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METRIC AND PERIODIC LINES IN THE POINCARÉ BALL MODEL OF HYPERBOLIC GEOMETRY

O. DEMIREL*, E. SOYTÜRK SEYRANTEPE
AND N. SÖNMEZ

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ABSTRACT. In this paper, we prove that every metric line in the Poincaré ball model of hyperbolic geometry is exactly a classical line of it. We also prove nonexistence of periodic lines in the Poincaré ball model of hyperbolic geometry.

1. **Introduction**

A real distance space $\Delta = (\mathbb{S}, d)$ is a non-empty set $\mathbb{S}$ together with a mapping $d : \mathbb{S} \times \mathbb{S} \to \mathbb{R}$. The subset k of $\mathbb{S}$ is called a metric line of Δ if, and only if, there exists a bijection $f : k \to \mathbb{R}$ such that ([2])

$$d(x, y) = |f(x) - f(y)| \quad \text{for all } x, y \in k.$$

The subset k of $\mathbb{S}$ is called a ρ-periodic line of Δ if and only if, there exists a bijection $f : k \to [0, \rho[$ with

$$d(x,y) = \begin{cases} (|f(x) - f(y)|, & \text{if } |f(x) - f(y)| \le \rho/2 \\ \rho - |f(x) - f(y)|, & \text{if } |f(x) - f(y)| > \rho/2 \end{cases}$$

for all $x, y \in k$.

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*Corresponding author

In [1], Benz determined the metric lines of hyperbolic geometry, the metric and periodic lines of Euclidean geometry, the 2π-periodic lines of spherical geometry, and the π-periodic lines of elliptic geometry. The problems to determine all metric lines and periodic lines of Δ is given by Benz as follows:

Problem 1.1. *Determine all injective functions* $x : \mathbb{R} \to \mathbb{S}$ *such that*

$$d\left(x\left(\xi\right), x\left(\eta\right)\right) = \left|\xi - \eta\right| \tag{1.1}$$

holds true for all real ξ, η.

Problem 1.2. *Determine all injective functions* $x : [0, \rho[\to \mathbb{S}$ *such that*

$$d\left(x\left(\xi\right), x\left(\eta\right)\right) = \begin{cases} \left|\xi - \eta\right|, & \text{if } \left|\xi - \eta\right| \leq \rho/2 \\ \rho - \left|\xi - \eta\right|, & \text{if } \left|\xi - \eta\right| > \rho/2 \end{cases} \tag{1.2}$$

holds true for all $\xi, \eta \in [0, \rho[$.

Moreover, Benz proved in his paper that the distance space (k, d) is a metric space for every metric and ρ-periodic line k of $\Delta = (\mathbb{S}, d)$. Throughout the paper, we only deal with Poincaré ball model of hyperbolic geometry. In the Poincaré ball model, also known as the conformal ball model, a gyroline (hyperbolic line) is an Euclidean semicircular arc that intersects the boundary of the ball orthogonally.

2. Möbius Transformations of the disc

In complex analysis Möbius transformations are well known and fundamental. The most general Möbius transformation of the complex open unit disc

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

in the complex z-plane

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta} \left(z_0 \oplus z\right)$$

defines the Möbius addition $\oplus$ in the disc, allowing the Möbius transformation of the disc to be viewed as Möbius *left gyrotranslation*

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by rotation. Here θ is a real number, $z_0 \in \mathbb{D}$, and $\overline{z_0}$ is the complex conjugate of z_0. Möbius substraction "$\ominus$" is given by $a \ominus z = a \oplus (-z)$, clearly $z \ominus z = 0$ and $\ominus z = -z$. Möbius addition $\oplus$ is a binary operation in the disc $\mathbb{D}$, but clearly it is neither commutative nor associative. Möbius addition $\oplus$ gives rise to the groupoid $(\mathbb{D}, \oplus)$ studied by Ungar in several books including [5, 6, 9, 10]. Möbius addition is similar to the common vector addition $+$ in Euclidean plane geometry. Since Möbius addition $\oplus$ is neither commutative nor associative, the groupoid $(\mathbb{D}, \oplus)$ is not a group but it has a group-like structure that we present below.

The breakdown of commutativity in Möbius addition is "repaired" by the introduction of gyration,

$$\mathrm{gyr} : \mathbb{D} \times \mathbb{D} \to \mathrm{Aut}(\mathbb{D}, \oplus)$$

given by the equation

$$\mathrm{gyr}\,[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\bar{b}}{1 + \bar{a}b}, \tag{2.1}$$

where $\mathrm{Aut}(\mathbb{D}, \oplus)$ is the automorphism group of the groupoid $(\mathbb{D}, \oplus)$. Therefore, the *gyrocommutative law* of Möbius addition $\oplus$ follows from the definition of gyration in (2.1),

$$a \oplus b = \mathrm{gyr}\,[a, b]\,(b \oplus a)\,. \tag{2.2}$$

Coincidentally, the gyration $\mathrm{gyr}[a, b]$ that repairs the breakdown of the commutative law of $\oplus$ in (2.2), repairs the breakdown of the associative law of $\oplus$ as well, giving rise to the respective *left* and *right gyroassociative laws*

$$\begin{aligned} a \oplus (b \oplus c) &= (a \oplus b) \oplus \mathrm{gyr}\,[a, b]\,c \\ (a \oplus b) \oplus c &= a \oplus (b \oplus \mathrm{gyr}\,[b, a]\,c) \end{aligned}$$

for all $a, b, c \in \mathbb{D}$.

Definition 2.1. *A groupoid* $(\mathbb{G}, \oplus)$ *is a gyrogroup if its binary operation satisfies the following axioms*

(G1) $0 \oplus a = 0$, *left identity property*
(G2) $\ominus a \oplus a = 0$, *left inverse property*
(G3) $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}\,[a, b]\,c$, *left gyroassociative law*
(G4) $\mathrm{gyr}\,[a, b] \in \mathrm{Aut}(\mathbb{G}, \oplus)$, *gyroautomorphism*
(G5) $\mathrm{gyr}\,[a, b] = \mathrm{gyr}\,[a \oplus b, b]$, *left loop property*

for all $a, b, c \in \mathbb{G}$.

Additionally, if the binary operation "$\oplus$" obeys the gyrocommutative law

(G6) $a \oplus b = \mathrm{gyr}\,[a, b]\,(b \oplus a)$, gyrocommutative law for all $a, b, c \in \mathbb{G}$, then $(\mathbb{G}, \oplus)$ called a gyrocommutative gyrogroup. It is easy to see that $-a = \ominus a$, for all elements a of $\mathbb{G}$.

Clearly, with these properties, one can now readily check that the Möbius complex disc groupoid $(\mathbb{D}, \oplus)$ is a gyrocommutative gyrogroup.

The axioms in Definition 2.1 imply the right identity property, the right inverse property, the right gyyroassociative law and the right loop property. We refer readers to [5] and [6] for more details about gyrogroups.

Now define the secondary binary operation $\boxplus$ in $\mathbb{G}$ by

$$a \boxplus b = a \oplus \operatorname{gyr}[a, \ominus b]\, b.$$

The primary and secondary operations of $\mathbb{G}$ are collectively called the dual operations of gyrogroups.

Let a, b be two elements of a gyrogroup $(\mathbb{G}, \oplus)$. Then the unique solution of the equation

$$a \oplus x = b$$

for the unknown x is

$$x = \ominus a \oplus b,$$

and the unique solution of the equation

$$x \oplus a = b$$

for the unknown x is

$$x = b \boxminus a.$$

3. Möbius gyrogroups: from the disc to the ball

Let us identify complex numbers of the complex plane $\mathbb{C}$ with vectors of the Euclidean plane $\mathbb{R}^2$ in the usual way:

$$\mathbb{C} \ni u = u_1 + iu_2 = (u_1, u_2) = \mathbf{u} \in \mathbb{R}^2.$$

Then the equations

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \operatorname{Re}(\overline{u}v) \\ \|\mathbf{u}\| &= |u|\,. \end{aligned} \tag{3.1}$$

give the inner product and the norm in $\mathbb{R}^2$, so that Möbius addition in the disc $\mathbb{D}$ of $\mathbb{C}$ becomes Möbius addition in the disc

$\mathbb{R}^2_1 = \left\{ \mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\| < 1 \right\}$ of $\mathbb{R}^2$. In fact we get from Eq.(3.1) that

$$
\begin{aligned}
u \oplus v =& \frac{u+v}{1+\overline{u}v} \\
=& \frac{(1+u\overline{v})(u+v)}{(1+\overline{u}v)(1+u\overline{v})} \\
=& \frac{\left(1+\overline{u}v+u\overline{v}+|v|^2\right)u+\left(1-|u|^2\right)v}{1+\overline{u}v+u\overline{v}+|u|^2|v|^2} \\
=& \frac{\left(1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2\right)\mathbf{u}+\left(1-\|\mathbf{u}\|^2\right)\mathbf{v}}{1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\
=& \mathbf{u} \oplus \mathbf{v}
\end{aligned}
\tag{3.2}
$$

for all $u, v \in \mathbb{D}$ and all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2_1$.

4. Möbius addition in the ball

Let $\mathbb{V}$ be any inner-product space and

$$\mathbb{V}_s = \{v \in \mathbb{V} : \|v\| < s\}$$

be the open ball of $\mathbb{V}$ with radius $s > 0$. Möbius addition in $\mathbb{V}_s$ is motivated by Eq.(3.2). It is given by the equation

(4.1)

$$\mathbf{u} \oplus \mathbf{v} = \frac{\left(1+\left(2/s^2\right)\mathbf{u}\cdot\mathbf{v}+\left(1/s^2\right)\|\mathbf{v}\|^2\right)\mathbf{u}+\left(1-\left(1/s^2\right)\|\mathbf{u}\|^2\right)\mathbf{v}}{1+(2/s^2)\,\mathbf{u}\cdot\mathbf{v}+(1/s^4)\,\|\mathbf{u}\|^2\|\mathbf{v}\|^2},$$

where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$ and where, ambiguously, $+$ denotes both addition of real numbers on the real line and addition of vectors in $\mathbb{V}$. Without loss of generality, we may assume that $s = 1$ in (4.1). However we prefer to keep s as a free positive parameter in order to exhibit the results that in the limit as $s \to \infty$, the ball $\mathbb{V}_s$ expands the whole of its real inner product space $\mathbb{V}$, and Möbius addition $\oplus$ reduces to vector addition $+$ in $\mathbb{V}$, i.e.,

$$\lim_{s\to\infty} \mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v}$$

and

$$\lim_{s\to\infty} \mathbb{V}_s = \mathbb{V}.$$

Möbius scalar multiplication is given by the equation

$$\begin{aligned} r \otimes \mathbf{v} =& s\frac{(1+\|\mathbf{v}\|/s)^r-(1-\|\mathbf{v}\|/s)^r}{(1+\|\mathbf{v}\|/s)^r+(1-\|\mathbf{v}\|/s)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ =& s\tanh\left(r\tanh^{-1}\|\mathbf{v}\|/s\right)\frac{\mathbf{v}}{\|\mathbf{v}\|}, \end{aligned}$$

where $r\in\mathbb{R}$, $\mathbf{u},\mathbf{v}\in\mathbb{V}_c$, $\mathbf{v}\neq 0$ and $r\otimes 0=0$. Möbius scalar multiplication possesses the following properties:

- (P1) $n\otimes\mathbf{v}=v\oplus v\oplus\cdots\oplus v$, ($n$-terms)
- (P2) $(r_1+r_2)\otimes\mathbf{v}=r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}$, scalar distribute law
- (P3) $(r_1r_2)\otimes\mathbf{v}=r_1\otimes(r_2\otimes\mathbf{v})$, scalar associative law
- (P4) $r\otimes(r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v})=r\otimes(r_1\otimes\mathbf{v})\oplus r\otimes(r_2\otimes\mathbf{v})$, monodistribute law
- (P5) $\|r\otimes\mathbf{v}\|=|r|\otimes\|\mathbf{v}\|$, homogeneity property
- (P6) $\frac{|r|\otimes\mathbf{v}}{\|r\otimes\mathbf{v}\|}=\frac{\mathbf{v}}{\|\mathbf{v}\|}$, scaling property
- (P7) $\operatorname{gyr}[\mathbf{a},\mathbf{b}](r\otimes\mathbf{v})=r\otimes\operatorname{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v}$, gyroautomorphism property
- (P8) $1\otimes\mathbf{v}=\mathbf{v}$, multiplicative unit property

Definition 4.1 (Möbius Gyrovector Spaces)**.** *Let* $(\mathbb{V}_s,\oplus)$ *be a Möbius gyrogroup equipped with scalar multiplication* $\otimes$*. The triple* $(\mathbb{V}_s,\oplus,\otimes)$ *is called a Möbius gyrovector space.*

5. Möbius Geodesics and Angles

As it is well known from Euclidean geometry, the straight line passing through two given points A and B of a vector space $\mathbb{R}^n$ can be represented by the expression

$$A+(-A+B)\,\xi \quad \text{for } \xi\in\mathbb{R}.$$

Obviously it passes through A when $\xi=0$, and through B when $\xi=1$.

In full analogy with Euclidean geometry, the unique Möbius geodesic passing through two given points A and B of a Möbius gyrovector space $(\mathbb{V}_s,\oplus,\otimes)$ is represented by the parametric gyrovector equation

$$L_{AB}=A\oplus(\ominus A\oplus B)\otimes\xi$$

with parameter $\xi\in\mathbb{R}$. It passes through A when $\xi=0$, and through B when $\xi=1$. The gyroline L_{AB} turns out to be a circular arc that intersects the boundary of the ball $\mathbb{V}_s$ orthogonally. The gyromidpoint M_{AB} of the points A and B corresponds to the parameter $\xi=1/2$ of the gyroline L_{AB}, see [6],

$$M_{AB} = A \oplus (\ominus A \oplus B) \otimes \frac{1}{2}.$$

The measure of a Mobius angle between two intersecting geodesic rays equals the measure of the Euclidean angle between corresponding intersecting tangent lines, as shown in *Fig 1* below. The gyrodistance

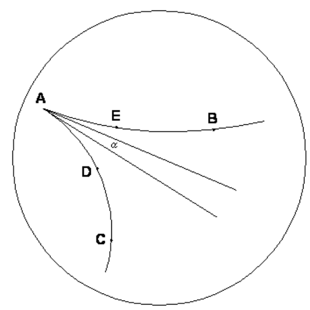

FIGURE 1. The unique 2-dimensional geodesics that pass through two given points and the hyperbolic angle between two intersecting geodesics rays in a Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$. For the non-zero gyrovectors $\ominus A \oplus B$ and $\ominus A \oplus C$ or equivalently $\ominus A \oplus E$ and $\ominus A \oplus D$ the measure of the gyroangle α given by the equation $\cos\alpha = \frac{\ominus A \oplus B}{\|\ominus A \oplus B\|} \cdot \frac{\ominus A \oplus C}{\|\ominus A \oplus C\|}$ or equivalently by the equation $\cos\alpha = \frac{\ominus A \oplus E}{\|\ominus A \oplus E\|} \cdot \frac{\ominus A \oplus D}{\|\ominus A \oplus D\|}$

remains invariant under automorphisms and left gyrotranslations (see [6]).

Definition 5.1. *The hyperbolic distance function in* $(\mathbb{V}_s, \oplus, \otimes)$*, is given by the equation*

$$d(A, B) = \|A \ominus B\| \quad \textit{for } A, B \in \mathbb{V}_s.$$

In [1], W. Benz stressed that in spherical and elliptic Geometries there do not exist metric lines, since for those geometrics the left-hand side of (1.1), is bounded, but not the right-hand side. By the same reason there do not exist metric lines in Poincaré ball model of hyperbolic geometry when we let

$$\mathbb{S} := \{X \in \mathbb{V} : \|X\| < 1\} \quad \text{and} \quad d(A, B) = \|A \ominus B\|.$$

6. Metric and Periodic Lines in Poincaré Ball Model of Hyperbolic Geometry

Let $\mathbb{V}$ be a real inner-product space of arbitrary finite or infinite dimension ≥ 2. Define the real distance space $\Delta = (\mathbb{S}, d)$ by

$$\mathbb{S} := \{X \in \mathbb{V} : \|X\| < 1\} \quad \text{and} \quad \tanh d(X, Y) = \|X \ominus Y\|$$

for all $X, Y \in \mathbb{S}$. The classical lines of Δ are given by

$$\{P \oplus Q \otimes \xi : \xi \in \mathbb{R}\}$$

for $P, Q \in \mathbb{S}$ such that $\|Q\| = \tanh 1$.

Theorem 6.1. *The metric lines of Δ are exactly the classical lines of Δ.*

Proof. Let P, Q be elements of $\mathbb{S}$ satisfying $\|Q\| = \tanh 1$. Then the function

$$x(\xi) = P \oplus Q \otimes \xi \tag{6.1}$$

is injective and

$$d(x(\xi), x(\eta)) = |\xi - \eta|$$

holds true for all $\xi, \eta \in \mathbb{R}$. Hence (6.1) is a metric line of Δ.

Now, suppose that the function $x : \mathbb{R} \to \mathbb{S}$ solves the functional equation (1.1) for all $\xi, \eta \in \mathbb{R}$. Put $P := x(0)$ and observe that

$$x'(\xi) := \ominus P \oplus x(\xi)$$

is also a solution since

$$\|(\ominus P \oplus x(\xi)) \ominus (\ominus P \oplus x(\eta))\| = \|x(\xi) \ominus x(\eta)\|$$

holds true for all $\xi, \eta \in \mathbb{R}$. Put $Q := x'(1)$, then $x'(0) = 0$ and observe, by (1.1),

$$\tanh|1 - 0| = \tanh 1 = \left\|x'(1) \ominus x'(0)\right\| = \|Q\|.$$

Since

$$\tanh|\xi - 0| = \tanh|\xi| = \left\|x'(\xi) \ominus x'(0)\right\| = \left\|x'(\xi)\right\|$$

for all $\xi \in \mathbb{R}$, and by (1.1), we get

$$\tanh|\xi - \eta| = \left\|x'(\xi) \ominus x'(\eta)\right\|,$$

i.e.,

$$\frac{\tanh^2 \xi + \tanh^2 \eta - 2 \tanh \xi \tanh \eta}{1 + \tanh^2 \xi \tanh^2 \eta - 2 \tanh \xi \tanh \eta} = \frac{\|x'(\xi)\|^2 + \|x'(\eta)\|^2 - 2\langle x'(\xi), x'(\eta)\rangle}{1 + \|x'(\xi)\|^2 \|x'(\eta)\|^2 - 2\langle x'(\xi), x'(\eta)\rangle},$$

and this implies $\tanh \xi \tanh \eta = \langle x'(\xi), x'(\eta)\rangle$. Hence

$$\left\langle x'(\xi), x'(\eta)\right\rangle^2 = \left\langle x'(\xi), x'(\xi)\right\rangle \left\langle x'(\eta), x'(\eta)\right\rangle$$

and by Cauchy-Schwarz we get

$$x'(\xi) = \varphi(\xi) \otimes Q \quad \text{for all } \xi \in \mathbb{R},$$

with

$$\varphi(\xi) = \xi,$$

in view of $\tanh \xi \tanh \eta = \langle x'(\xi), x'(\eta)\rangle$. Thus $x(\xi) = P \oplus Q \otimes \xi$ must be a classical line. □

Theorem 6.2. *For all $\rho > 0$, then there do not exist ρ-periodic lines in Δ.*

Proof. Assume that $x : [0, \rho[\to \mathbb{S}$ is a solution of (1.2), for a certain $\rho > 0$. Put $A := x(0)$ and observe $x'(\xi) := \ominus A \oplus x(\xi)$ is also a solution. Obviously $x'(0) = 0$ and put $P := x'(\rho/2)$. For all $0 \leq \xi \leq \rho/2$, by (1.2),

$$\tanh|\xi - 0| = \tanh \xi = \left\|x'(\xi) \ominus x'(0)\right\| = \left\|x'(\xi)\right\|.$$

It follows that for all $0 \leq \xi, \eta \leq \rho/2$,

$$\tanh|\xi - \eta| = \left\|x'(\xi) \ominus x'(\eta)\right\|$$

i.e.,

$$\frac{\tanh^2 \xi + \tanh^2 \eta - 2 \tanh \xi \tanh \eta}{1 + \tanh^2 \xi \tanh^2 \eta - 2 \tanh \xi \tanh \eta} = \frac{\|x'(\xi)\|^2 + \|x'(\eta)\|^2 - 2\langle x'(\xi), x'(\eta)\rangle}{1 + \|x'(\xi)\|^2 \|x'(\eta)\|^2 - 2\langle x'(\xi), x'(\eta)\rangle}.$$

and this implies $\tanh \xi \tanh \eta = \langle x'(\xi), x'(\eta)\rangle$. Hence

$$\left\langle x'(\xi), x'(\eta)\right\rangle^2 = \left\langle x'(\xi), x'(\xi)\right\rangle \left\langle x'(\eta), x'(\eta)\right\rangle$$

and by Cauchy-Schwarz we get

$$x'(\xi) = \varphi(\xi) \otimes P \quad \text{for all } \xi \in [0, \rho/2],$$

with

$$\varphi(\xi) = \frac{2\xi}{\rho},$$

in view of $\tanh\xi\tanh\eta = \langle x'(\xi), x'(\eta)\rangle$. If $\rho/2 < \varsigma < \rho$, by (1.2),

$$\tanh(\rho - |\varsigma - 0|) = \left\| x'(\varsigma) \ominus x'(0) \right\|.$$

Moreover, by (1.2),

$$\tanh(\varsigma - \rho/2) = \left\| x'(\varsigma) \ominus x'(\rho/2) \right\|.$$

This implies $\langle x'(\varsigma), P\rangle^2 = \langle x'(\varsigma), x'(\varsigma)\rangle \langle P, P\rangle$ and hence, by Cauchy-Schwarz

$$x'(\varsigma) = \delta(\varsigma) \otimes P \quad \text{for all } \varsigma \in]\rho/2, \rho[,$$

with

$$\delta(\varsigma) = \frac{2}{\rho}(\rho - \varsigma),$$

in view of $\tanh\frac{\rho}{2}\tanh(\rho - \varsigma) = \langle x'(\varsigma), P\rangle$. This yields $x'(\rho/4) = x'(3\rho/4)$, which contradicts

$$\left| \frac{3\rho}{4} - \frac{\rho}{4} \right| = d\left(x'\left(\frac{3\rho}{4}\right), x'\left(\frac{\rho}{4}\right) \right).$$

□

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Oğuzhan Demirel
Department of Mathematics, Faculty of Science and Literature, ANS Campus,
Afyon Kocatepe University, 03200 Afyonkarahisar, Turkey
Email: odemirel@aku.edu.tr

Emine Soytürk Seyrantepe
Department of Mathematics, Faculty of Science and Literature, ANS Campus,
Afyon Kocatepe University, 03200 Afyonkarahisar, Turkey
Email: soyturk@aku.edu.tr

Nilgün Sönmez
Department of Mathematics, Faculty of Science and Literature, ANS Campus,
Afyon Kocatepe University, 03200 Afyonkarahisar, Turkey
Email: nceylan@aku.edu.tr