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ERROR BOUNDS IN APPROXIMATING n-TIME DIFFERENTIABLE FUNCTIONS OF SELF-ADJOINT OPERATORS IN HILBERT SPACES VIA A TAYLOR'S TYPE EXPANSION

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Abstract. On utilizing the spectral representation of self-adjoint operators in Hilbert spaces, some error bounds in approximating n-time differentiable functions of self-adjoint operators in Hilbert Spaces via a Taylor's type expansion are given.

1. **Introduction**

Let U be a self-adjoint operator on the complex Hilbert space $(H, \langle ., . \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda, \tag{1.1}$$

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which in terms of vectors can be written as

$$\langle f(U)x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y\rangle, \tag{1.2}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and $g_{x,y}(m-0) = 0$ and $g_{x,y}(M) = \langle x, y\rangle$ for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

For a recent monograph devoted to various inequalities for continuous functions of self-adjoint operators, see [10] and the references therein.

For other recent results see [1, 11, 12, 13], [14] and the author's papers in preprint [2] - [9].

Utilising the spectral representation from (1.2) we have established the following Ostrowski type vector inequality [6]:

Theorem 1. *Let A be a self-adjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality*

$$\begin{aligned} |f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| &\le \langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} \bigvee_m^s (f) \\ &+ \langle (1_H - E_s)x, x\rangle^{1/2} \langle (1_H - E_s)y, y\rangle^{1/2} \bigvee_s^M (f) \\ &\le \|x\|\,\|y\| \left(\frac{1}{2}\bigvee_m^M (f) + \frac{1}{2}\left| \bigvee_m^s (f) - \bigvee_s^M (f) \right| \right) \le \|x\|\,\|y\| \bigvee_m^M (f) \end{aligned} \tag{1.3}$$

for any $x, y \in H$ and for any $s \in [m, M]$.

The trapezoid version of the above result has been obtained in [5] and is as follows:

Theorem 2. *With the assumptions in Theorem 1 we have the inequalities*

$$\begin{aligned} \left| \frac{f(M) + f(m)}{2} \cdot \langle x, y\rangle - \langle f(A)x, y\rangle \right| &\le \frac{1}{2} \max_{\lambda \in [m,M]} \Big[\langle E_\lambda x, x\rangle^{1/2} \langle E_\lambda y, y\rangle^{1/2} \\ &+ \langle (1_H - E_\lambda)x, x\rangle^{1/2} \langle (1_H - E_\lambda)y, y\rangle^{1/2} \Big] \bigvee_m^M (f) \le \frac{1}{2}\|x\|\,\|y\| \bigvee_m^M (f) \end{aligned} \tag{1.4}$$

for any $x, y \in H$.

In this paper, by utilizing the spectral representation of self-adjoint operators in Hilbert spaces, some error bounds in approximating n-time differentiable functions of self-adjoint operators in Hilbert Spaces via a Taylor's type expansion are given. Applications for some elementary functions of interest including the exponential and logarithmic functions are also provided.

2. **Main Results**

The following result provides a Taylor's type representation for a function of self-adjoint operators in Hilbert spaces with integral remainder.

Theorem 3. *Let* A *be a self-adjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M$, $\{E_\lambda\}_\lambda$ *be its spectral family,* I *be a closed subinterval on* $\mathbb{R}$ *with* $[m, M] \subset \mathring{I}$ *(the interior of* I*) and let* n *be an integer with* $n \geq 1$*. If* $f : I \to \mathbb{C}$ *is such that the* n*-th derivative* $f^{(n)}$ *is of bounded variation on the interval* $[m, M]$*, then for any* $c \in [m, M]$ *we have the equalities*

$$f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(A - c1_H)^k + R_n(f, c, m, M) \tag{2.1}$$

where

$$R_n(f, c, m, M) = \frac{1}{n!} \int_{m-0}^{M} \left(\int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t)\right) \right) dE_\lambda. \tag{2.2}$$

Proof. We utilize the Taylor formula for a function $f : I \to \mathbb{C}$ whose n-th derivative $f^{(n)}$ is locally of bounded variation on the interval I to write the equality

$$f(\lambda) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(\lambda - c)^k + \frac{1}{n!} \int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t)\right) \tag{2.3}$$

for any $\lambda, c \in [m, M]$, where the integral is taken in the Riemann-Stieltjes sense.

If we integrate the equality on $[m,M]$ in the Riemann-Stieltjes sense with the integrator E_λ we get

$$\int_{m-0}^{M} f(\lambda)\,dE_\lambda = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \int_{m-0}^{M} (\lambda - c)^k \,dE_\lambda + \frac{1}{n!} \int_{m-0}^{M} \left(\int_{c}^{\lambda} (\lambda - t)^n \,d\left(f^{(n)}(t)\right)\right) dE_\lambda$$

which, by the spectral representation (1.1), produces the equality (2.1) with the representation of the remainder from (2.2). □

The following particular instances are of interest for applications:

Corollary 4. *With the assumptions of the above Theorem 3, we have the equalities*

$$f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(m) (A - m1_H)^k + L_n(f,c,m,M) \tag{2.4}$$

where

$$L_n(f,c,m,M) = \frac{1}{n!} \int_{m-0}^{M} \left(\int_{m}^{\lambda} (\lambda - t)^n \,d\left(f^{(n)}(t)\right)\right) dE_\lambda$$

and

$$f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}\left(\frac{m+M}{2}\right) \left(A - \frac{m+M}{2} 1_H\right)^k + M_n(f,c,m,M) \tag{2.5}$$

where

$$M_n(f,c,m,M) = \frac{1}{n!} \int_{m-0}^{M} \left(\int_{\frac{m+M}{2}}^{\lambda} (\lambda - t)^n \,d\left(f^{(n)}(t)\right)\right) dE_\lambda$$

and

$$f(A) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(M) (M1_H - A)^k + U_n(f,c,m,M) \tag{2.6}$$

where

$$U_n(f,c,m,M) = \frac{(-1)^{n+1}}{n!} \int_{m-0}^{M} \left(\int_{\lambda}^{M} (t - \lambda)^n \,d\left(f^{(n)}(t)\right)\right) dE_\lambda, \tag{2.7}$$

respectively.

We start with the following result that provides an approximation for an n-time differentiable function of self-adjoint operators in Hilbert spaces:

Theorem 5. *Let A be a self-adjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then for any $c \in [m, M]$ we have the inequality*

(2.8)

$$\begin{aligned}
&\left|\left\langle R_n(f, c, m, M) x, y\right\rangle\right| \\
&= \left|\left\langle f(A) x, y\right\rangle - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \left\langle (A - c1_H)^k x, y\right\rangle\right| \\
&\leq \frac{1}{n!}\left[(c-m)^n \bigvee_m^c \left(f^{(n)}\right) \bigvee_m^c \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)\right. \\
&\left. + (M-c)^n \bigvee_c^M \left(f^{(n)}\right) \bigvee_c^M \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)\right] \\
&\leq \frac{1}{n!} \max\left\{(M-c)^n \bigvee_c^M \left(f^{(n)}\right), (c-m)^n \bigvee_c^M \left(f^{(n)}\right)\right\} \bigvee_m^M \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \\
&\leq \frac{1}{n!}\left(\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right)^n \bigvee_m^M \left(f^{(n)}\right) \bigvee_m^M \left(\left\langle E_{(\cdot)} x, y\right\rangle\right),
\end{aligned}$$

for any $x, y \in H$.

Proof. From the identities (2.1) and (2.2) we have

$$\begin{aligned}
(2.9) \qquad &\left\langle R_n(f, c, m, M) x, y\right\rangle \\
&= \frac{1}{n!} \int_{m-0}^{M} \left(\int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t)\right)\right) d\left\langle E_\lambda x, y\right\rangle \\
&= \frac{1}{n!} \int_{m-0}^{c} \left(\int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t)\right)\right) d\left\langle E_\lambda x, y\right\rangle \\
&+ \frac{1}{n!} \int_{c}^{M} \left(\int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t)\right)\right) d\left\langle E_\lambda x, y\right\rangle
\end{aligned}$$

for any $x, y \in H$.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function, $v : [a, b] \to \mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds $\left|\int_a^b p(t)\, dv(t)\right| \leq \max_{t\in[a,b]} |p(t)| \bigvee_a^b (v)$, where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Taking the modulus in (2.9) and utilizing the above property, we have

$$
\begin{aligned}
&|\langle R_n(f, c, m, M)\, x, y\rangle| \\
&\leq \frac{1}{n!}\left|\int_{m-0}^{c}\left(\int_c^\lambda (\lambda - t)^n\, d\left(f^{(n)}(t)\right)\right) d\langle E_\lambda x, y\rangle\right| \\
&+ \frac{1}{n!}\left|\int_{c}^{M}\left(\int_c^\lambda (\lambda - t)^n\, d\left(f^{(n)}(t)\right)\right) d\langle E_\lambda x, y\rangle\right| \\
&\leq \frac{1}{n!}\max_{\lambda\in[m,c]}\left|\int_c^\lambda (\lambda - t)^n\, d\left(f^{(n)}(t)\right)\right| \bigvee_m^c \left(\langle E_{(\cdot)}x, y\rangle\right) \\
&+ \frac{1}{n!}\max_{\lambda\in[c,M]}\left|\int_c^\lambda (\lambda - t)^n\, d\left(f^{(n)}(t)\right)\right| \bigvee_c^M \left(\langle E_{(\cdot)}x, y\rangle\right)
\end{aligned} \tag{2.10}
$$

for any $x, y \in H$.

By the same property for the Riemann-Stieltjes integral we have

$$
\max_{\lambda\in[m,c]}\left|\int_c^\lambda (\lambda - t)^n\, d\left(f^{(n)}(t)\right)\right| \leq (c - m)^n \bigvee_m^c \left(f^{(n)}\right) \tag{2.11}
$$

and

$$
\max_{\lambda\in[c,M]}\left|\int_c^\lambda (\lambda - t)^n\, d\left(f^{(n)}(t)\right)\right| \leq (M - c)^n \bigvee_c^M \left(f^{(n)}\right). \tag{2.12}
$$

Now, on making use of (2.10)-(2.12) we deduce

$$\begin{aligned}
&\left|\left\langle R_n\left(f,c,m,M\right)x,y\right\rangle\right| \\
&\leq \frac{1}{n!}\left[(c-m)^n \bigvee_m^c \left(f^{(n)}\right) \bigvee_m^c \left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right. \\
&\left.+\left(M-c\right)^n \bigvee_c^M \left(f^{(n)}\right) \bigvee_c^M \left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right] \\
&\leq \frac{1}{n!}\max\left\{(c-m)^n \bigvee_m^c \left(f^{(n)}\right), (M-c)^n \bigvee_c^M \left(f^{(n)}\right)\right\} \\
&\times \left[\bigvee_m^c \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) + \bigvee_c^M \left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right] \\
&\leq \frac{1}{n!}\max\left\{(c-m)^n, (M-c)^n\right\} \bigvee_m^M \left(f^{(n)}\right) \bigvee_m^M \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&= \frac{1}{n!}\left(\frac{1}{2}\left(M-m\right)+\left|c-\frac{m+M}{2}\right|\right)^n \bigvee_m^M \left(f^{(n)}\right) \bigvee_m^M \left(\left\langle E_{(\cdot)}x,y\right\rangle\right)
\end{aligned}$$

for any $x,y\in H$ and the proof is complete. □

The following particular cases are of interest for applications

Corollary 6. *With the assumption of Theorem 5 we have the inequalities*

$$\begin{aligned}
&\left|\left\langle f\left(A\right)x,y\right\rangle - \sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(m\right)\left\langle \left(A-m1_H\right)^k x,y\right\rangle\right| \\
&\leq \frac{1}{n!}\left(M-m\right)^n \bigvee_m^M \left(f^{(n)}\right) \bigvee_m^M \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\leq \frac{1}{n!}\left(M-m\right)^n \bigvee_m^M \left(f^{(n)}\right) \left\|x\right\|\left\|y\right\|,
\end{aligned} \tag{2.13}$$

$$\left|\langle f(A)x,y\rangle - \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(M) \left\langle (M1_H - A)^k x, y \right\rangle\right| \tag{2.14}$$
$$\leq \frac{1}{n!}(M-m)^n \bigvee_m^M \left(f^{(n)}\right) \bigvee_m^M \left(\langle E_{(\cdot)}x,y\rangle\right)$$
$$\leq \frac{1}{n!}(M-m)^n \bigvee_m^M \left(f^{(n)}\right) \|x\| \|y\|$$

and

(2.15)

$$\left|\langle f(A)x,y\rangle - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}\left(\frac{m+M}{2}\right) \left\langle \left(A - \frac{m+M}{2}1_H\right)^k x, y \right\rangle\right|$$
$$\leq \frac{1}{2^n n!}(M-m)^n \max\left\{ \bigvee_{\frac{m+M}{2}}^M \left(f^{(n)}\right), \bigvee_m^{\frac{m+M}{2}} \left(f^{(n)}\right) \right\} \bigvee_m^M \left(\langle E_{(\cdot)}x,y\rangle\right)$$
$$\leq \frac{1}{2^n n!}(M-m)^n \max\left\{ \bigvee_{\frac{m+M}{2}}^M \left(f^{(n)}\right), \bigvee_m^{\frac{m+M}{2}} \left(f^{(n)}\right) \right\} \|x\| \|y\|$$

respectively, for any $x, y \in H$.

Proof. The first part in the inequalities follow from (2.8) by choosing $c = m, c = M$ and $c = \frac{m+M}{2}$ respectively.

If P is a nonnegative operator on H, i.e., $\langle Px, x\rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle \tag{2.16}$$

for any $x, y \in H$.

Now, if $d : m = t_0 < t_1 < ... < t_{n-1} < t_n = M$ is an arbitrary partition of the interval $[m, M]$, then we have by Schwarz's inequality for nonnegative operators (2.16) that

$$\begin{aligned}
&\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\
&=\sup_{d}\left\{\sum_{i=0}^{n-1}\left|\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,y\right\rangle\right|\right\}\\
&\leq\sup_{d}\left\{\sum_{i=0}^{n-1}\left[\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle^{1/2}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle^{1/2}\right]\right\}:=B.
\end{aligned}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}
B&\leq\sup_{d}\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\}\\
&\leq\sup_{d}\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\}\\
&=\left[\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x,x\right\rangle\right)\right]^{1/2}\left[\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}y,y\right\rangle\right)\right]^{1/2}=\|x\|\,\|y\|
\end{aligned}$$

for any $x,y\in H$. These prove the last part of the above inequalities (2.13)-(2.15). □

The following result also holds:

Theorem 7. *Let A be a self-adjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m,M]\subset\mathring{I}$ (the interior of I) and let n be an integer with $n\geq 1$. If $f:I\to\mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is Lipschitzian with the constant $L_n>0$ on* *the interval $[m, M]$, then for any $c \in [m, M]$ we have the inequality*

(2.17)

$$\begin{aligned}
&\left|\left\langle R_n(f, c, m, M) x, y\right\rangle\right| \\
&\leq \frac{1}{(n+1)!} L_n\left[(c-m)^{n+1} \bigvee_m^c\left(\left\langle E_{(\cdot)} x, y\right\rangle\right)+(M-c)^{n+1} \bigvee_c^M\left(\left\langle E_{(\cdot)} x, y\right\rangle\right)\right] \\
&\leq \frac{1}{(n+1)!} L_n\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n+1} \bigvee_m^M\left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \\
&\leq \frac{1}{(n+1)!} L_n\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n+1}\|x\|\|y\|
\end{aligned}$$

for any $x, y \in H$.

Proof. First of all, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e., $|f(s) - f(t)| \leq L|s - t|$ for any $t, s \in [a, b]$, then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds $\left|\int_a^b p(t)\, dv(t)\right| \leq L \int_a^b |p(t)|\, dt$.

Now, on applying this property of the Riemann-Stieltjes integral we have

$$\max_{\lambda \in[m, c]}\left|\int_\lambda^c(t-\lambda)^n d\left(f^{(n)}(t)\right)\right| \leq \frac{L_n}{n+1}(c-m)^{n+1} \tag{2.18}$$

and

$$\max_{\lambda \in[c, M]}\left|\int_c^\lambda(\lambda-t)^n d\left(f^{(n)}(t)\right)\right| \leq \frac{L_n}{n+1}(M-c)^{n+1}. \tag{2.19}$$

Now, on utilizing the inequality (2.10), then we have from (2.18) and (2.19) that

(2.20)

$$\begin{aligned}
&\left|\left\langle R_n\left(f,c,m,M\right)x,y\right\rangle\right| \\
&\leq \frac{1}{(n+1)!}L_n\left(c-m\right)^{n+1}\bigvee_m^c\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&+\frac{1}{(n+1)!}L_n\left(M-c\right)^{n+1}\bigvee_c^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\leq \frac{1}{(n+1)!}L_n\max\left\{\left(c-m\right)^{n+1},\left(M-c\right)^{n+1}\right\}\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&=\frac{1}{(n+1)!}L_n\left(\frac{1}{2}\left(M-m\right)+\left|c-\frac{m+M}{2}\right|\right)^{n+1}\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right),
\end{aligned}$$

and the proof is complete. □

The following particular cases are of interest for applications:

Corollary 8. *With the assumption of Theorem 7 we have the inequalities*

$$\begin{aligned}
&\left|\left\langle f\left(A\right)x,y\right\rangle-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(m\right)\left\langle\left(A-m1_H\right)^k x,y\right\rangle\right| \\
&\leq\frac{1}{(n+1)!}\left(M-m\right)^{n+1}L_n\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)
\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}
&\left|\left\langle f\left(A\right)x,y\right\rangle-\sum_{k=0}^{n}\frac{(-1)^k}{k!}f^{(k)}\left(M\right)\left\langle\left(M1_H-A\right)^k x,y\right\rangle\right| \\
&\leq\frac{1}{(n+1)!}\left(M-m\right)^{n+1}L_n\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)
\end{aligned} \tag{2.22}$$

and

$$\left|\langle f(A)x,y\rangle-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(\frac{m+M}{2}\right)\left\langle\left(A-\frac{m+M}{2}1_H\right)^k x,y\right\rangle\right| \leq \frac{1}{2^{n+1}(n+1)!}(M-m)^{n+1}L_n\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right) \tag{2.23}$$

respectively, for any $x,y\in H$.

Let $u:[a,b]\to\mathbb{R}$ and $\varphi,\Phi\in\mathbb{R}$ be such that $\Phi>\varphi$. The following statements are equivalent:

(i) The function $u-\frac{\varphi+\Phi}{2}\cdot e$, where $e(t)=t$, $t\in[a,b]$, is $\frac{1}{2}(\Phi-\varphi)-$Lipschitzian;

(ii) We have the inequality: $\varphi\leq\frac{u(t)-u(s)}{t-s}\leq\Phi$ for each $t,s\in[a,b]$ with $t\neq s$;

(iii) We have the inequality: $\varphi(t-s)\leq u(t)-u(s)\leq\Phi(t-s)$ for each $t,s\in[a,b]$ with $t>s$.

Following [15], we can say that the function $u:[a,b]\to\mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) is said to be $(\varphi,\Phi)-$Lipschitzian on $[a,b]$.

Notice that in [15], the definition was introduced on utilizing the statement (iii) and only the equivalence (i) $\Leftrightarrow$ (iii) was considered.

The following corollary that provides a perturbed version of Taylor's expansion holds:

Corollary 9. *Let A be a self-adjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m,M]\subset\mathring{I}$ (the interior of I) and let n be an integer with $n\geq 1$. If $g:I\to\mathbb{R}$ is such that the n-th derivative $g^{(n)}$ is $(l_n,L_n)-$Lipschitzian with the constants $L_n>l_n>0$ on the interval $[m,M]$, then for any $c\in[m,M]$ we have* *the inequality*

(2.24)

$$\Big|\langle g(A)x,y\rangle - g(c)\langle x,y\rangle - \sum_{k=1}^{n}\frac{1}{k!}g^{(k)}(c)\left\langle (A-c1_H)^k x,y\right\rangle - \frac{l_n+L_n}{2}$$

$$\times\left[\frac{1}{(n+1)!}\left\langle A^{n+1}x,y\right\rangle - \frac{c^{n+1}}{(n+1)!}\langle x,y\rangle\right.$$

$$\left.-\sum_{k=1}^{n}\frac{c^{n-k+1}}{k!(n-k+1)!}\left\langle (A-c1_H)^k x,y\right\rangle\right]\Big|$$

$$\le \frac{1}{2(n+1)!}(L_n-l_n)$$

$$\times\left[(c-m)^{n+1}\bigvee_m^c\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) + (M-c)^{n+1}\bigvee_c^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]$$

$$\le \frac{1}{2(n+1)!}(L_n-l_n)\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n+1}\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$

for any $x,y\in H$.

Proof. Consider the function $f: I \to \mathbb{R}$ defined by

$$f(t) := g(t) - \frac{1}{(n+1)!}\frac{L_n+l_n}{2}\cdot t^{n+1}.$$

Observe that

$$f^{(k)}(t) := g^{(k)}(t) - \frac{1}{(n-k+1)!}\frac{L_n+l_n}{2}\cdot t^{n-k+1}$$

for any $k = 0, ..., n$.

Since $g^{(n)}$ is (l_n, L_n) −Lipschitzian it follows that $f^{(n)}(t) := g^{(n)}(t) - \frac{L_n+l_n}{2}\cdot t$ is $\frac{1}{2}(L_n - l_n)$-Lipschitzian and applying Theorem 7 for the function f, we deduce after required calculations the desired result (2.8). □

3. **Applications**

By utilizing Theorems 5 and 7 for the exponential function, we can state the following result:

Proposition 10. *Let A be a self-adjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family, then for any $c \in [m, M]$ we have the inequality*

(3.1)

$$\begin{aligned} &\left| \left\langle e^A x, y \right\rangle - e^c \sum_{k=0}^{n} \frac{1}{k!} \left\langle (A - c1_H)^k x, y \right\rangle \right| \\ &\le \frac{1}{n!} \left[(c-m)^n (e^c - e^m) \bigvee_m^c \left(\langle E_{(\cdot)} x, y \rangle \right) \right. \\ &\left. + (M-c)^n \left(e^M - e^c\right) \bigvee_c^M \left(\langle E_{(\cdot)} x, y \rangle \right) \right] \\ &\le \frac{1}{n!} \max\left\{ (M-c)^n \left(e^M - e^c\right), (c-m)^n (e^c - e^m) \right\} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \\ &\le \frac{1}{n!} \left(\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right)^n \left(e^M - e^m\right) \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \\ &\le \frac{1}{n!} \left(\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right)^n \left(e^M - e^m\right) \|x\| \|y\| \end{aligned}$$

and

(3.2)

$$\begin{aligned} &\left| \left\langle e^A x, y \right\rangle - e^c \sum_{k=0}^{n} \frac{1}{k!} \left\langle (A - c1_H)^k x, y \right\rangle \right| \\ &\le \frac{1}{(n+1)!} e^M \left[(c-m)^{n+1} \bigvee_m^c \left(\langle E_{(\cdot)} x, y \rangle \right) + (M-c)^{n+1} \bigvee_c^M \left(\langle E_{(\cdot)} x, y \rangle \right) \right] \\ &\le \frac{1}{(n+1)!} e^M \left(\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \\ &\le \frac{1}{(n+1)!} e^M \left(\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} \|x\| \|y\| \end{aligned}$$

for any $x, y \in H$.

The same Theorems 5 and 7 applied for the logarithmic function produce:

Proposition 11. *Let A be a positive definite operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M] \subset (0, \infty)$ and $\{E_\lambda\}_\lambda$ be its spectral family, then for any $c \in [m, M]$ we have the inequalities*

(3.3)

$$\begin{aligned}
&\left| \langle \ln Ax, y\rangle - \langle x, y\rangle \ln c - \sum_{k=1}^{n} \frac{(-1)^{k-1} \left\langle (A - c1_H)^k x, y \right\rangle}{kc^k} \right| \\
&\leq \frac{1}{n} \left[\frac{(c-m)^n (c^n - m^n)}{c^n m^n} \bigvee_m^c \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \right. \\
&\left. + \frac{(M-c)^n (M^n - c^n)}{M^m c^m} \bigvee_c^M \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \right] \\
&\leq \frac{1}{n} \max\left\{ \frac{(c-m)^n (c^n - m^n)}{c^n m^n}, \frac{(M-c)^n (M^n - c^n)}{M^m c^m} \right\} \bigvee_m^M \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \\
&\leq \frac{1}{n} \left(\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right)^n \frac{(M^n - m^n)}{M^m m^m} \bigvee_m^M \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \\
&\leq \frac{1}{n} \left(\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right)^n \frac{(M^n - m^n)}{M^m m^m} \|x\| \|y\|
\end{aligned}$$

and

(3.4)

$$\begin{aligned}
&\left| \langle \ln Ax, y\rangle - \langle x, y\rangle \ln c - \sum_{k=1}^{n} \frac{(-1)^{k-1} \left\langle (A - c1_H)^k x, y \right\rangle}{kc^k} \right| \\
&\leq \frac{1}{(n+1)\, m^{n+1}} \left[(c-m)^{n+1} \bigvee_m^c \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) + (M-c)^{n+1} \bigvee_c^M \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \right] \\
&\leq \frac{1}{(n+1)\, m^{n+1}} \left(\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} \bigvee_m^M \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)
\end{aligned}$$

for any $x, y \in H$.

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