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IMPROVED INFEASIBLE-INTERIOR-POINT ALGORITHM FOR LINEAR COMPLEMENTARITY PROBLEMS

M. ZANGIABADI* AND H. MANSOURI

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ABSTRACT. We present a modified version of the infeasible-interior-point algorithm for monotone linear complementary problems introduced by Mansouri et al. (Nonlinear Anal. Real World Appl. 12(2011) 545–561). Each main step of the algorithm consists of a feasibility step and several centering steps. We use a different feasibility step, which targets at the μ^+-center. It results a better iteration bound.

1. **Introduction**

For a comprehensive learning about interior-point methods (IPMs), we refer to Roos et al. [5] and Wright [7]. In [4], a full-Newton step infeasible interior-point method (IIPM) for linear optimization (LO) was presented and later this algorithm extended to linear complementarity problems (LCP) by Mansouri et al. [1]. In this paper we present a slightly different algorithm which uses a more natural feasibility step, which targets at the μ^+-center.

This paper is organized as follows. First, we review some results which

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∗Corresponding author

are due to [1], and then, apply them to analyze the feasibility and the centering steps of our algorithm. Then we present our algorithm. Each main step of the algorithm consists of a feasibility step and several centering steps. Recall that in [1] the feasibility step targets at the μ-center of the next pair of perturbed problems. Since the aim of each main iteration is to get a good approximation of the μ^+-center of the next pair of perturbed problems, we take a more natural approach to let the feasibility step target at the μ^+-center of the next pair of perturbed problems. Finally, we give some concluding remarks.

Notations

The notations used throughout the paper is rather standard: capital letters denote matrices, lower case letters denote vectors, script capital letters denote sets, and Greek letters denote scalars. All vectors are considered to be column vectors. The components of a vector $u \in \mathbf{R}^n$ will be denoted by u_i, $i = 1, \cdots, n$. The relation $u > 0$ is equivalent to $u_i > 0$, $i = 1, \cdots, n$, while $u \geq 0$ means $u_i \geq 0$, $i = 1, \cdots, n$. We denote $\mathbf{R}^n_+ = \{u \in \mathbf{R}^n \ : \ u \geq 0\}$, $\mathbf{R}^n_{++} = \{u \in \mathbf{R}^n \ : \ u > 0\}$. For any vector $x \in \mathbf{R}^n$, $x_{\min} = \min(x_1;\ x_2;\ \cdots;\ x_n)$ and $x_{\max} = \max(x_1;\ x_2;\ \cdots;\ x_n)$. If $u \in \mathbf{R}^n$ then $U := \text{diag}\ (u)$ denotes the diagonal matrix having the components of u as diagonal entries. If $x,\ s \in \mathbf{R}^n$, then xs denotes the componentwise (Hadamard) product of the vectors x and s. Furthermore, e denotes the all-one vector of length n. The 2-norm and the infinity norm for vectors are denoted by $\|\cdot\|$ and $\|\cdot\|_\infty$, respectively. The Frobenius matrix norm is given by

$$\|U\|^2 := \sum_{i=1}^{m} \sum_{j=1}^{n} U_{ij}^2 = \mathbf{Tr}\left(U^T U\right).$$

2. Preliminaries

The monotone linear complementarity problem (LCP) is to find vector pair $(x,\ s) \in \mathbf{R}^{2n}$ that satisfies the following conditions

$$s = Mx + q, \quad (x,\ s) \geq 0, \quad x^T s = 0, \qquad (P)$$

where $q \in \mathbf{R}^n$ and M is an $n \times n$ matrix supposed positive semidefinite. We denote the feasible set of the problem (P) by

$$\mathcal{F} := \left\{(x,\ s) \in \mathbf{R}^{2n}_+ : \quad s = Mx + q\right\}$$

and its solution set by

$$\mathcal{F}^* := \left\{ (x^*,\, s^*) \in \mathcal{F} : \quad (x^*)^T s^* = 0 \right\}.$$

Throughout this paper it will be assumed that $\mathcal{F}^*$ is not empty, i.e., (P) has at least one solution. As usual for infeasible interior-point methods (IIPMs), we use the starting point as in [1] that one knows some positive scalars ρ_p and ρ_d such that

$$\|x^*\|_\infty \le \rho_p, \quad \max\left\{\|s^*\|_\infty,\, \rho_p \|Me\|_\infty,\, \|q\|_\infty\right\} \le \rho_d, \tag{2.1}$$

for some $(x^*,\, s^*) \in \mathcal{F}^*$, and the initial iterates are

$$x^0 = \rho_p\, e, \quad s^0 = \rho_d\, e, \quad \mu^0 = \rho_p\, \rho_d. \tag{2.2}$$

Using $\left(x^0\right)^T s^0 = n\rho_p\rho_d$, the total number of iterations in the algorithm of [1] is bounded above by

$$72\, n\, \log \frac{\max\left\{n\rho_p\rho_d,\, \left\|r^0\right\|\right\}}{\varepsilon}, \tag{2.3}$$

where r^0 is the initial value of the residual:

$$r^0 = s^0 - Mx^0 - q. \tag{2.4}$$

Up to a constant factor, the iteration bound (2.3) was first obtained by Potra [2], and it is still the best-known iteration bound for IIPMs.

To describe the aim of this article, we need to recall the main ideas underlying the algorithm in [1]. For any ν with $0 < \nu \le 1$, we consider the perturbed problem (P_ν), defined by

$$s - Mx - q \;=\; \nu r^0, \qquad (x,\, s) \ge 0. \tag{P_ν}$$

Note that if $\nu = 1$ then $(x,\, s) = \left(x^0,\, s^0\right)$ yields a strictly feasible solution of (P_ν). Owing to the choice of initial iterates, we may conclude that if $\nu = 1$ then (P_ν) has a strictly feasible solution, which means that the perturbed problem then satisfies the well-known interior-point condition (IPC). More generally one has the following lemma.

Lemma 2.1. *If the original problem (P) is feasible then the perturbed problem (P_ν) satisfies the IPC.*

We assume that (P) is feasible, it follows from Lemma 2.1 that the problem (P_ν) satisfies the IPC for each $\nu \in (0,\, 1]$. But then its central path exists. This means that the following system has a unique solution for every $\mu > 0$

$$\begin{array}{rcl} s - Mx - q & = & \nu r^0, \quad x \geq 0, \quad s \geq 0, \\ xs & = & \mu e. \end{array} \tag{2.5}$$

If $\nu \in (0, 1]$ and $\mu = \nu \rho_p \rho_d$, we denote this unique solution in the sequel as $(x(\nu), s(\nu))$. Using this notation, we have, by taking $\nu = 1$, $(x(1), s(1)) = \left(x^0, s^0\right) = (\rho_p e, \rho_d e)$.

We measure proximity of iterates (x, s) to the μ-center of the perturbed problem (P_ν) by quantity $\delta(x, s; \mu)$, which is defined as follows:

$$\delta(x, s; \mu) = \frac{1}{\sqrt{2}} \left\| v - v^{-1} \right\|, \quad \text{where} \quad v := \sqrt{\frac{xs}{\mu}}. \tag{2.6}$$

Initially, we have $x = \rho_p e$ and $s = \rho_d e$ and $\mu^0 = \rho_p \rho_d$, where $v = e$ and $\delta(x, s; \mu) = 0$. In the sequel, we assume that at the start of each iteration, $\delta(x, s; \mu)$ is smaller than or equal to a small threshold value $\tau > 0$. So certainly this is true at the start of the first iteration.

3. **An iteration of the algorithm**

In this section we describe one iteration of our algorithm. As we established above, if $\nu = 1$ and $\mu = \mu^0$, then $(x, s) = \left(x^0, s^0\right)$ is the μ-center of the perturbed problem (P_ν). This is our initial iterate.

We measure proximity to the μ-center of the perturbed problem by the quantity $\delta(x, s; \mu)$ as defined in (2.6). Initially we thus have

$$\delta(x, s; \mu) = 0.$$

In what follows we assume that at the start of each iteration, just before the feasibility step, $\delta(x, s; \mu)$ is smaller than or equal to a small threshold value $\tau > 0$. So this is certainly true at the start of the first iteration.

Suppose that for some $\nu \in (0, 1]$, we have (x, s) satisfying the feasibility condition (2.5) and for $\nu = \frac{\mu}{\mu^0}$, and $x^T s \leq \left(n + \delta^2\right) \mu$ and $\delta(x, s; \mu) \leq \tau$. We reduce μ to $\mu^+ = (1 - \theta) \mu$, with $\theta \in (0, 1)$, and find new iterates (x^+, s^+) that satisfy (2.5), with μ replaced by μ^+ and ν by $\nu^+ = \frac{\mu^+}{\mu^0}$, and such that $x^T s \leq \left(n + \delta^2\right) \mu^+$ and $\delta(x^+, s^+; \mu^+) \leq \tau$. Note that $\nu^+ = (1 - \theta) \nu$.

To be more precise, this is achieved as follows. Each main iteration consists of feasibility step and a few centering steps. The feasibility step

serves to get iterates $\left(x^f,\, s^f\right)$ that are strictly feasible for (P_{ν^+}), and close to its μ-center $(x(\mu^+, \nu^+)\,,\, s(\mu^+, \nu^+))$.
In fact, the feasibility step is designed in a such a way that

$$\delta\left(x^f,\, s^f;\; \mu^+\right) \leq \frac{1}{\sqrt{2}}.$$

Since $\left(x^f,\, s^f\right)$ is strictly feasible for (P_{ν^+}), we can easily get iterates $(x^+,\, s^+)$ that are strictly feasible for (P_{ν^+}), and such that

$$\delta\left(x^+,\, s^+;\; \mu^+\right) \leq \tau,$$

just by performing a few centering steps starting at $\left(x^f,\, s^f\right)$ and targeting at the μ^+-center of (P_{ν^+}).

Before describing the search directions used in the feasibility step and the centering step, we give a more formal description of the algorithm in figure 1.

Infeasible full-Newton-step algorithm

Input:
  Accuracy parameter $\varepsilon > 0$;
  barrier update parameter θ, $0 < \theta < 1$;
  threshold parameter $\tau > 0$;
  x^0, $s^0 > 0$ and $\mu^0 > 0$ such that $x^0 s^0 = \mu^0 e$.
begin
  $x := x^0 > 0,\; s := s^0 > 0;\; \mu := \mu^0;$
  while $\max(n\mu,\, \|r\|) \geq \varepsilon$ **do**
  begin
    feasibility step:
      $(x,\, s) := (x,\, s) + \left(\Delta^f x,\, \Delta^f s\right);$
    μ-update:
      $\mu := (1-\theta)\mu;$
    centering steps:
    while $\delta(x,\, s;\; \mu) \geq \tau$ **do**
    begin
      $(x,\, s) := (x,\, s) + (\Delta x,\, \Delta s)\,;$
    end
  end
end

FIGURE 1. Infeasible full-Newton-step algorithm

According to the definition of (P_ν) the feasibility equation for (P_ν) is given by

$$s - Mx - q = \nu r^0, \;\; (x, s) \geq 0,$$

and this of (P_{ν^+}) by

$$s - Mx - q = \nu^+ r^0, \;\; (x,\, s) \geq 0.$$

To get iterates that are feasible for (P_{ν^+}) we need search directions $\Delta^f x$ and $\Delta^f s$ such that

$$(s + \Delta^f s) - M(x + \Delta^f x) - q = \nu^+ r^0, \;\; \left(x + \Delta^f x,\, s + \Delta^f s\right) > 0.$$

Since $(x,\, s)$ is feasible for (P_ν), it follows that $\Delta^f x$ and $\Delta^f s$ should satisfy

$$M\Delta^f x - \Delta^f s = \theta \nu r^0.$$

Therefore, the following system is used to define $\Delta^f x$ and $\Delta^f s$:

$$M\Delta^f x - \Delta^f s \;=\; \theta \nu r^0, \tag{3.1}$$

$$s\Delta^f x + x\Delta^f s \;=\; (1-\theta)\,\mu e - xs. \tag{3.2}$$

It is easy to see that if $(x,\, s)$ is feasible for the perturbed problem (P_ν), then after the feasibility step the iterates satisfy the feasibility condition for (P_{ν^+}), provided that they are nonnegative. Assuming that before the step $\delta\,(x,\, s;\, \mu) \leq \tau$ holds, and by taking θ small enough, it can be guaranteed that after the step, the iterates

$$x^f \;=\; x + \Delta^f x, \tag{3.3}$$

$$s^f \;=\; s + \Delta^f s. \tag{3.4}$$

are nonnegative and moreover $\delta\left(x^f,\, s^f;\, \mu^+\right) \leq \frac{1}{\sqrt{2}}$, where

$$\mu^+ = (1-\theta)\,\mu.$$

So, after the μ-update, the iterates are feasible for (P_{ν^+}), and μ is such that $\delta\left(x^f,\, s^f;\, \mu^+\right) \leq \frac{1}{\sqrt{2}}$.

Remark 3.1. *For (3.2), we use the linearization of $x^f s^f = (1-\theta)\,\mu e$, which means that we target at the μ^+-center. While in* [1]*, the linearization of $x^f s^f = \mu e$ is used (targeting at the μ-center). As our aim is to calculate a feasible solution to (P_{ν^+}), which should also lie in the quadratic convergence neighborhood to it's μ^+-center, the direction used here is more natural and intuitively better.*

In the centering steps, starting at iterates $(x, s) = \left(x^f, s^f\right)$ and targeting at the μ-center, the search directions $\Delta x,$ and Δs are the unique directions defined by

$$\begin{aligned} \Delta s - M\Delta x &= 0, \\ s\Delta x + x\Delta s &= \mu e - xs. \end{aligned} \tag{3.5}$$

Denoting the iterates after a centering step as x^+ and s^+, we recall the following from [1].

Lemma 3.2. *If $\delta := \delta(x, s; \mu) \leq 1$, then the new iterates are feasible, i.e. x^+ and s^+ are nonnegative, and $\left(x^+\right)^T s^+ \leq \left(n + \delta^2\right)\mu$. Moreover, if $\delta = \delta(x, s; \mu) \leq \frac{1}{\sqrt{2}}$, then $\delta = \delta(d, s; \mu) \leq \delta^2$.*

The centering steps serve to get iterates that satisfy

$$x^T s \leq \left(n + \delta^2\right)\mu^+ \quad \text{and} \quad \delta\left(x, s; \mu^+\right) \leq \tau,$$

where τ is much smaller than $\frac{1}{\sqrt{2}}$. By using Lemma 3.2, the required number of centering steps can easily be obtained. This goes as follows. After μ-update, we have $\delta\left(x^f, s^f; \mu^+\right) \leq \frac{1}{\sqrt{2}}$, and hence after k centering steps, the iterates (x, s) satisfy

$$\delta\left(x, s; \mu^+\right) \leq \left(\frac{1}{\sqrt{2}}\right)^{2^k}.$$

Just as in [1] this implies that no more than

$$\log_2\left(\log_2 \frac{1}{\tau^2}\right) \tag{3.6}$$

centering steps are needed.

4. Analysis of the feasibility step

Let (x, s) denote the iterates at the start of an iteration and assume $\delta(x, s; \mu) \leq \tau$. Recall that at the start of the first iteration, this is certainly true because then $\delta(x, s; \mu) = 0$.

4.1. **The effect of the feasibility step and the choice of θ.**

As we established in Section 3, the feasibility step generates new iterates (x^f, s^f) that satisfy the feasibility condition for (P_{ν^+}). A crucial element in the analysis is to show that after the feasibility step $\delta\left(x^f, s^f; \mu^+\right) \leq \frac{1}{\sqrt{2}}$, i.e., that the new iterates are within the region where the Newton process targeting at μ^+-center of (P_{ν^+}) is quadratically convergent.
Define

$$v = \sqrt{\frac{xs}{\mu}}, \quad d_x^f = \frac{v\Delta^f x}{x}, \quad d_s^f = \frac{v\Delta^f s}{s}, \tag{4.1}$$

by the use of (3.2) and (4.1), we get

$$\begin{aligned} x^f s^f &= xs + \left(x\Delta^f x + s\Delta^f s\right) + \Delta^f x \Delta^f s \\ &= (1-\theta)\,\mu e + \Delta^f x \Delta^f s \\ &= (1-\theta)\,\mu e + \tfrac{xs}{v^2} d_x^f d_s^f = \mu\left((1-\theta)\,e + d_x^f d_s^f\right). \end{aligned} \tag{4.2}$$

Lemma 4.1. *The iterates $\left(x^f, s^f\right)$ are strictly feasible if and only if $(1-\theta)\,e + d_x^f d_s^f > 0$*

Proof. The proof is similar to the proof of Lemma 3.1 in [1], and is omitted. □

Corollary 4.2. *The iterates $\left(x^f, s^f\right)$ are strictly feasible if $\left\|d_x^f d_s^f\right\|_\infty < 1-\theta$.*

Proof. By Lemma 4.1 x^f and s^f are strictly feasible if and only if $(1-\theta)\,e + d_x^f d_s^f > 0$. Since the last inequality holds if $\left\|d_x^f d_s^f\right\|_\infty < 1-\theta$, the corollary follows. □

We proceed by deriving an upper bound for $\delta\left(x^f, s^f; \mu^+\right)$. According to definition (2.6) one has

$$\delta\left(x^f, s^f; \mu^+\right) = \frac{1}{2}\left\|v^f - \left(v^f\right)^{-1}\right\|, \quad \text{where} \quad v^f = \sqrt{\frac{x^f s^f}{\mu^+}}.$$

In the sequel we denote $\delta\left(x^f, s^f; \mu^+\right)$ also shortly by $\delta\left(v^f\right)$, and we have the following result.

Lemma 4.3. *If* $\left\|d_x^f d_s^f\right\|_\infty < 1-\theta$, *then*

$$2\delta\left(v^f\right)^2 \le \frac{\left\|\frac{d_x^f d_s^f}{1-\theta}\right\|^2}{1-\left\|\frac{d_x^f d_s^f}{1-\theta}\right\|_\infty}.$$

Proof. To simplify the notations in this proof, let $z := \frac{d_x^f d_s^f}{1-\theta}$. After dividing both sides in (4.2) by μ^+ we get

$$\left(v^f\right)^2 = \frac{\mu\left[(1-\theta)\,e + d_x^f d_s^f\right]}{\mu^+} = \frac{\mu\left[(1-\theta)\,e + (1-\theta)\,z\right]}{(1-\theta)\,\mu} = e + z.$$

Hence we have

$$\begin{aligned} 2\delta\left(v^f\right)^2 &= \sum_{i=1}^n \left(\left(v_i^f\right)^2 + \left(v_i^f\right)^{-2} - 2\right) = \sum_{i=1}^n \left(1 + z_i + \frac{1}{1+z_i} - 2\right) \\ &= \sum_{i=1}^n \frac{z_i^2}{1+z_i} \le \sum_{i=1}^n \frac{z_i^2}{1-|z_i|} \le \sum_{i=1}^n \frac{z_i^2}{1-\|z\|_\infty} = \frac{\|z\|^2}{1-\|z\|_\infty}, \end{aligned}$$

where the inequalities are due to $\|z\|_\infty < 1$. This proves the lemma. □

4.2. **First upper bound for θ.**

Since we need to have $\delta\left(v^f\right) \le \frac{1}{\sqrt{2}}$, it follows from Lemma 4.3 that it suffices to have

$$\frac{\left\|\frac{d_x^f d_s^f}{1-\theta}\right\|^2}{1-\left\|\frac{d_x^f d_s^f}{1-\theta}\right\|_\infty} \le 1. \tag{4.3}$$

As we may easily verify that

$$\left\|d_x^f d_s^f\right\|^2 \le \left(\left\|d_x^f\right\|\left\|d_s^f\right\|\right)^2 \le \frac{1}{4}\left(\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2\right)^2, \tag{4.4}$$

$$\left\|d_x^f d_s^f\right\|_\infty \le \frac{1}{2}\left(\left\|d_x^f\right\|_\infty^2 + \left\|d_s^f\right\|_\infty^2\right) \le \frac{1}{2}\left(\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2\right). \tag{4.5}$$

For the moment we assume that $\frac{\left\|d_x^f\right\|^2+\left\|d_s^f\right\|^2}{1-\theta} < 2$. Then $\left\|\frac{d_x^f d_s^f}{1-\theta}\right\|_\infty < 1$, whence inequality (4.3) holds if

$$\frac{\frac{1}{4}\left(\frac{\left\|d_x^f\right\|^2+\left\|d_s^f\right\|^2}{1-\theta}\right)^2}{1-\frac{1}{2}\frac{\left\|d_x^f\right\|^2+\left\|d_s^f\right\|^2}{1-\theta}} \leq 1.$$

Considering $\frac{\left\|d_x^f\right\|^2+\left\|d_s^f\right\|^2}{1-\theta}$ as a single term, and by some elementary calculation, we obtain that (4.3) holds if

$$\frac{\left\|d_x^f\right\|^2+\left\|d_s^f\right\|^2}{1-\theta} \leq \sqrt{5}-1 \approx 1.237. \tag{4.6}$$

Also by Corollary 4.2 and inequality (4.5), the strict feasibility of $\left(x^f,\, s^f\right)$ can be derived from (4.6). In other words, the inequality (4.6) implies that after the feasibility step $\left(x^f,\, s^f\right)$ is strictly feasible and lies in the quadratic convergence neighborhood with respect to μ^+-center of (P_{ν^+}).

4.3. The scaled search direction d_x^f and d_s^f.

One may easily check that the system (3.1)-(3.2), which defines the search directions $\Delta^f x$ and $\Delta^f s$, can be expressed in term of the scaled search directions d_x^f and d_s^f as follows.

$$MS^{-1}Xd_x^f - d_s^f = \theta\nu v s^{-1} r^0, \tag{4.7}$$

$$d_x^f + d_s^f = (1-\theta)\, v^{-1} - v, \tag{4.8}$$

where $X = diag(x)$, $S = diag(s)$.

Lemma 4.4 (Corollary 2.3 in [2]). *Let $x > 0$ and $s > 0$ be two n-dimensional vectors, and let $M \in \mathbf{R}^{n\times n}$ be a positive semidefinite matrix. Then the solution $(u,\, z)$ of the linear system*

$$MS^{-1}Xu - z = \tilde{a}, \tag{4.9}$$

$$u + z = \tilde{b}, \tag{4.10}$$

satisfies the following relations:

$$
\begin{aligned}
&(4.11) \quad Du = (I + DMD)^{-1}(a+b), \quad & Dz &= b - Du, \\
&(4.12) & \|Du\| &\leq \|a+b\|, \\
&(4.13) & \|Du\|^2 + \|Dz\|^2 &\leq \|b\|^2 + 2\|a+b\|\|a\|,
\end{aligned}
$$

where $D = \left(S^{-1}X\right)^{\frac{1}{2}}$, $b = D\tilde{b}$ *and* $a = D\tilde{a}$.

We are now ready to find an upper bound for $\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2$. To this end we first apply Lemma 4.4 with $u = d_x^f$, $z = d_s^f$, $a = \theta\nu D v s^{-1} r^0$ and $b = D\left((1-\theta)v^{-1} - v\right)$, which implies that

$$
\begin{aligned}
&\left\|Dd_x^f\right\|^2 + \left\|Dd_s^f\right\|^2 \leq \left\|D\left((1-\theta)v^{-1} - v\right)\right\|^2 \\
(4.14) \qquad &+2\left\|\theta\nu D v s^{-1} r^0 + D\left((1-\theta)v^{-1} - v\right)\right\|\left\|\theta\nu D v s^{-1} r^0\right\|.
\end{aligned}
$$

By elementary properties of norms we have

$$
\left\|Dd_x^f\right\| \leq \|D\|\left\|d_x^f\right\|, \ \left\|Dd_s^f\right\| \leq \|D\|\left\|d_s^f\right\|,
$$

and

$$
\begin{aligned}
\left\|\theta\nu D v s^{-1} r^0\right\| &\leq \|D\|\left\|\theta\nu v s^{-1} r^0\right\|, \\
\left\|D\left((1-\theta)v^{-1} - v\right)\right\| &\leq \|D\|\left\|(1-\theta)v^{-1} - v\right\|.
\end{aligned}
$$

Substituting these bounds in (4.14) we obtain the following weaker condition

$$
\begin{aligned}
&\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2 \leq \left\|(1-\theta)v^{-1} - v\right\|^2 \\
(4.15) \qquad &+2\left(\left\|\theta\nu v s^{-1} r^0\right\| + \left\|(1-\theta)v^{-1} - v\right\|\right)\left\|\theta\nu v s^{-1} r^0\right\|.
\end{aligned}
$$

In order to obtain a bound for $\left\|\theta\nu vs^{-1}r^0\right\|$ we write, using $\nu=\frac{\mu}{\mu^0}$ and $v=\sqrt{\frac{xs}{\mu}}$,

$$\begin{aligned}
\left\|\theta\nu vs^{-1}r^0\right\| &= \theta\nu\left\|vs^{-1}\,r^0\right\| \\
&= \theta\frac{\sqrt{\mu}}{\mu^0}\left\|\sqrt{\frac{x}{s}}\,r^0\right\| \\
&\le \theta\frac{\sqrt{\mu}}{\mu^0}\left\|\sqrt{\frac{x}{s}}\,r^0\right\|_1 \\
&= \frac{\theta}{\mu^0}\left\|\sqrt{\frac{\mu}{xs}}\,x\,r^0\right\|_1 \\
&\le \frac{\theta}{\mu^0\,v_{\min}}\left\|x\,r^0\right\|_1 \\
&\le \frac{\theta}{\mu^0\,v_{\min}}\left\|\left(S^0\right)^{-1}\,r^0\right\|_\infty\left\|s^0\right\|_\infty\|x\|_1\,. \qquad (4.16)
\end{aligned}$$

To proceed we have to specify our initial iterates $\left(x^0,\,s^0\right)$. We assume that ρ_p and ρ_d are such that

$$\|x^*\|_\infty\le\rho_p,\qquad \max\left\{\|s^*\|_\infty\,,\,\rho_p\,\|Me\|_\infty\,,\,\|q\|_\infty\right\}\le\rho_d, \qquad (4.17)$$

for some $(x^*,\,s^*)\in\mathcal{F}^*$, and as usual we start the algorithm with

$$x^0=\rho_p\,e,\qquad s^0=\rho_d\,e,\qquad \mu^0=\rho_p\,\rho_d. \qquad (4.18)$$

For such starting points we have clearly

$$\left\|\left(S^0\right)^{-1}\,r^0\right\|_\infty\le 1+\frac{\rho_p}{\rho_d}\,\|Me\|_\infty+\frac{1}{\rho_d}\,\|q\|_\infty\le 3. \qquad (4.19)$$

By substituting (4.18) and (4.19) into (4.16) we obtain

$$\left\|\theta\nu vs^{-1}r^0\right\|\le\frac{3\,\theta}{\rho_p\,v_{\min}}\,\|x\|_1\,. \qquad (4.20)$$

By using Lemma 3.2 and $\|v\|^2=\frac{x^Ts}{\mu}$ we easily obtain the following inequality

$$\left\|(1-\theta)\,v^{-1}-v\right\|^2\le 2\,(1-\theta)\,\delta^2+\left(n+\delta^2\right)\theta^2. \qquad (4.21)$$

Using (4.20) and (4.21) in (4.15) we get

$$\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2 \leq 2\,(1-\theta)\,\delta^2 + \left(n+\delta^2\right)\theta^2$$

$$(4.22)\qquad +2\left(\frac{3\theta}{\rho_p\, v_{\min}}\,\|x\|_1 + \sqrt{2\,(1-\theta)\,\delta^2 + (n+\delta^2)\,\theta^2}\right)\frac{3\theta}{\rho_p\, v_{\min}}\,\|x\|_1\,.$$

Recall that $(x,\, s)$ is feasible for (P_ν) and $\delta\,(x,\, s;\ \mu) \leq \tau$; i.e., this iterate is close to the μ-center of (P_ν). Based on this information, we present the following three lemmas to estimate an upper bound for $\|x\|_1$ and a lower bound for $v_{\min}$.

Lemma 4.5. *Let $\delta = \delta\,(v)$ be given by (2.6). Then*

$$(4.23)\qquad \frac{1}{q\,(\delta)} \leq v_i \leq q\,(\delta)\,,$$

where

$$(4.24)\qquad q\,(\delta) := \frac{\sqrt{2}}{2}\delta + \sqrt{\frac{1}{2}\delta^2 + 1}.$$

Proof. The proof of this lemma is exactly the same as that of Lemma II.60 in [5]. ☐

Lemma 4.6 (Lemma 5.7 in [1])**.** *Let $(x,\, s)$ be feasible for the perturbed problem (P_ν) and $\left(x^0,\, s^0\right)$ as defined in (4.18). Then for any $(x^*,\, s^*) \in \mathcal{F}^*$, we have*

$$\nu\left(\left(s^0\right)^T x + \left(x^0\right)^T s\right) \leq \nu^2\left(x^0\right)^T s^0 + x^T s$$
$$+\nu\,(1-\nu)\left(\left(s^0\right)^T x^* + \left(x^0\right)^T s^*\right) - (1-\nu)\left(s^T x^* + x^T s^*\right).$$

Lemma 4.7 (Lemma 5.8 in [1])**.** *Let $(x,\, s)$ be feasible for the perturbed problem (P_ν) and $\delta\,(v)$ is defined as in (2.6) and $\left(x^0,\, s^0\right)$ as defined in (4.18). Then we have*

$$(4.25)\qquad \|x\|_1 \;\leq\; \left(2 + q\,(\delta)^2\right) n\rho_p,$$

$$(4.26)\qquad \|s\|_1 \;\leq\; \left(2 + q\,(\delta)^2\right) n\rho_p,$$

where $q\,(\delta)$ as defined in (4.24).

By substituting (4.23) and (4.25) into (4.22) we obtain

$$\begin{aligned}\left\|d_x^f\right\|^2+\left\|d_s^f\right\|^2 \le\; & 2\left(1-\theta\right)\delta^2+\left(n+\delta^2\right)\theta^2\\ & +2\left(3n\,\theta q\left(\delta\right)\left(q\left(\delta\right)^2+2\right)\right)^2\\ & +6\left(\sqrt{2\left(1-\theta\right)\delta^2+\left(n+\delta^2\right)\theta^2}\right)n\,\theta q\left(\delta\right)\left(q\left(\delta\right)^2+2\right).\end{aligned} \tag{4.27}$$

4.4. **Value for θ.** We have found that $\delta\left(v^f\right)\le\frac{1}{\sqrt{2}}$ holds if the inequality (4.6) is satisfied. Then by (4.27), inequality (4.6) holds if

$$\begin{aligned}&2\left(1-\theta\right)\delta^2+\left(n+\delta^2\right)\theta^2+2\left(3n\,\theta q\left(\delta\right)\left(q\left(\delta\right)^2+2\right)\right)^2\\ &+6\left(\sqrt{2\left(1-\theta\right)\delta^2+\left(n+\delta^2\right)\theta^2}\right)n\,\theta q\left(\delta\right)\left(q\left(\delta\right)^2+2\right)\le 1.237\left(1-\theta\right).\end{aligned}$$

Obviously, the left-hand side of the above inequality is increasing in δ, due to the definition $q\left(\delta\right):=\frac{\sqrt{2}}{2}\delta+\sqrt{\frac{1}{2}\delta^2+1}$. Using this one may easily verify that the above inequality is satisfied if

$$\tau=\frac{1}{8},\quad \theta=\frac{1}{14n}. \tag{4.28}$$

Then, according to (3.6), with τ as given, after the feasibility step at most 3 centering steps suffices to get iterates $(x^+,\, s^+)$ that satisfy $\delta\left(x^+,\, s^+;\, \mu^+\right)\le\tau$.

4.5. **Complexity analysis.** In the previous sections we have found that if at the start of an iteration the iterates satisfy $\delta\left(x,\, s;\, \mu\right)\le\tau$, with τ as defined in (4.28), then after the feasibility step, with θ as defined in (4.28), the iterates satisfy $\delta\left(x^f,\, s^f;\, \mu\right)\le\frac{1}{\sqrt{2}}$.

According to (3.6), at most 3 centering steps then suffice to get iterates $(x^+,\, y^+,\, s^+)$ that satisfy $\delta\left(x^+,\, s^+;\, \mu^+\right)\le\tau$ again. So each main iteration consists of at most 4 so-called inner iterations, in each iteration we need to compute a search direction (for either a feasibility step or a centering step).

usually one measures the complexity of an IPM by inner iterations as many times as needed. In each main iteration both the values of $n\mu$ and the norm of the residual are reduced by the factor $1-\theta$. Hence, the total number of the main iterations is bounded above by

$$\frac{1}{\theta}\log\frac{\max\left\{\left(x^0\right)^T s^0,\, \left\|r^0\right\|\right\}}{\varepsilon}.$$

Due to (4.28) we may take

$$\theta = \frac{1}{14n}.$$

Hence the total number of inner iterations is bounded above by

$$56\, n \log \frac{\max\left\{ \left(x^0\right)^T s^0,\, \left\|r^0\right\| \right\}}{\varepsilon}.$$

Thus we may state without any more proof the main result of the paper.

Theorem 4.8. *If (P) has optimal solution $(x^*, s^*) \in \mathcal{F}^*$ such that $\|x^*\|_\infty \le \rho_p$ and $\|s^*\|_\infty \le \rho_d$, then after at most*

$$56\, n \log \frac{\max\left\{ \left(x^0\right)^T s^0,\, \left\|r^0\right\| \right\}}{\varepsilon},$$

iterations the algorithm finds an ε-solution of LCP.

Remark 4.9. *The above iteration bound is derived under the assumption that there exists an optimal solution with $(x^*, s^*) \in \mathcal{F}^*$ such that $\|x^*\|_\infty \le \rho_p$ and $\|s^*\|_\infty \le \rho_d$. One might ask what happens if this is not satisfied. In that case, during the course of the algorithm it may happen that after some main steps the proximity measure δ (after the feasibility step) exceeds $\frac{1}{\sqrt{2}}$, because otherwise there is no reason why the algorithm would not generate an ε-solution. So if this happens, either (P) does not have optimal solution in $\mathcal{F}^*$ or the values of ρ_p and ρ_d have been too small. In the latter case one might run the algorithm once more with some larger ρ_p and ρ_d.*

5. **Numerical results**

In this section we present some numerical results. We consider the following examples:

Example 5.1. [6]

$$M = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ -1 & -1 & -2 & 0 \end{bmatrix}, q = \begin{bmatrix} -8 \\ -6 \\ -4 \\ 3 \end{bmatrix}.$$

Example 5.2. [3]

$$M = \begin{bmatrix} 1 & 0 & -0.5 & 0 & 1 & 3 & 0 \\ 0 & 0.5 & 0 & 0 & 2 & 1 & -1 \\ -0.5 & 0 & 1 & 0.5 & 1 & 2 & -4 \\ 0 & 0 & 0.5 & 0.5 & 1 & -1 & 0 \\ -1 & -2 & -1 & -1 & 0 & 0 & 0 \\ -3 & -1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 & 0 \end{bmatrix}, q = \begin{bmatrix} -1 \\ -3 \\ 1 \\ -1 \\ 5 \\ 4 \\ -1.5 \end{bmatrix}.$$

We solve examples 5.1 and 5.2 by using both the short updating algorithm [1] and the algorithm in Figure 1. For both algorithms, the initialization parameters ρ_p and ρ_d are assumed as described in Section 2, and the accuracy parameter ε is set to 10^{-4}. Table 1 shows the number of iterations to obtain ε-solutions for the above two examples with the short updating algorithm and the algorithm in Figure 1. From the table we see that the algorithm in Figure 1 reduced the number of iterations. Since for both algorithms the work in every iteration is almost the same, this is really a huge reduction.

Examples	Algorithm in [1]	Algorithm in Figure 1
Ex.5.1	138	51
Ex.5.2	180	86

Table 1. The number of iterations for examples 5.1 and 5.2

6. Concluding remarks

We presented a new IIPM for LCP; each main iteration consists of a feasibility step and three centering steps. Our new feasibility step is more natural, as it targets at the μ^+-center, which results a better iteration bound in compare with [1]. The ideas underlying this article can be used to extend the algorithm to second-order cone optimization and also to the symmetric cone optimization.

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M. Zangiabadi
Department of Applied Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P.O. Box 115, Shahrekord, Iran
Email: `Zangiabadi-m@sci.sku.ac.ir`

H. Mansouri
Department of Applied Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P.O. Box 115, Shahrekord, Iran
Email: `Mansouri@sci.sku.ac.ir`