It is straightforward to verify that $T$ is of rank one if and only if there exist $\phi \in X'$ and $u \in X$ such that $T = u \otimes \phi$, where $(u \otimes \phi)x := \phi(x)u$, for every $x \in X$. It is easy to verify that for every $u, v \in X; \phi, \psi \in X'$ and $T \in \mathcal{L}(X)$, we have

(1) $T(u \otimes \phi) = Tu \otimes \phi$,
(2) $(u \otimes \phi)T = u \otimes T^*\phi$,
(3) $r(u \otimes \phi) = |\phi(u)|$,
(4) $(u \otimes \phi)(v \otimes \psi) = \phi(v)u \otimes \psi$.

The authors of [1, 2, 3, 5, 6] employed a same method for finding solutions of some nonlinear partial differential equations (PDEs) in the family of KdV. The main idea of their method is as follows: Given a nonlinear PDE and a specific scalar solution to the equation, the first step in finding other solutions is to translate the given nonlinear equation to an operator equation. Using the specific solution as an aid, one then searches for a family of operator solutions to the operator equation. Having obtained the operator solutions, the second step is to transfer the operator valued solutions into a scalar solution by using a suitable scalarization technique. The first step is the most important one in the above operator method. In this step, for a given operator $A$ on the complex Banach space $\ell^2$, we need to find a specific operator $B$ on $\ell^2$ such that $I + e^{p(A)}B$ is invertible for some polynomial $p$.

In [1], the existence of $B$ is proved in general operator space. Here, we demonstrate this problem under different conditions and find a rank-one operator $B$ with more characteristics (related to their invariant subspaces). Moreover, we investigate the relations between the point spectrum of $A$ and existence of a rank-one operator $B$ such that $\mathrm{Lat} f(A)B = \mathrm{Lat}B$.

## 2. Main results

**Theorem 2.1.** *Let $A \in \mathcal{L}(X)$, where $\dim(X) \geqslant 3$. If there is $\lambda \in \sigma_p(A)$ such that $\mathrm{codim}(\overline{R(A - \lambda I)}) \geqslant 2$, then there exists a rank-one operator $B$ such that*

(1) *$B \in \{A\}'$,*
(2) *$I + f(A)B$ is invertible for every function $f \in \mathrm{Hol}(A)$, and*
(3) *for every $f \in \mathrm{Hol}(A)$, where $f(\lambda) \neq 0$,*

$$\mathrm{Lat}(I + f(A)B) = \mathrm{Lat}(f(A)B) = \mathrm{Lat}(B).$$

*Proof.* Let $0 \neq u \in N(A - \lambda I)$. Since $\overline{R(A - \lambda I)} + \mathrm{span}\{u\}$ is closed and $\mathrm{codim}(\overline{R(A - \lambda I)}) \geqslant 2$, by the Hahn-Banach Theorem, there exists nonzero $\phi \in X'$ such that $\phi(u) = 0$ and $\phi(R(A - \lambda I)) = 0$. This implies $(A^* - \lambda I)\phi = 0$. We show that $B := u \otimes \phi$ satisfies the conditions (1), (2), and (3).

Indeed,

$$\begin{aligned} BA - AB &= (u \otimes \phi)A - A(u \otimes \phi) \\ &= u \otimes A^*\phi - Au \otimes \phi \\ &= u \otimes A^*\phi - \lambda u \otimes \phi \\ &= u \otimes (A^* - \lambda I)\phi \\ &= 0. \end{aligned}$$

This proves (1).

Let $f$ be an analytic function on an open set containing the spectrum of $A$. Then, by the Functional Calculus, $f(A)B = Bf(A)$ and

$$r(f(A)B) \leqslant r(f(A))r(B) = r(f(A))|\phi(u)| = 0.$$

Therefore, $I + f(A)B$ is invertible.

A calculation shows that $f(A)u = f(\lambda)u$ and $\mathrm{Lat}B$ is the collection of all the subspaces of $N(\phi)$ and all of the subspaces of $X$ containing $u$. Also,

$$\begin{aligned} f(A)B &= f(A)(u \otimes \phi) \\ &= f(A)u \otimes \phi \\ &= f(\lambda)u \otimes \phi \\ &= f(\lambda)B. \end{aligned}$$

Let $\mathcal{M} \in \mathrm{Lat}(I + f(A)B)$. For every $x \in \mathcal{M}$,

$$\begin{aligned} (I + f(A)B)x &= x + f(\lambda)Bx \\ &= x + f(\lambda)\phi(x)u, \end{aligned}$$

and so $f(\lambda)\phi(x)u \in \mathcal{M}$. Thus, if $f(\lambda) \neq 0$ for a subspace $\mathcal{M}$ of $X$, we have $\mathcal{M} \in \mathrm{Lat}(I + f(A)B)$ if and only if either $u \in \mathcal{M}$ or $\mathcal{M}$ is a subset of $N(\phi)$. Therefore,

$$\operatorname{Lat}(I + f(A)B) = \operatorname{Lat}(f(A)B) = \operatorname{Lat}(B),$$

and the proof is complete. $\square$

**Remark 2.2.** *The condition* $\operatorname{codim}(\overline{R(A - \lambda I)}) \geqslant 2$ *states that* $R(A - \lambda I)$ *is not dense in* $X$. *So, by* [7, *Theorem* 4.12], $N(A^* - \lambda I) \neq (0)$ *and* $\lambda \in \sigma_p(A^*)$.

**Remark 2.3.** *The second condition in the above theorem is equivalent to* $f(A)B$ *being a quasinilpotent operator (i.e.,* $\sigma(f(A)B) = 0$*) on* $X$, *for every* $f \in \operatorname{Hol}(A)$.

**Remark 2.4.** *If* $X$ *is finite dimensional and* $A$ *has a real eigenvalue* $\lambda$ *such that the dimension of the eigenvector space of* $T$ *corresponding to* $\lambda$ *is bigger than 1, then* $\operatorname{codim}(R(A - \lambda I)) \geqslant 2$.

**Corollary 2.5.** *Let* $A$ *be a finite-rank linear operator on infinite dimensional Banach space* $X$. *Then, there exists a rank-one operator* $B$ *on* $X$ *such that*

(1) $B \in \{A\}'$,
(2) $I + f(A)B$ *is invertible for every* $f \in \operatorname{Hol}(A)$, *and*
(3) *for every* $f \in \operatorname{Hol}(A)$, *where* $f(0) \neq 0$,

$$\operatorname{Lat}(I + f(A)B) = \operatorname{Lat}(f(A)B) = \operatorname{Lat}(B).$$

*Proof.* Since $A$ is a finite-rank operator, $A$ is not one-to-one. So, $0 \in \sigma_p(A)$ and $\operatorname{codim}(R(A)) = \infty$. Therefore, the corollary is implied by Theorem 2.1. $\square$

In the preceding theorem, the hypothesis $\operatorname{codim}(\overline{R(A - \lambda I)}) \geqslant 2$ could not be replaced by a weaker one. For, let $X = \mathbb{C}^3$ and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

be a linear operator on $X$. Then, $\sigma(A) = \sigma_p(A) = \{1, 2, 3\}$, and for each $\lambda \in \sigma(A)$, we have $\dim(N(A - \lambda I)) = 1$ and $\dim(R(A - \lambda I)) = 2$. So, $\operatorname{codim}(R(A - \lambda I)) = 1$. We claim that there is no rank-one operator $B \in \{A\}'$ such that $I + f(A)B$ is invertible for every $f \in \operatorname{Hol}(A)$. Suppose there is a rank-one operator $B = u \otimes \phi \in \{A\}'$, where $u \in X$, $\phi \in X'$ and $I + f(A)B$ is invertible for every $f \in \operatorname{Hol}(A)$. Since $X =$

$\bigoplus_{\lambda=1}^{3} N(A-\lambda I)$, we assume that $u \in N(A-I)$ (the cases $u \in N(A-2I)$ and $u \in N(A-3I)$ can similarly be verified). Thus,

$$\begin{aligned} 0 = BA - AB &= (u \otimes \phi)A - A(u \otimes \phi) \\ &= u \otimes A^*\phi - Au \otimes \phi \\ &= u \otimes A^*\phi - u \otimes \phi \\ &= u \otimes (A^* - I)\phi. \end{aligned}$$

Then, $(A^* - I)\phi = 0$ and $\phi(R(A-I)) = 0$. But, we have $X = N(A - I) \bigoplus R(A-I)$ and $N(A-I) = \operatorname{span}\{u\}$, and so $u \notin R(A-I)$ and $\phi(u) \neq 0$. Now, let $f$ be a constant function $f(z) = \alpha$ for an arbitrary scalar $\alpha$ and every $z \in \mathbb{C}$. Then, $f(A) = \alpha I$ and $I + f(A)B = I + \alpha B$. Since $r(B) = |\phi(u)| \neq 0$, $B$ has a nonzero eigenvalue and so there is an operator

$$B' = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

similar to $B$, where $b$ is a nonzero scalar, i.e., there is an invertible linear operator $U$ on $X$ such that $B = UB'U^{-1}$. Then,

$$I + f(A)B' = \begin{pmatrix} 1+\alpha b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\det(I + f(A)B') = 1 + \alpha b$. Now, for $\alpha = \frac{-1}{b}$, we have $\det(I + f(A)B') = 0$, and thus $I + f(A)B'$ is not invertible. Since $f(A)U = Uf(A)$, we have

$$I + f(A)B = U(I + f(A)B')U^{-1},$$

and it follows that $I + f(A)B$ is not invertible, which is a contradiction.

Now, we are going to change the conditions of Theorem 2.1 such that $BA - AB$ will be of rank one. We will show that, when $\sigma_p(A)$ or $\sigma_p(A^*)$ is nonempty, there exists a bounded linear operator $B$ satisfying the conclusion of Theorem 2.1, but $BA - AB$ is of rank one.

**Lemma 2.6.** *Let $A \in \mathcal{L}(X)$ and let $\phi$ be an eigenvector of $A^*$ corresponding to an eigenvalue $\lambda$ of $A^*$. If $B = u \otimes \phi$, for some $u \in X$, then*

$$\operatorname{Lat}(I + Bf(A)) = \operatorname{Lat}(Bf(A)) = \operatorname{Lat}(B),$$

*for every $f \in \operatorname{Hol}(A)$ such that $f(\lambda) \neq 0$.*

*Proof.* We know that the equality $A^*\phi = \lambda\phi$ implies $f(A^*)\phi = f(\lambda)\phi$. Thus,

$$\begin{aligned} Bf(A) &= (u \otimes \phi)f(A) \\ &= u \otimes f(A)^*\phi \\ &= u \otimes f(A^*)\phi \\ &= u \otimes f(\lambda)\phi \\ &= f(\lambda)B. \end{aligned}$$

Therefore, the conclusion follows as in the proof of Theorem 2.1. □

**Theorem 2.7.** *Suppose $A \in \mathcal{L}(X)$ is not a scalar multiple of the identity and $\dim(X) \geqslant 3$. If the point spectrum of $A$ or of $A^*$ has an element $\lambda$, then there exists a rank-one operator $B$ such that*

(1) *$BA - AB$ is of rank-one,*
(2) *$I + f(A)B$ is invertible for every $f \in \mathrm{Hol}(A)$, and*
(3) *if $\lambda \in \sigma_p(A)$ $[\lambda \in \sigma_p(A^*)]$ and $f \in \mathrm{Hol}(A)$ such that $f(\lambda) \neq 0$, then*

$$\mathrm{Lat}(I + f(A)B) = \mathrm{Lat}(f(A)B) = \mathrm{Lat}(B)$$
$$[\mathrm{Lat}(I + Bf(A)) = \mathrm{Lat}(Bf(A)) = \mathrm{Lat}(B)].$$

Note that if $A = \alpha I$, for a scalar $\alpha$, then $BA - AB = 0$, for every $B \in \mathcal{L}(X)$.

*Proof.* First, assume that the point spectrum of $A$ is nonempty and let $u$ be an eigenvector of $A$ corresponding to an eigenvalue $\lambda$. By the hypothesis, we have $R(A - \lambda I) \neq \{0\}$. We consider two cases.

**Case I**. Suppose that $R(A - \lambda I) = \mathrm{span}\{u\}$.
In this case, we have $\dim(N(A - \lambda I)) \geqslant 2$ (if $X$ is finite dimensional, then by an elementary theorem of Linear Algebra, $\dim(N(A - \lambda I)) = \dim(X) - \dim(R(A - \lambda I)) \geqslant 2$. In the case that $\dim(X) = \infty$, consider the restriction of $A - \lambda I$ on a finite dimensional subspace $Y$ of $X$ which contains $u$ and $\dim(Y) \geqslant 3$). Choose $w \in N(A - \lambda I)$ being independent of $u$. So, there is $\phi \in X'$ vanishing on $w$ such that $\phi(u) \neq 0$. We shall show that $B := w \otimes \phi$ satisfies conditions (1) and (2).

Indeed,

$$\begin{aligned} BA - AB &= (w \otimes \phi)A - A(w \otimes \phi) \\ &= w \otimes A^*\phi - Aw \otimes \phi \\ &= w \otimes A^*\phi - \lambda w \otimes \phi \\ &= w \otimes (A^* - \lambda I)\phi. \end{aligned}$$

But, since $R(A - \lambda I) = \operatorname{span}\{u\}$ and $\phi(u) \neq 0$, we have $\phi(R(A - \lambda I)) \neq \{0\}$. The latter inequality implies that $(A^* - \lambda I)\phi \neq 0$ (by [7, Theorem 4.12]). This proves that $BA - AB$ is of rank one.

Let $f \in \operatorname{Hol}(A)$. We have $f(A)w = f(\lambda)w$ and

$$\begin{aligned} f(A)B &= f(A)(w \otimes \phi) \\ &= f(A)w \otimes \phi \\ &= f(\lambda)w \otimes \phi. \end{aligned}$$

Since $\phi(w) = 0$, it follows that

$$\begin{aligned} (f(A)B)^2 &= f(\lambda)^2(w \otimes \phi)(w \otimes \phi) \\ &= f(\lambda)^2(\phi(w)w \otimes \phi) \\ &= 0, \end{aligned}$$

and hence $I + f(A)B$ is invertible.

**Case II**. $R(A - \lambda I) \neq \operatorname{span}\{u\}$.
There exists a bounded linear functional $\phi$ on $X$ such that $\phi(u) = 0$, but $\phi(R(A - \lambda I)) \neq \{0\}$. Thus, $(A^* - \lambda I)\phi \neq 0$ and

$$\begin{aligned} BA - AB &= (u \otimes \phi)A - A(u \otimes \phi) \\ &= u \otimes A^*\phi - Au \otimes \phi \\ &= u \otimes (A^* - \lambda I)\phi. \end{aligned}$$

This implies $BA - AB$ is of rank one. The rest of the proof in this case proceeds as in Case I with $B = u \otimes \phi$. The proof of condition (3) is the same as that of Theorem 2.1.

Now, assume that the point spectrum of $A^*$ is nonempty and take an eigenvector $\phi \in X'$ corresponding to an eigenvalue $\lambda$. We shall make use of the weak$^*$ topology on $X'$. Recall that every weak$^*$-continuous linear functional on $X'$ is of the form $\hat{x} : \psi \mapsto \psi(x)$, for some $x \in X$. Also, note that $A^*$ is not a scalar multiple of the identity. If we assume, as in Case I above, that $R(A^* - \lambda I) = \operatorname{span}\{\phi\}$, then $N(A^* - \lambda I)$ is a

subspace of $X'$ with codimension one. So, $\dim(N(A^* - \lambda I)) \geqslant 2$ (since $\dim(X') \geqslant 3$). Let $\psi \in N(A^* - \lambda I)$ be independent of $\phi$. Then, there exists $u \in X$ such that $\psi(u) = 0$, but $\phi(u) \neq 0$. As in Case I of the first part, $B = u \otimes \psi$ satisfies conditions (1) and (2).
When $R(A^* - \lambda I) \neq \mathrm{span}\{\phi\}$, there exists $u \in X$ such that $\hat{u}(\phi) = 0$, but $\hat{u}(R(A^* - \lambda I)) \neq \{0\}$. Then, we have $\phi(u) = 0$ and $(A - \lambda I)u \neq 0$. Now, for $B := u \otimes \phi$, the proof proceeds as in Case II. Now, Lemma 2.6 completes the proof. □

Theorem 2.7 states that if $A$ has an eigenvalue $\lambda$, then there exists a rank-one operator $B$ such that $\mathrm{Lat}(f(A)B) = \mathrm{Lat}(B)$, for every $f \in \mathrm{Hol}(A)$, where $f(\lambda) \neq 0$. The converse of this result also holds. First, we state Lemma 1 of [4].

**Lemma 2.8.** *[4] Let $A, B \in \mathcal{L}(X)$ and $\mathrm{Lat}A = \mathrm{Lat}B$. If $B$ is of rank one, then $A = \alpha B + \beta I$, with $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq 0$.*

**Theorem 2.9.** *Let $X$ be a Banach space with $\dim(X) \geqslant 2$, and $A \in \mathcal{L}(X)$. If there exist a rank-one operator $B$ and a function $f$ such that $f$ is analytic and one-to-one on an open set containing $\sigma(A)$ and*

(1) $\mathrm{Lat}(f(A)B) = \mathrm{Lat}(B)$, *or*

(2) $\mathrm{Lat}(Bf(A)) = \mathrm{Lat}(B)$,

*then we have in Case (1), the point spectrum of $A$ is nonempty, and in Case (2), the point spectrum of $A^*$ is nonempty.*

*Proof.* Case (1). By the preceding lemma, $f(A)B = \alpha B + \beta I$, with $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq 0$. Then, $(f(A) - \alpha I)B = \beta I$, and since $B$ is of rank one, it follows that $\beta = 0$.
Let $B = u \otimes \phi$, for some $u \in X$ and $\phi \in X'$. So, there is $x \in X$ such that $\phi(x) \neq 0$, and we have the followings:

$$\begin{aligned} f(A)Bx &= \alpha Bx, \\ \phi(x)f(A)u &= \alpha\phi(x)u, \\ f(A)u &= \alpha u. \end{aligned}$$

Thus, by the Spectral Mapping Theorem, $\alpha \in \sigma(f(A)) = f(\sigma(A))$. Let $V$ be a bounded open set containing $\sigma(A)$ and such that $\overline{V}$ is contained in the given open set. Then, $f(V)$ is an open set, and $f(V)$ contains $\sigma(f(A))$ $(= f(\sigma(A)))$. Also, since $f|_{\overline{V}}$ is a homeomorphism, $f(\overline{V}) \supseteq \sigma(f(A))$. Let $f^{-1}$ be the inverse of $f$; i.e., $f^{-1}$ is an analytic function taking $f(V)$ into $V$ such that $f^{-1}(f(z)) = z$, for all $z \in V$. Then

$$\begin{aligned} Au &= f^{-1} \circ f(A)u \\ &= f^{-1}(f(A))u \\ &= f^{-1}(\alpha)u, \end{aligned}$$

and thus $f^{-1}(\alpha) \in \sigma_p(A)$.

Case (2). Let $\mathrm{Lat}(Bf(A)) = \mathrm{Lat}(B)$. By Lemma 2.8, there is a nonzero scalar $\alpha$ such that $Bf(A) = \alpha B$. Let $B = u \otimes \phi$, for some $u \in X$ and $\phi \in X'$. Therefore, since $\sigma(A) = \sigma(A^*)$, we have the followings:

$$\begin{aligned} (u \otimes \phi)f(A) &= \alpha(u \otimes \phi) \\ u \otimes f(A^*)\phi &= u \otimes \alpha\phi \\ f(A^*)\phi &= \alpha\phi \\ A^*\phi &= f^{-1}(\alpha)\phi, \end{aligned}$$

and so $f^{-1}(\alpha) \in \sigma_p(A^*)$. □

By combining Theorems 2.7 and 2.9 we obtain the following result.

**Corollary 2.10.** *Suppose $A \in \mathcal{L}(X)$, where $\dim(X) \geqslant 3$, and $A$ is not a scalar multiple of the identity. If there are rank-one operator $B'$ and function $g$ such that $g$ is analytic and one-to-one on an open set containing $\sigma(A)$ and $\mathrm{Lat}(g(A)B') = \mathrm{Lat}(B')$, then there exists a rank-one operator $B$ such that*

(1) *$BA - AB$ is of rank one,*
(2) *$I + f(A)B$ is invertible for every $f \in \mathrm{Hol}(A)$, and*
(3) *for every $f \in \mathrm{Hol}(A)$, which is nonzero on $\sigma_p(A)$, we have*

$$\mathrm{Lat}(I + f(A)B) = \mathrm{Lat}(f(A)B) = \mathrm{Lat}(B).$$

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