ERROR BOUNDS IN APPROXIMATING $n$-TIME DIFFERENTIABLE FUNCTIONS OF SELF-ADJOINT OPERATORS IN HILBERT SPACES VIA A TAYLOR’S TYPE EXPANSION

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Abstract. On utilizing the spectral representation of self-adjoint operators in Hilbert spaces, some error bounds in approximating $n$-time differentiable functions of self-adjoint operators in Hilbert Spaces via a Taylor’s type expansion are given.

1. Introduction

Let $U$ be a self-adjoint operator on the complex Hilbert space $(H, (.,.))$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any continuous function $f : [m, M] \rightarrow \mathbb{C}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieltjes integral:

\begin{equation}
 f(U) = \int_{m}^{M} f(\lambda) dE_\lambda,
\end{equation}


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which in terms of vectors can be written as

\[(1.2) \quad \langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda) \, d\langle E_{\lambda}x, y \rangle, \]

for any \(x, y \in H\). The function \(g_{x,y}(\lambda) := \langle E_{\lambda}x, y \rangle\) is of bounded variation on the interval \([m, M]\) and \(g_{x,y}(m-0) = 0\) and \(g_{x,y}(M) = \langle x, y \rangle\) for any \(x, y \in H\). It is also well known that \(g_{x}(\lambda) := \langle E_{\lambda}x, x \rangle\) is monotonic nondecreasing and right continuous on \([m, M]\).

For a recent monograph devoted to various inequalities for continuous functions of self-adjoint operators, see [10] and the references therein.

For other recent results see [1, 11, 12, 13, 14] and the author’s papers in preprint [2] - [9].

Utilising the spectral representation from (1.2) we have established the following Ostrowski type vector inequality [6]:

**Theorem 1.** Let \(A\) be a self-adjoint operator in the Hilbert space \(H\) with the spectrum \(Sp(A) \subseteq [m, M]\) for some real numbers \(m < M\) and let \(\{E_{\lambda}\}_{\lambda}\) be its spectral family. If \(f : [m, M] \to \mathbb{C}\) is a continuous function of bounded variation on \([m, M]\), then we have the inequality

\[|f(s)\langle x, y \rangle - \langle f(A)x, y \rangle| \leq \langle E_{s}x, x \rangle^{1/2} \langle E_{s}y, y \rangle^{1/2} \sqrt{\int_{m}^{s} f(t) dt} \]

\[+ \langle (1_{H} - E_{s})x, x \rangle^{1/2} \langle (1_{H} - E_{s})y, y \rangle^{1/2} \sqrt{\int_{m}^{s} f(t) dt} \]

\[\leq \|x\| \|y\| \left( \frac{1}{2} \sqrt{\int_{m}^{s} f(t) dt} + \frac{1}{2} \sqrt{\int_{s}^{M} f(t) dt} \right) \leq \|x\| \|y\| \max_{m \leq \lambda \leq M} \langle E_{\lambda}x, x \rangle^{1/2} \langle E_{\lambda}y, y \rangle^{1/2} \sqrt{\int_{m}^{s} f(t) dt} \]

for any \(x, y \in H\) and for any \(s \in [m, M]\).

The trapezoid version of the above result has been obtained in [5] and is as follows:

**Theorem 2.** With the assumptions in Theorem 1 we have the inequalities

\[|f(M) + f(m) - 2\langle f(A)x, y \rangle| \leq \frac{1}{2} \max_{m \leq \lambda \leq M} \left[ \langle E_{\lambda}x, x \rangle^{1/2} \langle E_{\lambda}y, y \rangle^{1/2} \right] \sqrt{\int_{m}^{M} f(t) dt} \]

\[+ \langle (1_{H} - E_{\lambda})x, x \rangle^{1/2} \langle (1_{H} - E_{\lambda})y, y \rangle^{1/2} \sqrt{\int_{m}^{M} f(t) dt} \leq \frac{1}{2} \|x\| \|y\| \max_{m \leq \lambda \leq M} \langle E_{\lambda}x, x \rangle^{1/2} \langle E_{\lambda}y, y \rangle^{1/2} \sqrt{\int_{m}^{M} f(t) dt} \]
for any \( x, y \in H \).

In this paper, by utilizing the spectral representation of self-adjoint operators in Hilbert spaces, some error bounds in approximating \( n \)-time differentiable functions of self-adjoint operators in Hilbert Spaces via a Taylor’s type expansion are given. Applications for some elementary functions of interest including the exponential and logarithmic functions are also provided.

2. Main Results

The following result provides a Taylor’s type representation for a function of self-adjoint operators in Hilbert spaces with integral remainder.

\textbf{Theorem 3.} Let \( A \) be a self-adjoint operator in the Hilbert space \( H \) with the spectrum \( \text{Sp}(A) \subseteq [m, M] \) for some real numbers \( m < M \), \( \{E_{\lambda}\}_\lambda \) be its spectral family, \( I \) be a closed subinterval on \( \mathbb{R} \) with \( [m, M] \subset I \) (the interior of \( I \)) and let \( n \) be an integer with \( n \geq 1 \). If \( f : I \to \mathbb{C} \) is such that the \( n \)-th derivative \( f^{(n)} \) is of bounded variation on the interval \([m, M]\), then for any \( c \in [m, M] \) we have the equalities

\[
(2.1) \quad f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) (A - c1_H)^k + R_n(f, c, m, M)
\]

where

\[
(2.2) \quad R_n(f, c, m, M) = \frac{1}{n!} \int_{m}^{M} \left( \int_{c}^{\lambda} (\lambda - t)^n d\left( f^{(n)}(t) \right) \right) dE_{\lambda}.
\]

\textit{Proof.} We utilize the Taylor formula for a function \( f : I \to \mathbb{C} \) whose \( n \)-th derivative \( f^{(n)} \) is locally of bounded variation on the interval \( I \) to write the equality

\[
(2.3) \quad f(\lambda) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) (\lambda - c)^k + \frac{1}{n!} \int_{c}^{\lambda} (\lambda - t)^n d\left( f^{(n)}(t) \right)
\]

for any \( \lambda, c \in [m, M] \), where the integral is taken in the Riemann-Stieltjes sense.
If we integrate the equality on \([m, M]\) in the Riemann-Stieltjes sense with the integrator \(E\) we get

\[
\int_{m-0}^{M} f(\lambda) \, d\lambda = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \int_{m-0}^{M} (\lambda - c)^k \, d\lambda + \frac{1}{n!} \int_{m-0}^{M} \left( \int_{c}^{\lambda} (\lambda - t)^n \, d\left(f^{(n)}(t)\right) \right) \, d\lambda
\]

which, by the spectral representation (1.1), produces the equality (2.1) with the representation of the remainder from (2.2). \(\square\)

The following particular instances are of interest for applications:

**Corollary 4.** With the assumptions of the above Theorem 3, we have the equalities

\(\tag{2.4}\)
\[f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(m) (A - m1_H)^k + L_n(f, c, m, M)\]

where

\[L_n(f, c, m, M) = \frac{1}{n!} \int_{m-0}^{M} \left( \int_{c}^{\lambda} (\lambda - t)^n \, d\left(f^{(n)}(t)\right) \right) \, d\lambda\]

and

\(\tag{2.5}\)
\[f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}\left(\frac{m + M}{2}\right) \left( A - \frac{m + M}{2}1_H \right)^k + M_n(f, c, m, M)\]

where

\[M_n(f, c, m, M) = \frac{1}{n!} \int_{m-0}^{M} \left( \int_{m + M/2}^{\lambda} (\lambda - t)^n \, d\left(f^{(n)}(t)\right) \right) \, d\lambda\]

and

\(\tag{2.6}\)
\[f(A) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(M) (M1_H - A)^k + U_n(f, c, m, M)\]

where

\(\tag{2.7}\)
\[U_n(f, c, m, M) = \frac{(-1)^{n+1}}{n!} \int_{m-0}^{M} \left( \int_{\lambda}^{M} (t - \lambda)^n \, d\left(f^{(n)}(t)\right) \right) \, d\lambda\]

respectively.
We start with the following result that provides an approximation for an \( n \)-time differentiable function of self-adjoint operators in Hilbert spaces:

**Theorem 5.** Let \( A \) be a self-adjoint operator in the Hilbert space \( H \) with

the spectrum \( \text{Sp}(A) \subseteq [m, M] \) for some real numbers \( m < M \), \( \{E_\lambda\}_\lambda \)

be its spectral family, \( I \) be a closed subinterval on \( \mathbb{R} \) with \([m, M] \subset \bar{I} \)

(the interior of \( I \)) and let \( n \) be an integer with \( n \geq 1 \). If \( f : I \to \mathbb{C} \) is

such that the \( n \)-th derivative \( f^{(n)} \) is of bounded variation on the interval

\([m, M]\), then for any \( c \in [m, M] \) we have the inequality

\[
|\langle R_n(f, c, m, M)x, y \rangle| \\
= |\langle f(A)x, y \rangle - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \langle (A - c1_H)^k x, y \rangle| \\
\leq \frac{1}{n!} \left[ (c - m)^n \bigvee_{m}^{c} \left( f^{(n)} \right) \bigvee_{m}^{c} \left( \langle E_{\cdot} x, y \rangle \right) \\
+ (M - c)^n \bigvee_{c}^{M} \left( f^{(n)} \right) \bigvee_{m}^{M} \left( \langle E_{\cdot} x, y \rangle \right) \right] \\
\leq \frac{1}{n!} \max \left\{ (M - c)^n \bigvee_{c}^{M} \left( f^{(n)} \right), (c - m)^n \bigvee_{m}^{M} \left( f^{(n)} \right) \bigvee_{m}^{M} \left( \langle E_{\cdot} x, y \rangle \right) \right\} \\
\leq \frac{1}{n!} \left( \frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right)^n \bigvee_{m}^{M} \left( f^{(n)} \right) \bigvee_{m}^{M} \left( \langle E_{\cdot} x, y \rangle \right),
\]

for any \( x, y \in H \).

**Proof.** From the identities (2.1) and (2.2) we have

\[
\langle R_n(f, c, m, M)x, y \rangle \\
= \frac{1}{n!} \int_{m-0}^{M} \left( \int_{c}^{\lambda} (\lambda - t)^n d \left( f^{(n)}(t) \right) \right) d \langle E_{\lambda} x, y \rangle \\
= \frac{1}{n!} \int_{m-0}^{c} \left( \int_{c}^{\lambda} (\lambda - t)^n d \left( f^{(n)}(t) \right) \right) d \langle E_{\lambda} x, y \rangle \\
+ \frac{1}{n!} \int_{c}^{M} \left( \int_{c}^{\lambda} (\lambda - t)^n d \left( f^{(n)}(t) \right) \right) d \langle E_{\lambda} x, y \rangle
\]
for any $x, y \in H$.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function, $v : [a, b] \to \mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) \, dv(t) \right| \leq \max_{t \in [a, b]} |p(t)| \, \bigvee_a^b (v),$$

where $\bigvee_a^b (v)$ denotes the total variation of $v$ on $[a, b]$.

Taking the modulus in (2.9) and utilizing the above property, we have

\begin{align}
(2.10) \quad & |\langle R_n (f, c, m, M) \, x, y \rangle| \\
& \leq \frac{1}{n!} \left| \int_m^c \left( \int_c^\lambda (\lambda - t)^n \, d \left( f^{(n)} (t) \right) \right) \, d \langle E(x, y) \rangle \right| \\
& + \frac{1}{n!} \left| \int_c^M \left( \int_c^\lambda (\lambda - t)^n \, d \left( f^{(n)} (t) \right) \right) \, d \langle E(x, y) \rangle \right| \\
& \leq \frac{1}{n!} \max_{\lambda \in [m, c]} \left| \int_c^\lambda (\lambda - t)^n \, d \left( f^{(n)} (t) \right) \right| \bigvee_m^c \left( \langle E(x, y) \rangle \right) \\
& + \frac{1}{n!} \max_{\lambda \in [c, M]} \left| \int_c^\lambda (\lambda - t)^n \, d \left( f^{(n)} (t) \right) \right| \bigvee_c^M \left( \langle E(x, y) \rangle \right)
\end{align}

for any $x, y \in H$.

By the same property for the Riemann-Stieltjes integral we have

\begin{align}
(2.11) \quad & \max_{\lambda \in [m, c]} \left| \int_c^\lambda (\lambda - t)^n \, d \left( f^{(n)} (t) \right) \right| \leq (c - m)^n \bigvee_m^c \left( f^{(n)} \right) \\
\end{align}

and

\begin{align}
(2.12) \quad & \max_{\lambda \in [c, M]} \left| \int_c^\lambda (\lambda - t)^n \, d \left( f^{(n)} (t) \right) \right| \leq (M - c)^n \bigvee_c^M \left( f^{(n)} \right). 
\end{align}
Now, on making use of (2.10)-(2.12) we deduce

\[ |\langle R_n (f, c, m, M) x, y \rangle| \]

\[ \leq \frac{1}{n!} \left[ (c - m)^n \sum_{m}^{c} \left( f^{(m)} \right) \sum_{m}^{c} \left( \langle E(x) x, y \rangle \right) \right] \]

\[ + (M - c)^n \sum_{m}^{c} \left( f^{(n)} \right) \sum_{m}^{c} \left( \langle E(x) x, y \rangle \right) \]

\[ \leq \frac{1}{n!} \max \left\{ (c - m)^n \sum_{m}^{c} \left( f^{(m)} \right), (M - c)^n \sum_{m}^{c} \left( f^{(n)} \right) \right\} \]

\[ \times \left[ \sum_{m}^{c} \left( \langle E(x) x, y \rangle \right) + \sum_{m}^{c} \left( \langle E(x) x, y \rangle \right) \right] \]

\[ \leq \frac{1}{n!} \max \{ (c - m)^n, (M - c)^n \} \sum_{m}^{c} \left( f^{(n)} \right) \sum_{m}^{c} \left( \langle E(x) x, y \rangle \right) \]

\[ = \frac{1}{n!} \left( \frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right) \sum_{m}^{c} \left( f^{(n)} \right) \sum_{m}^{c} \left( \langle E(x) x, y \rangle \right) \]

for any \( x, y \in H \) and the proof is complete. \( \square \)

The following particular cases are of interest for applications

**Corollary 6.** With the assumption of Theorem 5 we have the inequalities

\[
\langle f (A) x, y \rangle - \frac{1}{k!} f^{(k)} (m) \langle (A - m1_H)^k x, y \rangle \leq \frac{1}{n!} \left( M - m \right)^n \sum_{m}^{c} \left( f^{(n)} \right) \sum_{m}^{c} \left( \langle E(x) x, y \rangle \right) \]

\[
\leq \frac{1}{n!} \sum_{m}^{c} \left( f^{(n)} \right) \sum_{m}^{c} \left( \langle E(x) x, y \rangle \right) \]

\[
\leq \frac{1}{n!} \sum_{m}^{c} \left( f^{(n)} \right) \parallel x \parallel \parallel y \parallel ,
\]
\begin{equation}
\begin{aligned}
\langle f(A) x, y \rangle - \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(M) \langle (M1_H - A)^k x, y \rangle \\
\leq \frac{1}{n!} (M - m)^n \bigg| \bigg| f^{(n)} \bigg| \bigg| \sum_{m} (E_m x, y) \\
\leq \frac{1}{n!} (M - m)^n \bigg| \bigg| f^{(n)} \bigg| \bigg| x \bigg| \bigg| y \\
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\langle f(A) x, y \rangle - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)} \left( \frac{m + M}{2} \right) \langle (A - \frac{m + M}{2})^k x, y \rangle \\
\leq \frac{1}{2^n n!} (M - m)^n \max \left\{ \sum_{m}^{M} \left| f^{(m)} \right| \right\} \sum_{m}^{M} \left| (E_m x, y) \right| \\
\leq \frac{1}{2^n n!} (M - m)^n \max \left\{ \sum_{m}^{M} \left| f^{(m)} \right| \right\} \left| x \right| \left| y \right|
\end{aligned}
\end{equation}

respectively, for any \( x, y \in H \).

**Proof.** The first part in the inequalities follow from (2.8) by choosing \( c = m, c = M \) and \( c = \frac{m + M}{2} \) respectively.

If \( P \) is a nonnegative operator on \( H \), i.e., \( \langle Px, x \rangle \geq 0 \) for any \( x \in H \), then the following inequality is a generalization of the Schwarz inequality in \( H \)

\begin{equation}
|\langle Px, y \rangle|^2 \leq \langle Px, x \rangle \langle Py, y \rangle
\end{equation}

for any \( x, y \in H \).

Now, if \( d : m = t_0 < t_1 < ... < t_{n-1} < t_n = M \) is an arbitrary partition of the interval \([m, M]\), then we have by Schwarz’s inequality
for nonnegative operators (2.16) that

\[
\begin{align*}
\sup_d \left\{ \sum_{i=0}^{n-1} \langle (E_{t+1} - E_t) x, y \rangle \right\} \\
\leq \sup_d \left\{ \sum_{i=0}^{n-1} \left[ \langle (E_{t+1} - E_t) x, x \rangle \right]^{1/2} \langle (E_{t+1} - E_t) y, y \rangle \right\}^{1/2} := B.
\end{align*}
\]

By the Cauchy-Buniakowski-Schwarz inequality for sequences of real numbers we also have that

\[
\begin{align*}
B & \leq \sup_d \left\{ \sum_{i=0}^{n-1} \left[ \langle (E_{t+1} - E_t) x, x \rangle \right] \right\}^{1/2} \left\{ \sum_{i=0}^{n-1} \langle (E_{t+1} - E_t) y, y \rangle \right\}^{1/2} \\
& \leq \sup_d \left\{ \sum_{i=0}^{n-1} \langle (E_{t+1} - E_t) x, x \rangle \right\}^{1/2} \left\{ \sum_{i=0}^{n-1} \langle (E_{t+1} - E_t) y, y \rangle \right\}^{1/2} \\
& = \left[ \mathcal{V}_M \left( \langle E_{(\cdot)} x, x \rangle \right) \right]^{1/2} \left[ \mathcal{V}_M \left( \langle E_{(\cdot)} y, y \rangle \right) \right]^{1/2} = \|x\| \|y\|
\end{align*}
\]

for any \(x, y \in H\). These prove the last part of the above inequalities (2.13)-(2.15).

The following result also holds:

**Theorem 7.** Let \(A\) be a self-adjoint operator in the Hilbert space \(H\) with the spectrum \(Sp(A) \subseteq [m, M]\) for some real numbers \(m < M\), \(\{E_\lambda\}_\lambda\) be its spectral family, \(I\) be a closed subinterval on \(\mathbb{R}\) with \([m, M] \subset I\) (the interior of \(I\)) and let \(n\) be an integer with \(n \geq 1\). If \(f : I \to \mathbb{C}\) is such that the \(n\)-th derivative \(f^{(n)}\) is Lipschitzian with the constant \(L_n > 0\) on
the interval $[m, M]$, then for any $c \in [m, M]$ we have the inequality

\begin{equation}
|\langle R_n(f, c, m, M) x, y \rangle| 
\end{equation}

\begin{align*}
&\leq \frac{1}{(n+1)!} L_n \left[ (c-m)^{n+1} \bigg( \frac{c}{m} \langle E_c x, y \rangle \bigg) + (M-c)^{n+1} \bigg( \frac{M}{c} \langle E_c x, y \rangle \bigg) \right] \\
&\leq \frac{1}{(n+1)!} L_n \left( \frac{1}{2} (M-m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} M \bigg( \langle E_c x, y \rangle \bigg) \\
&\leq \frac{1}{(n+1)!} L_n \left( \frac{1}{2} (M-m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} \|x\| \|y\|
\end{align*}

for any $x, y \in H$.

**Proof.** First of all, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e., $|f(s) - f(t)| \leq L |s - t|$ for any $t, s \in [a, b]$, then the Riemann-Stieltjes integral $\int_a^b p(t) dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) dv(t) \right| \leq L \int_a^b |p(t)| \, dt.$$

Now, on applying this property of the Riemann-Stieltjes integral we have

\begin{align*}
&\max_{\lambda \in [m, c]} \left| \int_{\lambda}^{c} (t-\lambda)^n d \left( f^{(n)}(t) \right) \right| 
\end{align*}

\begin{equation}
\leq \frac{L_n}{n+1} (c-m)^{n+1}
\end{equation}

and

\begin{align*}
&\max_{\lambda \in [c, M]} \left| \int_{c}^{\lambda} (\lambda-t)^n d \left( f^{(n)}(t) \right) \right| 
\end{align*}

\begin{equation}
\leq \frac{L_n}{n+1} (M-c)^{n+1}.
\end{equation}
Now, on utilizing the inequality (2.10), then we have from (2.18) and (2.19) that

\begin{equation}
|\langle R_n (f, c, m, M) x, y \rangle| \leq \frac{1}{(n+1)!} L_n (c - m)^{n+1} \sum_{m}^{c} \left( \langle E(c)x, y \rangle \right) \\
+ \frac{1}{(n+1)!} L_n (M - c)^{n+1} \sum_{c}^{M} \left( \langle E(c)x, y \rangle \right) \\
\leq \frac{1}{(n+1)!} L_n \max \left\{ (c - m)^{n+1}, (M - c)^{n+1} \right\} \sum_{m}^{M} \left( \langle E(c)x, y \rangle \right) \\
= \frac{1}{(n+1)!} L_n \left( \frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right)^{n+1} \sum_{m}^{M} \left( \langle E(c)x, y \rangle \right),
\end{equation}

and the proof is complete.

The following particular cases are of interest for applications:

**Corollary 8.** With the assumption of Theorem 7 we have the inequalities

\begin{equation}
\left| \langle f (A) x, y \rangle - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)} (m) \langle (A - m1_H)^k x, y \rangle \right| \leq \frac{1}{(n+1)!} (M - m)^{n+1} L_n \sum_{m}^{M} \left( \langle E(c)x, y \rangle \right)
\end{equation}

and

\begin{equation}
\left| \langle f (A) x, y \rangle - \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)} (M) \langle (M1_H - A)^k x, y \rangle \right| \leq \frac{1}{(n+1)!} (M - m)^{n+1} L_n \sum_{m}^{M} \left( \langle E(c)x, y \rangle \right)
\end{equation}
and

\[
\left| \langle f(A)x, y \rangle - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)} \left( \frac{m + M}{2} \right) \left\langle \left( A - \frac{m + M}{2}1_H \right)^{k} x, y \right\rangle \right| \leq \frac{1}{2^{n+1}(n+1)!} (M - m)^{n+1} L_n \sum_{m}^{M} \langle \langle E_{(\cdot)}x, y \rangle \rangle
\]

(2.23)

respectively, for any \( x, y \in H \).

Let \( u : [a, b] \to \mathbb{R} \) and \( \varphi, \Phi \in \mathbb{R} \) be such that \( \Phi > \varphi \). The following statements are equivalent:

(i) The function \( u - \frac{\varphi + \Phi}{2} \cdot e \), where \( e(t) = t, t \in [a, b] \), is \( \frac{1}{2} (\Phi - \varphi) \) Lipschitzian;

(ii) We have the inequality: \( \varphi \leq \frac{u(t) - u(s)}{t-s} \leq \Phi \) for each \( t, s \in [a, b] \) with \( t \neq s \);

(iii) We have the inequality: \( \varphi (t - s) \leq u(t) - u(s) \leq \Phi (t - s) \) for each \( t, s \in [a, b] \) with \( t > s \).

Following [15], we can say that the function \( u : [a, b] \to \mathbb{R} \) which satisfies one of the equivalent conditions (i) – (iii) is said to be \( \varphi, \Phi \) Lipschitzian on \([a, b] \).

Notice that in [15], the definition was introduced on utilizing the statement (iii) and only the equivalence (i) \( \Leftrightarrow \) (iii) was considered.

The following corollary that provides a perturbed version of Taylor’s expansion holds:

**Corollary 9.** Let \( A \) be a self-adjoint operator in the Hilbert space \( H \) with the spectrum \( \text{Sp}(A) \subseteq [m, M] \) for some real numbers \( m < M \), \( \{E_{\lambda}\}_{\lambda} \) be its spectral family, \( I \) be a closed subinterval on \( \mathbb{R} \) with \( [m, M] \subset \overline{I} \) (the interior of \( I \)) and let \( n \) be an integer with \( n \geq 1 \). If \( g : I \to \mathbb{R} \) is such that the \( n \)-th derivative \( g^{(n)} \) is \((l_n, L_n)\) Lipschitzian with the constants \( L_n > l_n > 0 \) on the interval \([m, M] \), then for any \( c \in [m, M] \) we have
the inequality

\[
\langle g(A) x, y \rangle - g(x, y) - \sum_{k=1}^{n} \frac{1}{k!} g^{(k)}(c) \langle (A - c1_H)^k x, y \rangle - \frac{l_n + L_n}{2}
\]

\[
\times \left[ \frac{1}{(n+1)!} \langle A^{n+1} x, y \rangle - \frac{c^{n+1}}{(n+1)!} \langle x, y \rangle \right.
\]

\[
- \sum_{k=1}^{n} \frac{c^{n-k+1}}{k! (n-k+1)!} \left( (A - c1_H)^k x, y \right) \right]
\]

\[
\leq \frac{1}{2 (n+1)!} (L_n - l_n)
\]

\[
\times \left[ (c - m)^{n+1} \sum_{m} c \left( \langle E(c)x, y \rangle \right) + (M - c)^{n+1} \sum_{m} M \left( \langle E(c)x, y \rangle \right) \right]
\]

\[
\leq \frac{1}{2 (n+1)!} (L_n - l_n) \left( \frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right)^{n+1} \sum_{m} M \left( \langle E(c)x, y \rangle \right)
\]

for any \( x, y \in H \).

**Proof.** Consider the function \( f : I \to \mathbb{R} \) defined by

\[
f(t) := g(t) - \frac{1}{(n+1)!} \frac{L_n + l_n}{2} \cdot t^{n+1}.
\]

Observe that

\[
f^{(k)}(t) := g^{(k)}(t) - \frac{1}{(n-k+1)!} \frac{L_n + l_n}{2} \cdot t^{n-k+1}
\]

for any \( k = 0, \ldots, n \).

Since \( g^{(n)} \) is \((l_n, L_n)\)-Lipschitzian it follows that \( f^{(n)}(t) := g^{(n)}(t) - \frac{L_n + l_n}{2} \cdot t \) is \( \frac{1}{2} (L_n - l_n)\)-Lipschitzian and applying Theorem 7 for the function \( f \), we deduce after required calculations the desired result (2.8). \( \square \)

3. **Applications**

By utilizing Theorems 5 and 7 for the exponential function, we can state the following result:
Proposition 10. Let $A$ be a self-adjoint operator in the Hilbert space $H$ with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family, then for any $c \in [m,M]$ we have the inequality

\[(3.1) \quad \left| \langle e^A x, y \rangle - e^c \sum_{k=0}^n \frac{1}{k!} \langle (A - c 1_H)^k x, y \rangle \right| \]

\[\leq \frac{1}{n!} \left[ (c - m)^n (e^c - e^m) \int_m^c (\langle E_\lambda x, y \rangle) \right] + (M - c)^n (e^M - e^c) \frac{M}{c} \int_m^c (\langle E_\lambda x, y \rangle) \]

\[\leq \frac{1}{n!} \max \{ (M - c)^n (e^M - e^c), (c - m)^n (e^c - e^m) \} \int_m^c (\langle E_\lambda x, y \rangle) \]

\[\leq \frac{1}{n!} \left( \frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right)^n (e^M - e^m) \frac{M}{c} \int_m^c (\langle E_\lambda x, y \rangle) \]

\[\leq \frac{1}{n!} \left( \frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right)^n (e^M - e^m) \|x\| \|y\|\]

and

\[(3.2) \quad \left| \langle e^A x, y \rangle - e^c \sum_{k=0}^n \frac{1}{k!} \langle (A - c 1_H)^k x, y \rangle \right| \]

\[\leq \frac{1}{(n+1)!} e^M \left[ (c - m)^{n+1} \int_m^c (\langle E_\lambda x, y \rangle) + (M - c)^{n+1} \int_m^c (\langle E_\lambda x, y \rangle) \right] \]

\[\leq \frac{1}{(n+1)!} e^M \left( \frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right)^{n+1} \frac{M}{c} \int_m^c (\langle E_\lambda x, y \rangle) \]

\[\leq \frac{1}{(n+1)!} e^M \left( \frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right)^{n+1} \|x\| \|y\|\]

for any $x, y \in H$.

The same Theorems 5 and 7 applied for the logarithmic function produce:
Proposition 11. Let $A$ be a positive definite operator in the Hilbert space $H$ with the spectrum $\text{Sp}(A) \subseteq [m, M] \subset (0, \infty)$ and $\{E_\lambda\}_\lambda$ be its spectral family, then for any $c \in [m, M]$ we have the inequalities

\begin{equation}
\left| \langle \ln Ax, y \rangle - \langle x, y \rangle \ln c - \sum_{k=1}^n \frac{(-1)^{k-1} \langle (A - c1_H)^k x, y \rangle}{kc^k} \right| \leq \frac{1}{n} \left[ \frac{(c - m)^n (c^n - m^n)}{c^nm^n} \sum_{m}^{c} \left( \langle E_\lambda x, y \rangle \right) \right] + \frac{(M - c)^n (M^n - c^n)}{M^mc^m} \sum_{c}^{M} \left( \langle E_\lambda x, y \rangle \right) \leq \frac{1}{n} \max \left\{ \frac{(c - m)^n (c^n - m^n)}{c^nm^n}, \frac{(M - c)^n (M^n - c^n)}{M^mc^m} \right\} \sum_{m}^{M} \left( \langle E_\lambda x, y \rangle \right) \leq \frac{1}{n} \left( \frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right) \right)^n \frac{(M^n - m^n)}{M^m} \sum_{m}^{M} \left( \langle E_\lambda x, y \rangle \right) \leq \frac{1}{n} \left( \frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right) \right)^n \frac{(M^n - m^n)}{M^m} \left\| x \right\| \left\| y \right\| \end{equation}

and

\begin{equation}
\left| \langle \ln Ax, y \rangle - \langle x, y \rangle \ln c - \sum_{k=1}^n \frac{(-1)^{k-1} \langle (A - c1_H)^k x, y \rangle}{kc^k} \right| \leq \frac{1}{(n + 1) m^{n+1}} \left[ \frac{(c - m)^{n+1} \sum_{m}^{c} \left( \langle E_\lambda x, y \rangle \right)}{m} + (M - c)^{n+1} \sum_{c}^{M} \left( \langle E_\lambda x, y \rangle \right) \right] \leq \frac{1}{(n + 1) m^{n+1}} \left( \frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right)^{n+1} \sum_{m}^{M} \left( \langle E_\lambda x, y \rangle \right) \end{equation}

for any $x, y \in H$.

References


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