METRIC AND PERIODIC LINES IN THE POINCARÉ
BALL MODEL OF HYPERBOLIC GEOMETRY

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ABSTRACT. In this paper, we prove that every metric line in the
Poincaré ball model of hyperbolic geometry is exactly a classical line
of it. We also prove nonexistence of periodic lines in the Poincaré
ball model of hyperbolic geometry.

1. Introduction

A real distance space $\Delta = (\mathbb{S}, d)$ is a non-empty set $\mathbb{S}$ together with
a mapping $d : \mathbb{S} \times \mathbb{S} \to \mathbb{R}$. The subset $k$ of $\mathbb{S}$ is called a metric line of $\Delta$
if, and only if, there exists a bijection $f : k \to \mathbb{R}$ such that ([2])

$$d(x, y) = |f(x) - f(y)| \quad \text{for all } x, y \in k.$$ 

The subset $k$ of $\mathbb{S}$ is called a $\rho$-periodic line of $\Delta$ if and only if, there
exists a bijection $f : k \to [0, \rho]$ with

$$d(x, y) = \begin{cases} 
|f(x) - f(y)|, & \text{if } |f(x) - f(y)| \leq \rho/2 \\
\rho - |f(x) - f(y)|, & \text{if } |f(x) - f(y)| > \rho/2 
\end{cases}$$

for all $x, y \in k$. 

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In [1], Benz determined the metric lines of hyperbolic geometry, the metric and periodic lines of Euclidean geometry, the $2\pi$-periodic lines of spherical geometry, and the $\pi$-periodic lines of elliptic geometry. The problems to determine all metric lines and periodic lines of $\Delta$ is given by Benz as follows:

**Problem 1.1.** Determine all injective functions $x : \mathbb{R} \to \mathbb{S}$ such that

\[
(1.1) \quad d(x(\xi), x(\eta)) = |\xi - \eta|
\]
holds true for all real $\xi, \eta$.

**Problem 1.2.** Determine all injective functions $x : [0, \rho] \to \mathbb{S}$ such that

\[
(1.2) \quad d(x(\xi), x(\eta)) = \begin{cases} |\xi - \eta|, & \text{if } |\xi - \eta| \leq \rho/2 \\ \rho - |\xi - \eta|, & \text{if } |\xi - \eta| > \rho/2 \end{cases}
\]
holds true for all $\xi, \eta \in [0, \rho]$.

Moreover, Benz proved in his paper that the distance space $(k, d)$ is a metric space for every metric and $\rho$-periodic line $k$ of $\Delta = (\mathbb{S}, d)$. Throughout the paper, we only deal with Poincaré ball model of hyperbolic geometry. In the Poincaré ball model, also known as the conformal ball model, a gyroline (hyperbolic line) is an Euclidean semicircular arc that intersects the boundary of the ball orthogonally.

### 2. Möbius Transformations of the disc

In complex analysis Möbius transformations are well known and fundamental. The most general Möbius transformation of the complex open unit disc

\[ \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \]
in the complex $z$-plane

\[ z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0}z} = e^{i\theta} (z_0 \oplus z) \]
defines the Möbius addition $\oplus$ in the disc, allowing the Möbius transformation of the disc to be viewed as Möbius left gyrotranslation

\[ z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0}z} \]
followed by rotation. Here $\theta$ is a real number, $z_0 \in \mathbb{D}$, and $\overline{z_0}$ is the complex conjugate of $z_0$. Möbius substraction ”$\ominus$” is given by $a \ominus z = a \oplus (-z)$, clearly $z \ominus z = 0$ and $\ominus z = -z$. Möbius addition $\oplus$ is a
binary operation in the disc $\mathbb{D}$, but clearly it is neither commutative nor associative. Möbius addition $\oplus$ gives rise to the groupoid $(\mathbb{D}, \oplus)$ studied by Ungar in several books including [5, 6, 9, 10]. Möbius addition is similar to the common vector addition $+$ in Euclidean plane geometry. Since Möbius addition $\oplus$ is neither commutative nor associative, the groupoid $(\mathbb{D}, \oplus)$ is not a group but it has a group-like structure that we present below.

The breakdown of commutativity in Möbius addition is ”repaired” by the introduction of gyration,

$$\text{gyr} : \mathbb{D} \times \mathbb{D} \to \text{Aut}(\mathbb{D}, \oplus)$$

given by the equation

$$\text{gyr} [a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + ab}{1 + \overline{ab}},$$

where $\text{Aut}(\mathbb{D}, \oplus)$ is the automorphism group of the groupoid $(\mathbb{D}, \oplus)$. Therefore, the gyrocommutative law of Möbius addition $\oplus$ follows from the definition of gyration in (2.1),

$$(2.2) \quad a \oplus b = \text{gyr} [a, b] (b \oplus a).$$

Coincidentally, the gyration $\text{gyr}[a, b]$ that repairs the breakdown of the commutative law of $\oplus$ in (2.2), repairs the breakdown of the associative law of $\oplus$ as well, giving rise to the respective left and right gyroassociative laws

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr} [a, b] c$$

and

$$(a \oplus b) \oplus c = a \oplus (b \oplus \text{gyr} [b, a] c)$$

for all $a, b, c \in \mathbb{D}$.

**Definition 2.1.** A groupoid $(\mathbb{G}, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms

- (G1) $0 \oplus a = 0$, left identity property
- (G2) $a \oplus a = 0$, left inverse property
- (G3) $a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr} [a, b] c$, left gyroassociative law
- (G4) $\text{gyr} [a, b] \in \text{Aut}(\mathbb{G}, \oplus)$, gyroautomorphism
- (G5) $\text{gyr} [a, b] = \text{gyr} [a \oplus b, b]$, left loop property

for all $a, b, c \in \mathbb{G}$.

Additionally, if the binary operation ”$\oplus$” obeys the gyrocommutative law

$$(G6) \quad a \oplus b = \text{gyr} [a, b] (b \oplus a), \text{ gyrocommutative law}$$
for all \( a, b, c \in G \), then \((G, \oplus)\) called a gyrocommutative gyrogroup. It is easy to see that \(-a = \ominus a\), for all elements \( a \) of \( G \).

Clearly, with these properties, one can now readily check that the Möbius complex disc groupoid \((D, \oplus)\) is a gyrocommutative gyrogroup.

The axioms in Definition 2.1 imply the right identity property, the right inverse property, the right gyroassociative law and the right loop property. We refer readers to [5] and [6] for more details about gyrogroups.

Now define the secondary binary operation \(\ominus\) in \( G \) by

\[
a \ominus b = a \oplus \text{gyr}[a, \ominus b] \ominus b.
\]

The primary and secondary operations of \( G \) are collectively called the dual operations of gyrogroups.

Let \( a, b \) be two elements of a gyrogroup \((G, \oplus)\). Then the unique solution of the equation

\[
a \oplus x = b
\]

for the unknown \( x \) is

\[
x = \ominus a \oplus b,
\]

and the unique solution of the equation

\[
x \oplus a = b
\]

for the unknown \( x \) is

\[
x = b \ominus a.
\]

3. Möbius gyrogroups: from the disc to the ball

Let us identify complex numbers of the complex plane \( \mathbb{C} \) with vectors of the Euclidean plane \( \mathbb{R}^2 \) in the usual way:

\[
\mathbb{C} \ni u = u_1 + iu_2 = (u_1, u_2) = u \in \mathbb{R}^2.
\]

Then the equations

\[
\begin{align*}
\mathbf{u} \cdot \mathbf{v} &= \text{Re}(\overline{uv}) \\
\| \mathbf{u} \| &= |u|.
\end{align*}
\]

(3.1)

give the inner product and the norm in \( \mathbb{R}^2 \), so that Möbius addition in the disc \( \mathbb{D} \) of \( \mathbb{C} \) becomes Möbius addition in the disc.
\[ \mathbb{R}^2_1 = \{ \mathbf{v} \in \mathbb{R}^2 : \| \mathbf{v} \| < 1 \} \] of \( \mathbb{R}^2 \). In fact we get from Eq.(3.1) that

\[ u \oplus v = \frac{u + v}{1 + u \cdot \mathbf{v}} \]

\[ = \frac{(1 + u \cdot \mathbf{v})(u + v)}{(1 + u \cdot \mathbf{v})(1 + u \cdot \mathbf{v})} \]

(3.2)

\[ = \frac{1 + \| \mathbf{v} \|^2 + u \cdot \mathbf{v} + |u|^2}{1 + \| \mathbf{v} \|^2 + |u|^2} u + \left( 1 - |u|^2 \right) v \]

\[ = \frac{1 + 2u \cdot \mathbf{v} + \| \mathbf{v} \|^2}{1 + 2u \cdot \mathbf{v} + \| \mathbf{v} \|^2} u + \left( 1 - \| \mathbf{u} \|^2 \right) v \]

\[ = u \oplus v \]

for all \( u, v \in \mathbb{D} \) and all \( u, v \in \mathbb{R}^2_1 \).

4. Möbius addition in the ball

Let \( \mathbb{V} \) be any inner-product space and

\[ \mathbb{V}_s = \{ \mathbf{v} \in \mathbb{V} : \| \mathbf{v} \| < s \} \]

be the open ball of \( \mathbb{V} \) with radius \( s > 0 \). Möbius addition in \( \mathbb{V}_s \) is motivated by Eq.(3.2). It is given by the equation

(4.1)

\[ u \oplus v = \frac{\left( 1 + \left( 2/s^2 \right) \mathbf{u} \cdot \mathbf{v} + \left( 1/s^2 \right) \| \mathbf{v} \|^2 \right) \mathbf{u} + \left( 1 - \left( 1/s^2 \right) \| \mathbf{u} \|^2 \right) \mathbf{v}}{1 + \left( 2/s^2 \right) \mathbf{u} \cdot \mathbf{v} + \left( 1/s^2 \right) \| \mathbf{u} \|^2 \| \mathbf{v} \|^2} , \]

where \( \cdot \) and \( \| \cdot \| \) are the inner product and norm that the ball \( \mathbb{V}_s \) inherits from its space \( \mathbb{V} \) and where, ambiguously, \( + \) denotes both addition of real numbers on the real line and addition of vectors in \( \mathbb{V} \). Without loss of generality, we may assume that \( s = 1 \) in (4.1). However we prefer to keep \( s \) as a free positive parameter in order to exhibit the results that in the limit as \( s \to \infty \), the ball \( \mathbb{V}_s \) expands the whole of its real inner product space \( \mathbb{V} \), and Möbius addition \( \oplus \) reduces to vector addition \( + \) in \( \mathbb{V} \), i.e.,

\[ \lim_{s \to \infty} u \oplus v = u + v \]

and

\[ \lim_{s \to \infty} \mathbb{V}_s = \mathbb{V} . \]
Möbius scalar multiplication is given by the equation

$$r \otimes v = \frac{s(1 + \|v\|/s) - (1 - \|v\|/s)}{1 + \|v\|/s + (1 - \|v\|/s)\|v\|} r - \frac{1 - \|v\|/s}{1 + \|v\|/s + (1 - \|v\|/s)\|v\|} r (1 + \|v\|/s) r + \frac{1 - \|v\|/s}{1 + \|v\|/s + (1 - \|v\|/s)\|v\|} r v \|v\| = \frac{s \tanh(r \tanh^{-1} \|v\|/s)}{\|v\|} v,$$

where $r \in \mathbb{R}$, $u, v \in V^c$, $v \neq 0$ and $r \otimes 0 = 0$. Möbius scalar multiplication possesses the following properties:

- (P1) $n \otimes v = v \oplus v \oplus \cdots \oplus v$, ($n$-terms)
- (P2) $(r_1 + r_2) \otimes v = r_1 \otimes v \oplus r_2 \otimes v$, scalar distribute law
- (P3) $(r_1 r_2) \otimes v = r_1 \otimes (r_2 \otimes v)$, scalar associative law
- (P4) $r \otimes (r_1 \otimes v \oplus r_2 \otimes v) = r \otimes (r_1 \otimes v) \oplus r \otimes (r_2 \otimes v)$, monodistribute law
- (P5) $\|r \otimes v\| = |r| \otimes \|v\|$, homogeneity property
- (P6) $\frac{|r| \otimes v}{\|r \otimes v\|} = \frac{v}{\|v\|}$, scaling property
- (P7) $\text{gyr}[a, b] (r \otimes v) = r \otimes \text{gyr}[a, b] v$, gyroautomorphism property
- (P8) $1 \otimes v = v$, multiplicative unit property

**Definition 4.1 (Möbius Gyrovector Spaces).** Let $(V_s, \oplus)$ be a Möbius gyrogroup equipped with scalar multiplication $\otimes$. The triple $(V_s, \oplus, \otimes)$ is called a Möbius gyrovector space.

5. Möbius Geodesics and Angles

As it is well known from Euclidean geometry, the straight line passing through two given points $A$ and $B$ of a vector space $\mathbb{R}^n$ can be represented by the expression

$$A + (-A + B) \xi \quad \text{for } \xi \in \mathbb{R}.$$

Obviously it passes through $A$ when $\xi = 0$, and through $B$ when $\xi = 1$.

In full analogy with Euclidean geometry, the unique Möbius geodesic passing through two given points $A$ and $B$ of a Möbius gyrovector space $(V_s, \oplus, \otimes)$ is represented by the parametric gyrovector equation

$$L_{AB} = A \oplus (\ominus A \oplus B) \otimes \xi$$

with parameter $\xi \in \mathbb{R}$. It passes through $A$ when $\xi = 0$, and through $B$ when $\xi = 1$. The gyroline $L_{AB}$ turns out to be a circular arc that intersects the boundary of the ball $V_s$ orthogonally. The gyromidpoint $M_{AB}$ of the points $A$ and $B$ corresponds to the parameter $\xi = 1/2$ of
the gyrolines $L_{AB}$, see [6],

$$M_{AB} = A \oplus (\ominus A \oplus B) \otimes \frac{1}{2}.$$ 

The measure of a Mobius angle between two intersecting geodesic rays equals the measure of the Euclidean angle between corresponding intersecting tangent lines, as shown in Fig 1 below. The gyrodistance remains invariant under automorphisms and left gyrotranslations (see [6]).

**Definition 5.1.** The hyperbolic distance function in $(\mathbb{V}_s, \oplus, \otimes)$, is given by the equation

$$d(A, B) = \|A \odot B\| \quad \text{for } A, B \in \mathbb{V}_s.$$ 

In [1], W. Benz stressed that in spherical and elliptic Geometries there do not exist metric lines, since for those geometries the left-hand side of (1.1), is bounded, but not the right-hand side. By the same reason there do not exist metric lines in Poincaré ball model of hyperbolic geometry when we let

$$\mathbb{S} := \{X \in \mathbb{V} : \|X\| < 1\} \quad \text{and} \quad d(A, B) = \|A \odot B\|.$$

Let \( V \) be a real inner-product space of arbitrary finite or infinite dimension \( \geq 2 \). Define the real distance space \( \Delta = (S, d) \) by
\[
S := \{ X \in V : \|X\| < 1 \} \text{ and } \tanh d(X, Y) = \|X \odot Y\|
\]
for all \( X, Y \in S \). The classical lines of \( \Delta \) are given by
\[
\{ P \oplus Q \otimes \xi : \xi \in \mathbb{R} \}
\]
for \( P, Q \in S \) such that \( \|Q\| = \tanh 1 \).

**Theorem 6.1.** The metric lines of \( \Delta \) are exactly the classical lines of \( \Delta \).

**Proof.** Let \( P, Q \) be elements of \( S \) satisfying \( \|Q\| = \tanh 1 \). Then the function
\[
(6.1) \quad x(\xi) = P \oplus Q \otimes \xi
\]
is injective and
\[
d(x(\xi), x(\eta)) = |\xi - \eta|
\]
holds true for all \( \xi, \eta \in \mathbb{R} \). Hence (6.1) is a metric line of \( \Delta \).

Now, suppose that the function \( x : \mathbb{R} \to S \) solves the functional equation (1.1) for all \( \xi, \eta \in \mathbb{R} \). Put \( P := x(0) \) and observe that
\[
x'(\xi) := \ominus P \oplus x(\xi)
\]
is also a solution since
\[
\|(\ominus P \oplus x(\xi)) \ominus (\ominus P \oplus x(\eta))\| = \|x(\xi) \ominus x(\eta)\|
\]
holds true for all \( \xi, \eta \in \mathbb{R} \). Put \( Q := x'(1) \), then \( x'(0) = 0 \) and observe, by (1.1),
\[
\tanh |1 - 0| = \tanh 1 = \|x'(1) \ominus x'(0)\| = \|Q\|.
\]
Since
\[
\tanh |\xi - 0| = \tanh |\xi| = \|x'(\xi) \ominus x'(0)\| = \|x'(\xi)\|
\]
for all \( \xi \in \mathbb{R} \), and by (1.1), we get
\[
\tanh |\xi - \eta| = \|x'(\xi) \ominus x'(\eta)\|,
\]
i.e.,
\[
\frac{\tanh^2 \xi + \tanh^2 \eta - 2 \tanh \xi \tanh \eta}{1 + \tanh^2 \xi \tanh^2 \eta - 2 \tanh \xi \tanh \eta} = \frac{\| x'(\xi) \|^2 + \| x'(\eta) \|^2 - 2 \langle x'(\xi), x'(\eta) \rangle}{1 + \| x'(\xi) \|^2 \| x'(\eta) \|^2 - 2 \langle x'(\xi), x'(\eta) \rangle},
\]
and this implies \( \tanh \xi \tanh \eta = \langle x'(\xi), x'(\eta) \rangle \). Hence
\[
\langle x'(\xi), x'(\eta) \rangle^2 = \langle x'(\xi), x'(\xi) \rangle \langle x'(\eta), x'(\eta) \rangle
\]
and by Cauchy-Schwarz we get
\[
x'(\xi) = \varphi(\xi) \otimes Q \quad \text{for all} \ \xi \in \mathbb{R},
\]
with
\[
\varphi(\xi) = \xi,
\]
in view of \( \tanh \xi \tanh \eta = \langle x'(\xi), x'(\eta) \rangle \). Thus \( x(\xi) = P \oplus Q \otimes \xi \) must be a classical line. \( \square \)

**Theorem 6.2.** For all \( \rho > 0 \), then there do not exist \( \rho \)-periodic lines in \( \Delta \).

**Proof.** Assume that \( x : [0, \rho] \to S \) is a solution of (1.2), for a certain \( \rho > 0 \). Put \( A := x(0) \) and observe \( x'(\xi) := \ominus A \oplus x(\xi) \) is also a solution. Obviously \( x'(0) = 0 \) and put \( P := x'(\rho/2) \). For all \( 0 \leq \xi \leq \rho/2 \), by (1.2),
\[
\tanh |\xi - 0| = \tanh \xi = \| x'(\xi) \oplus x'(0) \| = \| x'(\xi) \|.
\]
It follows that for all \( 0 \leq \xi, \eta \leq \rho/2 \),
\[
\tanh |\xi - \eta| = \| x'(\xi) \oplus x'(\eta) \|
\]
i.e.,
\[
\frac{\tanh^2 \xi + \tanh^2 \eta - 2 \tanh \xi \tanh \eta}{1 + \tanh^2 \xi \tanh^2 \eta - 2 \tanh \xi \tanh \eta} = \frac{\| x'(\xi) \|^2 + \| x'(\eta) \|^2 - 2 \langle x'(\xi), x'(\eta) \rangle}{1 + \| x'(\xi) \|^2 \| x'(\eta) \|^2 - 2 \langle x'(\xi), x'(\eta) \rangle},
\]
and this implies \( \tanh \xi \tanh \eta = \langle x'(\xi), x'(\eta) \rangle \). Hence
\[
\langle x'(\xi), x'(\eta) \rangle^2 = \langle x'(\xi), x'(\xi) \rangle \langle x'(\eta), x'(\eta) \rangle
\]
and by Cauchy-Schwarz we get
\[
x'(\xi) = \varphi(\xi) \otimes P \quad \text{for all} \ \xi \in [0, \rho/2],
\]
with
\[ \varphi(\xi) = \frac{2\xi}{\rho}, \]
in view of \( \tanh \xi \tanh \eta = \langle x'(\xi), x'(\eta) \rangle \). If \( \rho/2 < \varsigma < \rho \), by (1.2),
\[ \tanh (\rho - |\varsigma - 0|) = \|x'(\varsigma) \ominus x'(0)\|. \]
Moreover, by (1.2),
\[ \tanh (\varsigma - \rho/2) = \|x'(\varsigma) \ominus x'((\rho/2))\|. \]
This implies \( \langle x'(\varsigma), P \rangle^2 = \langle x'(\varsigma), x'(\varsigma) \rangle \langle P, P \rangle \) and hence, by Cauchy-Schwarz
\[ x'(\varsigma) = \delta(\varsigma) \otimes P \quad \text{for all } \varsigma \in ]\rho/2, \rho[, \]
with
\[ \delta(\varsigma) = \frac{2}{\rho}(\rho - \varsigma), \]
in view of \( \tanh \frac{\rho}{2} \tanh (\rho - \varsigma) = \langle x'(\varsigma), P \rangle \). This yields \( x'(\rho/4) = x'(3\rho/4) \), which contradicts
\[ \left| \frac{3\rho}{4} - \frac{\rho}{4} \right| = d\left( x'(\frac{3\rho}{4}), x'(\frac{\rho}{4}) \right). \]
\[ \square \]

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