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AN ELEMENTARY METHOD FOR COMPUTING THE KOSTKA COEFFICIENTS

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Abstract. We present a simple and elementary method for computing the Kostka numbers. We use this method to give compact formulas for $K_{\pi,\mu}$ in some special cases.

1. Introduction

Kostka coefficients appear in combinatorics and representation theory and they are very important from a physical point of view. The Kostka number $K_{\pi,\mu}$ is the number of semi-standard Young tableaux of shape π and content μ (see Section 2 below). It is also the multiplicity with which the weight μ appears in the irreducible representation of $\mathfrak{sl}_n(\mathbb{C})$ with highest weight π. Similarly, in the representation theory of symmetric groups we encounter the Kostka numbers as the multiplicity of irreducible character χ_μ in the induction of the principal character of the Young subgroup S_π to S_m. The Kostka coefficients are also important in the study of Schur functions. This work is devoted to computing these numbers in some special cases using an elementary method.

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2. **Generalities**

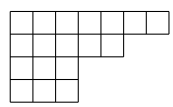
Let m be a positive integer. By a partition of m, we mean an s-tuple $\pi = [a_1, a_2, \ldots, a_s]$ of positive integers with

$$a_1 \geq a_2 \geq \cdots \geq a_s,$$

and

$$a_1 + a_2 + \cdots + a_s = m.$$

We say that s is the height of π and we denote it by $h(\pi)$. To any partition π of m we associate a *Ferrer's diagram* consisting of m boxes arranged in s rows in such a way that the ith row contains a_i boxes. For example, the following is the Ferrer's diagram associated with the partition $\pi = [7, 5, 3, 3]$:



Now, let $\mu = [b_1, b_2, \ldots, b_r]$ be another partition of m. We say that π majorizes μ, if for any i we have

$$b_1 + b_2 + \cdots + b_i \leq a_1 + a_2 + \cdots + a_i.$$

In this case, we write $\mu \trianglelefteq \pi$. This is a partial ordering on the set of all partitions of m. The partition $[m]$ is the maximum element and the partition $[1^m] = [1, 1, \ldots, 1]$ is the minimum element with respect to this ordering. If $\pi = [a_1, a_2, \ldots, a_s]$ and $\mu = [b_1, b_2, \ldots, b_r]$ are two arbitrary partitions of m, then by a semi-standard Young tableau of shape π and content μ, we mean any distribution of the numbers $1, 2, \ldots, m$ in the boxes of the associated Ferrer's diagram of π in such a way that

1. every row is non-decreasing;
2. every column is (strictly) increasing; and
3. for any $1 \leq i \leq m$, the multiplicity of i in the distribution is b_i.

Example 2.1. Let $m = 5$, $\pi = [3, 2]$ and $\mu = [2, 2, 1]$. Then the only semi-standard Young tableaux of shape π and content μ are the following two diagrams:

1	1	2
2	3	

1	1	3
2	2	

The number of all semi-standard Young tableaux of shape π and content μ is denoted by $K_{\pi,\mu}$ and it is called the *Kostka coefficient* or the *Kostka number*. This combinatorial notion has a wide range of interpretations in the representation theory of Lie groups and Lie algebras as well as the representation theory of the symmetric group (see [1] or [2]). The Kostka numbers are also very important in the study of symmetric functions, especially Schur functions. It is well-known that $K_{\pi,\mu} \neq 0$ if and only if $\mu \trianglelefteq \pi$; see [2] for details.

3. Computing Kostka coefficients

Let $\pi = [a_1, a_2, \ldots, a_s]$ and $\mu = [b_1, b_2, \ldots, b_r]$ be two arbitrary partitions of m with $\mu \trianglelefteq \pi$. In general, every semi-standard Young tableau of shape π and content μ has the following form:

$$\begin{array}{lllll} (1^{b_1}) & (2^{x_1}) & (3^{y_1}) & (4^{z_1}) & \cdots \\ (2^{x_2}) & (3^{y_2}) & (4^{z_2}) & \cdots & \\ (3^{y_3}) & (4^{z_3}) & \cdots & & \\ \vdots & \vdots & & & \end{array}$$

where, (t^k) denotes $\boxed{t}\boxed{t}\boxed{\cdots}\boxed{t}$ (k-times), and $x_1, y_1, \ldots$ are non-negative integers. Given a semi-standard Young tableau of shape π and content μ as above, we must have the following equalities:

$$\begin{array}{rcl} x_1 + x_2 & = & b_2 \\ y_1 + y_2 + y_3 & = & b_3 \\ z_1 + z_2 + z_3 + z_4 & = & b_4 \\ & \vdots & \\ x_1 + y_1 + z_1 + \cdots & = & a_1 - b_1 \\ x_2 + y_2 + z_2 + \cdots & = & a_2 \\ y_3 + z_3 + \cdots & = & a_3 \\ & \vdots & \end{array}$$

as well as the inequalities below:

$$\begin{array}{rcl} x_2 & \leq & b_1 \\ x_2+y_2 & \leq & b_1+x_1 \\ x_2+y_2+z_2 & \leq & b_1+x_1+y_1 \\ & \vdots & \\ y_3 & \leq & x_2 \\ y_3+z_3 & \leq & x_2+y_2 \\ & \vdots & \end{array}$$

Our strategy is to count the number of non-negative integer solutions of this system of equalities and inequalities. The resulting number obviously will be the Kostka number $K_{\pi,\mu}$. In what follows, we are going to solve the system in some special cases: if we denote by (s,r) the pair consisting of the heights of π and μ in that order, then the rest of this article concerns the following cases:

$$(s,r)=(2,3),\ (3,3),\ (3,4),\ (4,4).$$

3.1. **The case** $(s,r)=(2,3)$**.** Let $\pi=[a_1,a_2]$ and $\mu=[b_1,b_2,b_3]$. The general form of any semi-standard Young tableau of shape π and content μ is as follows:

$$\begin{array}{lll} (1^{b_1}) & (2^{x_1}) & (3^{y_1}) \\ (2^{x_2}) & (3^{y_2}) & \end{array}$$

Here, x_1,x_2,y_1,y_2 are non-negative integers and we have the following equalities:

$$\begin{array}{rcl} x_1+x_2 & = & b_2 \\ y_1+y_2 & = & b_3 \\ x_1+y_1 & = & a_1-b_1 \\ x_2+y_2 & = & a_2 \end{array}$$

as well as two inequalities:

$$\begin{array}{rcl} x_2 & \leq & b_1 \\ a_2 & \leq & b_1+x_1. \end{array}$$

We can write down the system of equations in matrix form:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_3 \\ a_1 - b_1 \\ a_2 \end{bmatrix}.$$

We name the matrix of coefficients to be H. It has rank 3, and so any solution of the system has the form $X = X_0 + X_1$, where X_1 is an arbitrary but fixed solution and X_0 is the one-parametric solution of the homogenous system $HX = 0$. We set:

$$X_1 = \begin{bmatrix} a_1 - b_1 \\ b_1 + b_2 - a_1 \\ 0 \\ b_3 \end{bmatrix}.$$

On the other hand, the solution for $HX = 0$ is:

$$X_0 = \begin{bmatrix} T \\ -T \\ -T \\ T \end{bmatrix},$$

where, $T \in \mathbb{Z}$. Hence, we have

$$\begin{aligned} x_1 &= T + a_1 - b_1 \\ x_2 &= -T + b_1 + b_2 - a_1 \\ y_1 &= -T \\ y_2 &= T + b_3. \end{aligned}$$

Applying the inequalities we obtain five restrictions on T:

$$\begin{array}{rcccc} b_1 - a_1 & \leq & T & \leq & 0 \\ & & T & \leq & b_1 + b_2 - a_1 \\ b_2 - a_1 & \leq & T & & \\ a_2 - a_1 & \leq & T & & \\ -b_3 & \leq & T. & & \end{array}$$

Now, define

$$\begin{aligned} \mathcal{L}_{\pi,\mu} &= \max(b_1 - a_1, a_2 - a_1, -b_3), \\ \mathcal{U}_{\pi,\mu} &= \min(0, b_1 + b_2 - a_1). \end{aligned}$$

So, we have proved the following result.

Theorem 3.1. *Let* $\pi = [a_1, a_2]$ *and* $\mu = [b_1, b_2, b_3]$. *Then, we have*

$$K_{\pi,\mu} = \mathcal{U}_{\pi,\mu} - \mathcal{L}_{\pi,\mu} + 1.$$

Remark 3.2. It is important to note that since we assumed $\mu \trianglelefteq \pi$, every number in the list $b_1 - a_1, a_2 - a_1, -b_3$ is less than or equal to the numbers in the list $0, b_1 + b_2 - a_1$ (for example, $a_2 - a_1 \leq b_1 + b_2 - a_1$, because $a_2 \leq \frac{m}{2}$, while $b_1 + b_2 \geq \frac{2m}{3}$). So, we never have the case $\mathcal{U}_{\pi,\mu} < \mathcal{L}_{\pi,\mu}$. The same is true for other cases in the next subsections.

3.2. **The case** $(s, r) = (3, 3)$**.** Now, suppose $\pi = [a_1, a_2, a_3]$ and $\mu = [b_1, b_2, b_3]$. Any semi-standard Young tableau of shape π and content μ has the form

$$\begin{array}{lll} (1^{b_1}) & (2^{x_1}) & (3^{y_1}) \\ (2^{x_2}) & (3^{y_2}) & \\ (3^{y_3}) & & \end{array}$$

with equalities

$$\begin{array}{rcl} x_1 + x_2 &=& b_2 \\ y_1 + y_2 + y_3 &=& b_3 \\ x_1 + y_1 &=& a_1 - b_1 \\ x_2 + y_2 &=& a_2 \\ y_3 &=& a_3, \end{array}$$

and inequalities

$$\begin{array}{rcl} 0 < x_2 &\leq& b_1 \\ x_2 + y_2 &\leq& b_1 + x_1 \\ a_3 &\leq& x_2. \end{array}$$

The solution for this system is:

$$\begin{array}{rcl} x_1 &=& T + a_1 - b_1 \\ x_2 &=& -T + b_1 + b_2 - a_1 \\ y_1 &=& -T \\ y_2 &=& T + b_3 - a_3. \end{array}$$

Applying the inequalities, we obtain the restrictions

$$\begin{array}{rcccc} b_1 - a_1 & \leq & T & \leq & 0 \\ a_2 - a_1 & \leq & T & \leq & b_1 + b_2 - a_1 \\ a_3 - b_3 & \leq & T & \leq & a_2 - b_3. \end{array}$$

Note that the inequality $b_2 - a_1 \leq T$ is removed rom the above list, because $b_2 \leq a_2$. Note also that, if we put $a_3 = 0$, we get the same inequalities as in the case $(s, r) = (2, 3)$. Hence, if we define

$$\begin{array}{rcl} \mathcal{L}_{\pi,\mu} & = & \max(b_1 - a_1, a_2 - a_1, a_3 - b_3), \\ \mathcal{U}_{\pi,\mu} & = & \min(0, a_2 - b_3), \end{array}$$

then we have proved the following result.

Theorem 3.3. *For* $\pi = [a_1, a_2, a_3]$ *and* $\mu = [b_1, b_2, b_3]$, *we have*

$$K_{\pi,\mu} = \mathcal{U}_{\pi,\mu} - \mathcal{L}_{\pi,\mu} + 1.$$

3.3. **The case** $(s, r) = (3, 4)$**.** Now, we assume that $\pi = [a_1, a_2, a_3]$ and $\mu = [b_1, b_2, b_3, b_4]$. Hence, our semi-standard Young tableau looks like

$$\begin{array}{llll} (1^{b_1}) & (2^{x_1}) & (3^{y_1}) & (4^{z_1}) \\ (2^{x_2}) & (3^{y_2}) & (4^{z_2}) & \\ (3^{y_3}) & (4^{z_3}) & & \end{array}$$

We have the equations

$$\begin{array}{rcl} x_1 + x_2 & = & b_2 \\ y_1 + y_2 + y_3 & = & b_3 \\ z_1 + z_2 + z_3 & = & b_4 \\ x_1 + y_1 + z_1 & = & a_1 - b_1 \\ x_2 + y_2 + z_2 & = & a_2 \\ y_3 + z_3 & = & a_3, \end{array}$$

as well as the following inequalities

$$\begin{array}{rcl} x_2 & \leq & b_1 \\ a_2 - b_1 & \leq & x_1 + z_2 \\ z_1 & \leq & a_1 - a_2 \\ z_2 & \leq & a_2 - a_3 \\ y_3 & \leq & x_2. \end{array}$$

Let H be the matrix of coefficients in the system of equations. Then, H is row equivalent to the following matrix K:

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, any solution to the system is in the form $X = X_0 + X_1$, where X_1 is an arbitrary but fixed solution and X_0 is the general solution of the homogenous system $KX = 0$. For X_1, we use

$$\begin{bmatrix} a_1 - b_1 \\ b_1 + b_2 - a_1 \\ 0 \\ a_1 + a_2 - b_1 - b_2 \\ -a_1 - a_2 + b_1 + b_2 + b_3 \\ 0 \\ 0 \\ b_4 \end{bmatrix}.$$

On the other hand, X_0, the general solution of the system $KX = 0$, has the following form:

$$\begin{bmatrix} T + S \\ -T - S \\ U - T \\ T \\ -U \\ -S - U \\ S \\ U \end{bmatrix},$$

with $T, S, U \in \mathbb{Z}$. Hence, we have

$$\begin{aligned}
x_1 &= T + S + a_1 - b_1 \\
x_2 &= -T - S + b_1 + b_2 - a_1 \\
y_1 &= U - T \\
y_2 &= T + a_1 + a_2 - b_1 - b_2 \\
y_3 &= -U + a_3 - b_4 \\
z_1 &= -S - U \\
z_2 &= S \\
z_3 &= U + b_4.
\end{aligned}$$

Applying and simplifying inequalities, we obtain the following restrictions on T, S, U:

$$\begin{array}{ccccc}
b_1 + b_2 - a_1 - a_2 & \leq & T & & \\
0 & \leq & S & \leq & a_2 - a_3 \\
-b_4 & \leq & U & \leq & a_3 - b_4 \\
b_1 - a_1 & \leq & T + S & \leq & b_1 + b_2 - a_1 \\
a_2 - a_1 & \leq & S + U & \leq & 0 \\
0 & \leq & U - T & & \\
 & & T + S - U & \leq & a_2 - b_3 \\
a_2 - a_1 & \leq & T + 2S. & &
\end{array}$$

The appendix contains a compact formula for the number of triples of integers (T, S, U) satisfying the above conditions.

3.4. **The case** $(s, r) = (4, 4)$**.** Suppose we have $\pi = [a_1, a_2, a_3, b_4]$ and $\mu = [b_1, b_2, b_3, b_4]$. In this case, we have the following pattern:

$$\begin{array}{llll}
(1^{b_1}) & (2^{x_1}) & (3^{y_1}) & (4^{z_1}) \\
(2^{x_2}) & (3^{y_2}) & (4^{z_2}) & \\
(3^{y_3}) & (4^{z_3}) & & \\
(4^{z_4}) & & &
\end{array}$$

Hence, obviously $z_4 = a_4$. For other indeterminates, we have the following system of equations:

$$\begin{aligned} x_1 + x_2 &= b_2 \\ y_1 + y_2 + y_3 &= b_3 \\ z_1 + z_2 + z_3 &= b_4 - a_4 \\ x_1 + y_1 + z_1 &= a_1 - b_1 \\ x_2 + y_2 + z_2 &= a_2 \\ y_3 + z_3 &= a_3. \end{aligned}$$

Also, the following inequalities hold:

$$\begin{aligned} x_2 &\leq b_1 \\ a_2 - b_1 &\leq x_1 + z_2 \\ z_1 &\leq a_1 - a_2 \\ z_2 &\leq a_2 - a_3 \\ a_4 \leq y_3 &\leq x_2. \end{aligned}$$

As in the case $(3,4)$, we obtain:

$$\begin{aligned} x_1 &= T + S + a_1 - b_1 \\ x_2 &= -T - S + b_1 + b_2 - a_1 \\ y_1 &= U - T \\ y_2 &= T + a_1 + a_2 - b_1 - b_2 \\ y_3 &= -U - a_1 - a_2 + b_1 + b_2 + b_3 \\ z_1 &= -S - U \\ z_2 &= S \\ z_3 &= U + b_4 - a_4, \end{aligned}$$

where, $T, S, U \in \mathbb{Z}$ and these numbers clearly must satisfy the following conditions:

$$\begin{array}{ccccc} b_1 + b_2 - a_1 - a_2 & \leq & T & & \\ 0 & \leq & S & \leq & a_2 - a_3 \\ a_4 - b_4 & \leq & U & \leq & a_3 - b_4 \\ b_1 - a_1 & \leq & T + S & \leq & b_1 + b_2 - a_1 \\ a_2 - a_1 & \leq & S + U & \leq & 0 \\ 0 & \leq & U - T & & \\ & & T + S - U & \leq & a_2 - b_3 \\ a_2 - a_1 & \leq & T + 2S. & & \end{array}$$

The number of all triples of integers satisfying the above conditions is computed in the appendix, and thus again we obtain a compact formula for the Kostka number.

4. **Appendix**

Here, we give a formula for the number of triples of integers satisfying the following seven conditions:

$$\begin{array}{ccccc} a & \leq & T & \leq & b \\ a' & \leq & S & \leq & b' \\ a'' & \leq & U & \leq & b'' \\ A & \leq & T+S & \leq & B \\ A' & \leq & S+U & \leq & B' \\ A'' & \leq & T+S-U & \leq & B'' \\ C & \leq & T+2S & & \\ 0 & \leq & U-T. & & \end{array}$$

Note that all constants in this system are integers or $\pm\infty$. In what follows, we will use the notation $< x >$ for $\max(0, x)$. We proceed step by step, first solving some simpler cases.

1. First, we obtain N, the number of solutions of the following system,

$$\begin{array}{ccccc} a & \leq & T & \leq & b \\ a' & \leq & S & \leq & b' \\ A & \leq & T+S & \leq & B. \end{array}$$

Let $A \leq i \leq B$. Define

$$X = \{(T, S) : a \leq T \leq b,\ a' \leq S \leq b'\}$$

and

$$X_i = \{(T, S) \in X : T + S = i\}.$$

It is clear that $(T, S) \in X_i$ if and only if $S = i - T$ and

$$\begin{array}{ccccc} a & \leq & T & \leq & b \\ -b' + i & \leq & T & \leq & -a' + i. \end{array}$$

Hence, we have

$$N = \sum_{i=A}^{B} < \min(b, -a' + i) - \max(a, -b' + i) + 1 > .$$

2. Now, we consider the following situation:

$$\begin{array}{ccccc} a & \leq & T & \leq & b \\ a^{'} & \leq & S & \leq & b^{'} \\ a^{''} & \leq & U & \leq & b^{''} \\ A & \leq & T+S & \leq & B \\ A^{'} & \leq & S+U & \leq & B^{'}. \end{array}$$

Let $A \leq i \leq B$ and $A^{'} \leq j \leq B^{'}$. Define

$$X = \{(T,S,U) : a \leq T \leq b,\ a^{'} \leq S \leq b^{'},\ a^{''} \leq U \leq b^{''}\}$$

and

$$X_{ij} = \{(T,S,U) \in X : T+S = i, S+U = j\}.$$

We have $(T,S,U) \in X_{ij}$ if and only if $T = i - S$ and $U = j - S$, and also

$$\begin{array}{ccccc} -b+i & \leq & S & \leq & -a+i \\ a^{'} & \leq & S & \leq & b^{'} \\ -b^{''}+j & \leq & S & \leq & -a^{''}+j. \end{array}$$

So, we obtain:

$$N = \sum_{i=A}^{B} \sum_{j=A'}^{B'} < \min(b^{'}, -a+i, -a^{''}+j) - \max(a^{'}, -b+i, -b^{''}+j) + 1 > .$$

3. Now, consider the system,

$$\begin{array}{ccccc} a & \leq & T & \leq & b \\ a^{'} & \leq & S & \leq & b^{'} \\ a^{''} & \leq & U & \leq & b^{''} \\ A & \leq & T+S & \leq & B \\ A^{'} & \leq & S+U & \leq & B^{'} \\ A^{''} & \leq & T+S-U & \leq & B^{''}. \end{array}$$

To handle this case, let

$$f_{ij} = \max(a^{'}, -b+i, -b^{''}+j),\ g_{ij} = \min(b^{'}, -a+i, -a^{''}+j).$$

We have $f_{ij} \leq S \leq g_{ij}$ and the condition $A^{''} \leq T+S-U \leq B^{''}$ implies

$$A^{''} \leq i - j + S \leq B^{''},$$

which is equivalent to:

$$A^{''} - i + j \leq S \leq B^{''} - i + j.$$

Hence, we have

$$N = \sum_{i=A}^{B} \sum_{j=A'}^{B'} < \min(g_{ij}, B^{''} - i + j) - \max(f_{ij}, A^{''} - i + j) + 1 >,$$

or in other words,

$$N = \sum_{i=A}^{B} \sum_{j=A'}^{B'} < \min(B^{''} - i + j, b^{'}, -a + i, -a^{''} + j)$$

$$- \max(A^{''} - i + j, a^{'}, -b + i, -b^{''} + j) + 1 > .$$

4. We can now discuss the main case. We only need to have the additional restrictions $T \leq U$ and $C \leq T + 2S$. But, $T = i - S$ and $U = j - S$, and so we must have $i \leq j$ and also $C - i \leq S$. Hence, our required number is:

$$N = \sum_{i=A}^{B} \sum_{j=\max(A',i)}^{B'} < \min(B^{''} - i + j, b^{'}, -a + i, -a^{''} + j)$$

$$- \max(A^{''} - i + j, a^{'}, -b + i, -b^{''} + j) + 1 > .$$

Remark 4.1. Note that in any summation operation $\sum_{j=x}^{y} F(j)$, the result is zero if $y < x$.

Example 4.2. Suppose $\pi = [4, 3, 2, 1]$ and $\mu = [3, 3, 2, 2]$. Then, the Kostka number $K_{\pi,\mu}$ is equal to the number of integer solutions of the following system of inequalities:

$$\begin{array}{ccccc}
-1 & \leq & T & \leq & +\infty \\
0 & \leq & S & \leq & 1 \\
-1 & \leq & U & \leq & 0 \\
-1 & \leq & T + S & \leq & 2 \\
-1 & \leq & S + U & \leq & 0 \\
-\infty & \leq & T + S - U & \leq & 1 \\
-1 & \leq & T + 2S & & \\
0 & \leq & U - T. & &
\end{array}$$

A direct check shows that $K_{\pi,\mu} = 4$. Using our formula, we also have

$$\begin{aligned} N &= \sum_{i=-1}^{2} \sum_{j=\max(-1,i)}^{0} < \min(1-i+j, 1, 1+i, 1+j) \\ &\qquad\qquad\qquad\qquad - \max(0, -1+j, -1-i) + 1 > \\ &= < 0 - 0 + 1 > + < 0 - 0 + 1 > + < 1 - 0 + 1 > \\ &= 4. \end{aligned}$$

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References

[1] W. Fulton and J. Harris, Representation Theory, A First Course, *Springer-Verlag*, 1991.

[2] B. Sagan, The Symmetric Group: Representation, Combinatorial Algorithms and Symmetric Functions, *Wadsworth and Brook/ Cole Math. Series*, 1991.

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