

COMPLETE CONVERGENCE OF MOVING-AVERAGE PROCESSES UNDER NEGATIVE DEPENDENCE SUB-GAUSSIAN ASSUMPTIONS

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ABSTRACT. The complete convergence is investigated for moving-average processes of doubly infinite sequence of negative dependence sub-Gaussian random variables with zero means, finite variances and absolutely summable coefficients. As a corollary, the rate of complete convergence is obtained under some suitable conditions on the coefficients.

1. Introduction

Let $\{Y_j, -\infty < j < \infty\}$ be a doubly infinite sequence of independent and identically distributed random variables with zero means and finite variances, $\{c_j, -\infty < j < \infty\}$ be an absolutely summable sequence of real numbers and $X_k = \sum_{j=-\infty}^{\infty} c_{j+k} Y_j$, $k \geq 1$. Under some suitable conditions on the coefficients, many scholars have been studied the limiting behavior of moving-average process $\{X_k, k \geq 1\}$. Among them, Li and Zhang [16] proved the complete moment convergence of the sequence $\{\frac{1}{n^{1/p}} \sum_{k=1}^n X_k, n \geq 1\}$ for all $1 \leq p < 2$, when $\{Y_j, -\infty < j < \infty\}$

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is a doubly infinite sequence of identically distributed negative associated random variables with zero means and finite variances. Chen et al. ([10], [11]) studied limiting behavior of moving average processes under ρ -mixing and negatively association assumptions respectively. The case of independent random variables dominated by random variable Y such that $E|Y|^{2p} < \infty$ for all $1 \leq p < 2$, Sadeghi and Bozorgnia [17] derived complete convergence of the sequence $\{\frac{1}{n^{1/p}} \sum_{k=1}^n X_k, n \geq 1\}$. Li [15] studied complete moment convergence of the sequence $\{X_k, k \geq 1\}$, when $\{Y_j, -\infty < j < \infty\}$ is a doubly infinite sequence of independent and identically distributed (see also, Zhou [20], Sung [18], Baek et al. [5], Yun–Xia [15], Kim and Ko [13] and Budsaba et al. [7]). Sub-Gaussianity properties of random variables are important features, since they allow us to derive results concerning, large deviations inequalities, strong limit theorems of weighted sums and convergence of series of dependence random variables (see Antonini, et al.([3], [4]) and Amini et al. ([1], [2])). In this paper, we study the complete convergence of sequence $\{\frac{1}{n^p} \sum_{k=1}^n X_k, n \geq 1\}$ under some suitable conditions on the coefficients $\{c_j\}$, when $\{Y_j, -\infty < j < \infty\}$ is a doubly infinite sequence of negative dependence sub-Gaussian random variables with $\tau(Y_j) \leq \alpha$ (α is a constant) for all j where $\tau(Y_j) = \inf\{\alpha \geq 0 : E(e^{tY_j}) \leq \exp[\frac{\alpha^2 t^2}{2}], t \in R\}$. We also, extend Theorem 2 in Sadeghi and Bozorgnia [17] to the case of negative dependence sub-Gaussian random variables and determine the rate of complete convergence of moving-average processes under some suitable conditions on the coefficients.

Definition 1.1. *A symmetric random variable X is said to be sub-Gaussian random variable, if there exists a nonnegative real number α such that for each real number t ,*

$$Ee^{tX} \leq \exp\left[\frac{\alpha^2 t^2}{2}\right].$$

The number $\tau(X)$, will be called the Gaussian standard of the random variable X . It is evident that X will be a sub-Gaussian random variable if and only if $\tau(X) < \infty$. Moreover, a sub-Gaussian random variable X always satisfies the relations $E(X^2) \leq \tau^2(X)$, $E(X) = 0$ and $P[|X| > \varepsilon] \leq 2 \exp[-\frac{\varepsilon^2}{2\alpha^2}]$ for all $\varepsilon > 0$. If $E(X^2) = \tau^2(X)$, then X is called strictly sub-Gaussian (see Buldying et al.[8] and Taylor and Hu [19] for additional properties.)

Definition 1.2. *The random variables X_1, \dots, X_n are said to be negative dependent (ND) if the following inequalities are hold:*

$$P\left[\bigcap_{j=1}^n (X_j \leq x_j)\right] \leq \prod_{j=1}^n P[X_j \leq x_j] \text{ and } P\left[\bigcap_{j=1}^n (X_j > x_j)\right] \leq \prod_{j=1}^n P[X_j > x_j],$$

for all $x_1, \dots, x_n \in R$.

i) *An infinite sequence $\{X_n, n \geq 1\}$ is said to be ND if every finite subset is ND.*

ii) *The sequence $\{X_n, n \geq 1\}$ is said pair-wise negative dependent (PND) if for all $i \neq j$. X_i and X_j are negative quadrant dependent (NQD), that is for all real numbers x and y ,*

$$P(X_i \leq x, Y_j \leq y) \leq P(X_i \leq x)P(Y_j \leq y).$$

Definition 1.3. *(Hsu and Robbins [12]) The sequence $\{X_n, n \geq 1\}$ of random variables converges to the constant a completely (denoted $\lim_{n \rightarrow \infty} X_n = a$ completely), if for every $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} P[|X_n - a| > \varepsilon] < \infty.$$

The following lemmas and corollary are important in the proof of our main results.

Lemma 1.4. *(Bozorgnia et al. [6]). Let X_1, \dots, X_n be ND nonnegative random variables. Then, $E[\prod_{j=1}^n X_j] \leq \prod_{j=1}^n E[X_j]$.*

Lemma 1.5. *(Burton and Dehling [9]) Let $\sum_{i=-\infty}^{\infty} c_i$ be an absolutely convergent series of real numbers with $c = \sum_{i=-\infty}^{\infty} c_i$, then for all $k \geq 1$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} \left(\left| \sum_{j=i+1}^{i+n} c_j \right| \right)^k = |c|^k.$$

Corollary 1.6. *Under the assumptions of Lemma 1.5, if $c_j = 0, j < 0$, then we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{\infty} \left(\sum_{j=i+1}^{i+n} c_j \right)^2 = c^2 \text{ and } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\sum_{j=0}^{n-i} c_j \right)^2 = c^2.$$

2. Main results

In this section, we drive the complete convergence of the sequence $\frac{1}{n^{1/p}} \sum_{k=1}^n X_k$ for each $p > 1/2$, under negative dependence sub-Gaussian assumption. Moreover, we obtain the rate of complete convergence under suitable conditions on the sequence $\{c_j\}$. Our main results is based on the following lemma that allows us upper bounds for the Gaussian standard of moving-average process $\{X_n, n \geq 1\}$ and $\sum_{k=1}^n X_k$, where $X_n = \sum_{j=-\infty}^{\infty} c_{n+j} Y_j$, $n \geq 1$.

Lemma 2.1. *Let $\{Y_j, -\infty < j < \infty\}$ be a doubly infinite sequence of negative dependence sub-Gaussian random variables with $\tau(Y_j) \leq \alpha$ for all j , and $\{c_j, -\infty < j < \infty\}$ be an absolutely summable sequence of real numbers. Then*

i) The random variable $X_n = \sum_{j=-\infty}^{\infty} c_{n+j} Y_j$, $n \geq 1$ is sub-Gaussian with

$$\tau(X_n) \leq 2\alpha \sqrt{\sum_{j=-\infty}^{\infty} c_{n+j}^2}.$$

ii) The random variable $\sum_{k=1}^n X_k, n \geq 1$ is sub-Gaussian with

$$\tau(\sum_{k=1}^n X_k) \leq 2\alpha \sqrt{\sum_{i=-\infty}^{\infty} a_{ni}^2}, \text{ where } a_{ni} = \sum_{k=i+1}^{n+i} c_k$$

iii) If $c_j = 0$ for all $j < 0$, and $X_n = \sum_{j=0}^{\infty} c_j Y_{n-j}$. Then X_n is a sub-Gaussian random variable with $\tau(X_n) \leq 2\alpha \sqrt{\sum_{i=0}^{\infty} c_j^2}$, $n \geq 1$, and $\sum_{k=1}^n X_k, n \geq 1$ is a sub-Gaussian random variable with

$$\tau(\sum_{k=1}^n X_k) \leq 2\alpha \sqrt{\sum_{i=0}^{\infty} a_{ni}^2 + \sum_{i=1}^n b_{ni}^2},$$

where $a_{ni} = \sum_{j=i+1}^{n+i} c_j$ and $b_{ni} = \sum_{j=0}^{n-i} c_j$.

Proof. *i)* For every $m > n \geq 1$ we define

$$X_{nm} = \sum_{j=0}^m c_{n+j} Y_j + \sum_{j=1}^m c_{n-j} Y_{-j} = W_{nm} + \bar{W}_{nm}.$$

Applying Cauchy–Schwarz inequality and Lemma 1.4, for each real number t , we have

$$\begin{aligned}
 Ee^{tX_{nm}} &= E[e^{tW_{nm}+t\bar{W}_{nm}}] \leq \sqrt{Ee^{2tW_{nm}}.Ee^{2t\bar{W}_{nm}}} \\
 &\leq \left(E \exp\left(4t \sum_{A^+} c_{n+j}Y_j\right) \times E \exp\left(4t \sum_{A^-} c_{n+j}Y_j\right) \right)^{\frac{1}{4}} \\
 &\times \left(E \exp\left(4t \sum_{B^+} c_{n-j}Y_{-j}\right) \times E \exp\left(4t \sum_{B^-} c_{n-j}Y_{-j}\right) \right)^{\frac{1}{4}} \\
 &\leq \left(\prod_{j \in A^+} E \exp(4tc_{n+j}Y_j) \prod_{j \in A^-} E \exp(4tc_{n+j}Y_j) \right)^{\frac{1}{4}} \\
 &\times \left(\prod_{j \in B^+} E \exp(4tc_{n-j}Y_j) \prod_{j \in B^-} E \exp(4tc_{n-j}Y_j) \right)^{\frac{1}{4}} \\
 &\leq \left(\exp\left(8t^2\alpha^2 \sum_{j=0}^m c_{n+j}^2 + 8t^2\alpha^2 \sum_{j=1}^m c_{n-j}^2\right) \right)^{\frac{1}{4}} \\
 &\leq \exp\left(2t^2\alpha^2 \sum_{j=0}^{\infty} c_{n+j}^2 + 2t^2\alpha^2 \sum_{j=1}^{\infty} c_{n-j}^2\right) \\
 &= \exp\left(2t^2\alpha^2 \sum_{j=-\infty}^{\infty} c_{n+j}^2\right)
 \end{aligned}$$

where $A^+ = \{j : c_{n+j} \geq 0\}$, $B^+ = \{j : c_{n-j} \geq 0\}$, $n \geq 1$ and A^-, B^- are complement of A^+ and B^+ respectively. Now using Fatou’s Lemma we obtain

$$Ee^{tX_n} \leq \exp\left(2t^2\alpha^2 \sum_{j=-\infty}^{\infty} c_{n+j}^2\right).$$

Therefore, X_n is a sub-Gaussian random variables with

$$\tau(X_n) \leq 2\alpha\sqrt{\sum_{j=-\infty}^{\infty} c_{n+j}^2}.$$

Similarly, we can prove part (ii).

iii) Applying Cauchy–Schwarz inequality, Property 1.4 (iii) in Bozorgnia et al.[6] and Lemma 1.4, it can be verified that the random variable X_n is a sub-Gaussian with $\tau(X_n) \leq 2\alpha\sqrt{\sum_{i=0}^{\infty} c_j^2}$. For sub-Gaussianity

$\sum_{k=1}^n X_k$, we can write,

$$\sum_{k=1}^n X_k = \sum_{k=1}^n \sum_{j=0}^{\infty} c_j Y_{k-j} = \sum_{i=1}^n b_{ni} Y_i + \sum_{i=0}^{\infty} a_{ni} Y_{-i},$$

where $a_{ni} = \sum_{j=i+1}^{n+i} c_j$ and $b_{ni} = \sum_{j=0}^{n-i} c_j$ and for all $m > n \geq 1$, and so,

$$X_{nm} = \sum_{j=1}^n b_{nj} Y_j + \sum_{j=0}^m a_{nj} Y_{-j}.$$

Now similar to the proof of part (i) for every real number t , we get

$$\begin{aligned} Ee^{tX_{nm}} &\leq \exp(2t^2\alpha^2\{\sum_{i=0}^m a_{ni}^2 + \sum_{j=1}^n b_{nj}^2\}) \\ &\leq \exp(2t^2\alpha^2\{\sum_{i=0}^{\infty} a_{ni}^2 + \sum_{j=1}^n b_{nj}^2\}), \end{aligned}$$

Therefore, Fatou's Lemma implies that

$$Ee^{tX_n} \leq \exp(2t^2\alpha^2\{\sum_{i=0}^{\infty} a_{ni}^2 + \sum_{j=1}^n b_{nj}^2\}).$$

This completes the proof. \square

The following theorem is an extension of Theorem 2 in Sadeghi and Bozorgnia [17].

Theorem 2.2. . Let $\{Y_j, -\infty < j < \infty\}$ be a doubly infinite sequence of negative dependent sub-Gaussian random variables with $\tau(Y_j) \leq \alpha$ for all integer index j and $\{c_j\}$ be an absolutely summable sequence of real numbers such that $\sum_{j=-\infty}^{\infty} c_j = c$. Then for all $p > 1/2$,

$$(2.1) \quad \frac{1}{n^p} \sum_{k=1}^n X_k \rightarrow 0, \text{ completely, as } n \rightarrow \infty,$$

where $X_k = \sum_{j=-\infty}^{\infty} c_{j+k} Y_j$, for all $k \geq 1$. Moreover, if $c_j = 0$, $j < 0$ and $\sum_{j=0}^{\infty} c_j = c$, then (2.1) holds.

Proof. We can write

$$\sum_{k=1}^n X_k = \sum_{k=1}^n \sum_{i=-\infty}^{\infty} c_{i+k} Y_i = \sum_{i=-\infty}^{\infty} a_{ni} Y_i,$$

where $a_{ni} = \sum_{j=i+1}^{n+i} c_j$. Applying Lemmas 1.4 and 2.1 and the fact that $\sum_{k=1}^n X_k$ is a sub-Gaussian random variable with $\tau(\sum_{k=1}^n X_k) \leq 2\alpha\sqrt{\sum_{i=-\infty}^{\infty} a_{ni}^2}$, for all $\varepsilon > 0$, we obtain

$$P\left[\left|\sum_{k=1}^n X_k\right| > \varepsilon n^p\right] \leq 2 \exp\left(-\frac{\varepsilon^2 n^{2p}}{4\alpha^2 \sum_{i=-\infty}^{\infty} a_{ni}^2}\right).$$

Now Corollary 1 implies that $\sum_{i=-\infty}^{\infty} a_{ni}^2 = O(n)$, therefore,

$$\sum_{n=1}^{\infty} P\left[\left|\sum_{k=1}^n X_k\right| > \varepsilon n^p\right] \leq \sum_{n=1}^{\infty} 2 \exp\left(-\frac{\varepsilon^2 n^{2p-1}}{4\alpha^2 M}\right) < \infty,$$

for all $p > 1/2$ and $0 < M < \infty$. Therefore, the proof is complete. \square

In the following proposition, as an application of Theorem 2.2 we construct the moving-average process $\{X_n, n \geq 1\}$ which is pair-wise negative dependent and so (2.1) also holds for all $p > \frac{1}{2}$.

Proposition 2.3. . *Let $\{Y_j, -\infty < j < \infty\}$ be a sequence of i.i.d. standard Gaussian with $c_0 = 1, c_j = -\beta^j, j \geq 1, 0 < \beta < \frac{1}{\sqrt{2}}$, and $X_n = \sum_{j=0}^{\infty} c_j Y_{n-j}$, then for every $p > 1/2$*

$$\frac{1}{n^p} \sum_{k=1}^n X_k \rightarrow 0, \text{ completely as } n \rightarrow \infty.$$

Proof. The random process $\{X_n, n \geq 1\}$ is a Gaussian stationary sequence with negative covariances given by

$$\begin{aligned} E(X_n \cdot X_{n+k}) &= \sum_{j=0}^{\infty} c_j c_{j+k} = -\beta^k \left(1 - \sum_{j=1}^{\infty} \beta^{2j}\right) \\ &= -\frac{\beta^k (1 - 2\beta^2)}{1 - \beta^2} = \gamma(k) < 0. \end{aligned}$$

The conditional distribution of X_n given $X_{n+k} = y$ is normal with mean $-\beta^k y$ and variance $\frac{(1-\beta^{2k})(1-2\beta^2)}{1-\beta^2}$. Hence this conditional distribution is increasing in y , this implies that X_n and X_{n+k} are NQD (see Lehmann [14]). Similarly we can show that the conditional distribution of X_{n+k} given to $X_n = x$ for all $k \geq 1$ is increasing in x . Moreover the random variable $\sum_{k=1}^n X_k = \sum_{i=0}^{\infty} a_{ni} Y_{-i} + \sum_{i=1}^n b_{ni} Y_i$ is strictly sub-Gaussian

with

$$\tau^2\left(\sum_{k=1}^n X_k\right) = \sum_{i=0}^{\infty} a_{ni}^2 + \sum_{i=1}^n b_{ni}^2,$$

where $a_{ni} = \sum_{j=i+1}^{n+i} c_j$ and $b_{ni} = \sum_{j=0}^{n-i} c_j$. By our assumption we obtain $\sum_{j=0}^{\infty} c_j^2 = \frac{1-2\beta^2}{1-\beta^2} < \infty$, $\sum_{j=0}^{\infty} c_j = \frac{1-2\beta}{1-\beta} = c < \infty$ and applying Corollary 1.6 we can show that,

$$\sum_{i=0}^{\infty} a_{ni}^2 + \sum_{i=1}^n b_{ni}^2 = O(n).$$

Thus for all $\varepsilon > 0$ and $p > 1/2$ we get

$$\sum_{n=1}^{\infty} P\left[\left|\sum_{k=1}^n X_k\right| > \varepsilon n^p\right] \leq \sum_{n=1}^{\infty} 2 \exp\left(-\frac{\varepsilon^2 n^{2p-1}}{2M_1}\right) < \infty,$$

this complete the proof. \square

The next theorem gives us the rate of complete convergence of moving-average process under some suitable conditions on the coefficients.

Theorem 2.4. *Let $\{Y_j, -\infty < j < \infty\}$ be a doubly infinite sequence of negative dependent sub-Gaussian random variables with $\tau(Y_j) \leq \alpha$ for all j and $\{c_j, -\infty < j < \infty\}$ be an absolutely summable sequence of real numbers with $\sum_{j=-\infty}^{\infty} c_j = c$. If $p > 1/2$, then for all $\beta > 0$ and $\varepsilon > 0$,*

$$(2.2) \quad \sum_{n=1}^{\infty} n^{\beta} P\left[\left|\sum_{k=1}^n X_k\right| > \varepsilon n^p\right] < \infty,$$

where $X_k = \sum_{j=-\infty}^{\infty} c_{j+k} Y_j$ for all $k \geq 1$. Moreover, if $c_j = 0, j < 0$ with $\sum_{j=0}^{\infty} c_j = c$ and p, β satisfy in the above conditions then, the statement (2.2) is valid.

Proof. i) Applying Lemmas 1.4 ,2.1 and Theorem 2.2 we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\beta} P\left[\left|\sum_{k=1}^n X_k\right| > \varepsilon n^p\right] &\leq 2 \sum_{n=1}^{\infty} n^{\beta} \exp\left(-\frac{\varepsilon^2 n^{2p}}{4\alpha^2 \sum_{i=-\infty}^{\infty} a_{ni}^2}\right) \\ &\leq 2 \sum_{n=1}^{\infty} n^{\beta} \exp\left(-\frac{\varepsilon^2 n^{2p-1}}{4\alpha^2 M}\right) < \infty. \end{aligned}$$

The last inequality holds because, for all $p > 1/2$ and $\beta > 0$ we have

$$\sum_{n=1}^{\infty} n^{\beta} \exp\left(-\frac{\varepsilon^2 n^{2p-1}}{4\alpha^2 M}\right) < \infty \Leftrightarrow \int_1^{\infty} x^{\beta} \exp\left(-\frac{\varepsilon^2 x^{2p-1}}{4\alpha^2 M}\right) dx < \infty,$$

this completes the proof. \square

3. Conclusion

Let $\{Y_j, -\infty < j < \infty\}$ be a doubly infinite sequence of independent sub-Gaussian random variables with $\tau(Y_j) \leq \alpha$ for all j , then all of the above theorems and lemmas are true in this case. Moreover if $\{Y_j, -\infty < j < \infty\}$ is a doubly infinite sequence of negative dependent random variables dominated by Y such that Y is a sub-Gaussian random variable with $\tau(Y) \leq \alpha$ under suitable conditions on the sequence $\{c_j, -\infty < j < \infty\}$, then our results are true. In particular, the results of Sadeghi and Bozorgnia [17] are valid in this case.

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