Bulletin of the Iranian Mathematical Society Vol. 36 *No.* 2 (*2010*), *pp 267-277.*

BOUNDS FOR THE REGULARITY OF EDGE IDEAL OF VERTEX DECOMPOSABLE AND SHELLABLE GRAPHS

S. MORADI* AND D. KIANI

Communicated by Siamak Yassemi

Abstract. We give upper bounds for the regularity of edge ideal of some classes of graphs in terms of invariants of graph. We introduce two numbers $a'(G)$ and $n(G)$ depending on graph G and show that for a vertex decomposable graph G, $\mathrm{reg}(R/I(G)) \leq \min\{a'(G), n(G)\}$ and for a shellable graph G, $\mathrm{reg}(R/I(G)) \leq n(G)$. Moreover, it is shown that for a graph G, where G^c is a d-tree, we have $\mathrm{pd}(R/I(G)) = \max_{v\in V(G)}\{\deg_G(v)\}$.

1. Introduction

Let G be a simple graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. The edge ideal of G in the polynomial ring $R = k[x_1, \ldots, x_n]$ is defined as $I(G) = (x_ix_j : \{x_i, x_j\} \in E(G))$. The edge ideal of a graph was first considered by Villarreal [14]. Finding connections between algebraic properties of an edge ideal and invariants of graph is of great interest. One question in this area is to explain the regularity of an edge ideal by some information from graph. For some classes of graphs, for example, chordal graphs and shellable bipartite graphs, this question

MSC(2010): Primary: 13F55, 13D02; Secondary: 05C75.
Keywords: Edge ideals, vertex decomposable, shellable complex, Castelnuovo-Mumford regularity, projective dimension.
Received: 25 August 2009, Accepted: 6 October 2009.
∗Corresponding author

was answered; see [6] and [12]. For these graphs it was shown that the regularity of $R/I(G)$ was equal to the maximum number of pairwise 3-disjoint edges in G, denoted by $a(G)$. Also, in [9, Lemma 2.2], it was shown that for any graph G, $\mathrm{reg}(R/I(G)) \geq a(G)$. Here, we give upper bounds for $\mathrm{reg}(R/I(G))$ for shellable and vertex decomposable graphs in terms of invariants of graph. First, we recall some definitions.

Let G be a graph. An independent set of G is a subset $F \subseteq V(G)$ such that $e \nsubseteq F$, for any $e \in E(G)$. The **independence complex** of G is the simplicial complex,

$$\Delta_G = \{F \subseteq V(G) : F \text{ is an independent set of } G\}.$$

For a simplicial complex Δ on X, the **Alexander dual simplicial complex** $\Delta^\vee$ to Δ is defined as follows:

$$\Delta^\vee = \{F \subseteq X; X \setminus F \notin \Delta\}.$$

Definition 1.1. A simplicial complex Δ is **shellable** if the facets (maximal faces) of Δ can be ordered as $F_1, \ldots, F_s$ such that for all $1 \leqslant i < j \leqslant s$, there exist some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$. We call $F_1, \ldots, F_s$ a shelling for Δ.

The above definition is referred to as *non-pure shellable* and is due to Björner and Wachs [1]. Here, we will drop the adjective "non-pure". A graph G is called shellable, if the independence complex Δ_G is shellable.

Definition 1.2. A monomial ideal $I = (f_1, \ldots, f_m)$ of the polynomial ring $R = k[x_1, \ldots, x_n]$ has **linear quotients**, if there exists an order $f_1 < \cdots < f_m$ on the generators of I such that the colon ideal $(f_1, \ldots, f_{i-1}) : f_i$ is generated by a subset of variables for, all $2 \leq i \leq m$.

Also, for any $1 \leq i \leq m$, $\mathrm{set}_I(f_i)$ is defined as:

$$\mathrm{set}_I(f_i) = \{x_k : \; x_k \in (f_1, \ldots, f_{i-1}) : f_i\}.$$

The following result relates squarefree monomial ideals with linear quotients and shellable simplicial complexes.

Theorem A. [7, Theorem 1.4] The simplicial complex Δ is shellable if and only if $I_\Delta^\vee$ has linear quotients.

For a simplicial complex Δ and $F \in \Delta$, link of F in Δ is defined as $\mathrm{lk}_\Delta(F) = \{G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta\}$ and the deletion of F is the simplicial complex $\mathrm{del}_\Delta(F) = \{G \in \Delta : G \cap F = \emptyset\}$.

Definition 1.3. Let Δ be a simplicial complex on the vertex set $V = \{x_1, \dots, x_n\}$. Then, Δ is **vertex decomposable** if:

1) The only facet of Δ is $\{x_1, \dots, x_n\}$, or $\Delta = \emptyset$.

2) There exists a vertex $x \in V$ such that $\mathrm{del}_\Delta(x)$ and $\mathrm{lk}_\Delta(x)$ are vertex decomposable, and such that every facet of $\mathrm{del}_\Delta(x)$ is a facet of Δ.

A graph G is called vertex decomposable, if the independence complex Δ_G is vertex decomposable.

The **Castelnuovo-Mumford regularity** (or simply regularity) of an R-module M is defined as:

$$\mathrm{reg}(M) := \max\{j - i |\ \beta_{i,j}(M) \neq 0\},$$

and

$$\mathrm{pd}(M) := \max\{i|\ \beta_{i,j}(M) \neq 0, \text{ for some } j\}.$$

For a monomial ideal $I = (x_{11} \cdots x_{1n_1}, \dots, x_{t1} \cdots x_{tn_t})$ of the polynomial ring R, the **Alexande dual ideal** of I, which is denoted by $I^\vee$, is defined as:

$$I^\vee = (x_{11}, \dots, x_{1n_1}) \cap \cdots \cap (x_{t1}, \dots, x_{tn_t}).$$

The following theorem was proved in [11].

Theorem B. Let I be an square-free monomial ideal. Then, $\mathrm{pd}(I^\vee) = \mathrm{reg}(R/I)$.

Two edges $\{x, y\}$ and $\{w, z\}$ of G are called 3**-disjoint**, if the induced subgraph of G on $\{x, y, w, z\}$ consists of exactly two disjoint edges or equivalently, in the complement graph G^c, the induced graph on $\{x, y, w, z\}$ is a four-cycle. A path of length n is the graph with $V(G) = \{x_1, \dots, x_{n+1}\}$ and $E(G) = \{\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_n, x_{n+1}\}\}$.

Here, we find upper bounds for $\mathrm{reg}(R/I(G))$ in the case of shellable and vertex decomposable graphs. In Theorem 2.5, we show that for a shellable graph G, $\mathrm{reg}(R/I(G)) \leq n(G)$ and in Corollary 2.9 it is shown that for a vertex decomposable graph G, $\mathrm{reg}(R/I(G)) \leq \min\{a'(G), n(G)\}$. In Theorem 2.10, it is shown that if G^c has no triangle, then $\mathrm{reg}(R/I(G)) \leq 2$, and finally Theorem 2.13 shows that for a graph G, where G^c is a d-tree, the projective dimention of $R/I(G)$ is equal to $\max_{v \in V(G)}\{\deg_G(v)\}$.

2. **Main results**

For a graph G, let $a'(G)$ be the maximum number of vertex disjoint paths of length at most two in G such that paths of lengths one are pairwise 3-disjoint in G. Also, by $\alpha'(G)$ we mean the matching number of G.

Theorem 2.1. *Let G be a vertex decomposable graph. Then,* $\mathrm{reg}(R/I(G)) \leq a'(G)$.

Proof. By Theorem B, we have $\mathrm{reg}(R/I(G)) = \mathrm{pd}(I(G)^\vee)$. So, it is enough to show that $\mathrm{pd}(I(G)^\vee) \leq a'(G)$. By induction on $|V(G)|$, we prove the assertion. For $|V(G)| = 2$, there is nothing to prove. Let $|V(G)| > 2$. From the definition of vertex decomposable, there exists a vertex $x \in V(G)$ such that $\mathrm{del}_\Delta(x)$ and $\mathrm{lk}_\Delta(x)$ are vertex decomposable. Let $H_1 = G \setminus \{x\}$ and $H_2 = G \setminus (\{x\} \cup N_G(x))$. It is easy to see that $\mathrm{del}_\Delta(x) = \Delta_{H_1}$ and $\mathrm{lk}_\Delta(x) = \Delta_{H_2}$. Thus, H_1 and H_2 are vertex decomposable and each facet of Δ_{H_1} is a facet of Δ_G. Since a minimal vertex cover of a graph is the complement of a facet of the independence complex, for any minimal vertex cover C of H_1, $C \cup \{x\}$ is a minimal vertex cover of G. Also, observe that for each minimal vertex cover C of G containing x, $C \setminus \{x\}$ is a minimal vertex cover of H_1. Therefore, all the minimal vertex covers of G containing x are $C_1 \cup \{x\}, \ldots, C_n \cup \{x\}$, where $C_1, \ldots, C_n$ are the minimal vertex covers of H_1. Let $N_G(x) = \{y_1, \ldots, y_t\}$ and let C be a minimal vertex cover of G such that $x \notin C$. Then, $\{y_1, \ldots, y_t\} \subseteq C$ and $C \setminus \{y_1, \ldots, y_t\}$ is a minimal vertex cover of H_2. Also, for a minimal vertex cover C of H_2, $C \cup \{y_1, \ldots, y_t\}$ is a minimal vertex cover of G. Thus, the minimal vertex covers of G, which do not contain x, are $C'_1 \cup \{y_1, \ldots, y_t\}, \ldots, C'_m \cup \{y_1, \ldots, y_t\}$, where $C'_1, \ldots, C'_m$ are the minimal vertex covers of H_2. Therefore, $I(G)^\vee = xI(H_1)^\vee + y_1 \cdots y_t I(H_2)^\vee$. We show that $xI(H_1)^\vee \cap y_1 \cdots y_t I(H_2)^\vee = xy_1 \cdots y_t I(H_2)^\vee$. Let $x^C \in I(H_2)^\vee$ be a minimal generator. Then, $C \cup \{y_1, \ldots, y_t\}$ is a vertex cover of H_1, and hence $x^C \in I(H_1)^\vee$. Thus, $xy_1 \cdots y_t I(H_2)^\vee \subseteq xI(H_1)^\vee \cap y_1 \cdots y_t I(H_2)^\vee$. Now, let $x^C \in xI(H_1)^\vee \cap y_1 \cdots y_t I(H_2)^\vee$. Then, $x, y_1, \ldots, y_t \in C$ and $C \setminus \{x\}$ is a vertex cover of H_1 and $C \setminus \{x, y_1, \ldots, y_t\}$ is a vertex cover of H_2. Thus, $x^C = xy_1 \cdots y_t x^{C \setminus \{x, y_1, \ldots, y_t\}} \in xy_1 \cdots y_t I(H_2)^\vee$. Thus, we have the following short exact sequence:

$$0 \to xy_1 \cdots y_t I(H_2)^\vee \to xI(H_1)^\vee \oplus y_1 \cdots y_t I(H_2)^\vee \to I(G)^\vee \to 0.$$

Therefore, $\mathrm{pd}(I(G)^\vee) \leq \max\{\mathrm{pd}(I(H_2)^\vee) + 1, \mathrm{pd}(I(H_1)^\vee)\}$. By induction hypothesis, we have $\mathrm{pd}(I(H_1)^\vee) \leq a'(H_1)$ and $\mathrm{pd}(I(H_2)^\vee) \leq a'(H_2)$. We consider two cases.

Case 1. Let $\deg_G(x) \geq 2$. Then, y_1, x, y_2 is a path of length two and $y_1, x, y_2 \notin V(H_2)$. Thus, $a'(H_2) + 1 \leq a'(G)$. Since $a'(H_1) \leq a'(G)$, we have $\mathrm{pd}(I(G)^\vee) \leq \max\{a'(H_2) + 1, a'(H_1)\} \leq a'(G)$.

Case 2. Let $\deg_G(x) = 1$ and $N_G(x) = \{y\}$, for some y. No minimal vertex cover of H_1 contains y, since if a minimal vertex cover of H_1, say C, contains y, then $C \cup \{x\}$ is a non-minimal vertex cover of G, which is a contradiction, as discussed above. This means that each minimal vertex cover of H_1 contains $N_{H_1}(y)$. Thus, $P_{N_{H_1}(y)} \subseteq \cap_{i=1}^n P_{C_i} = I(H_1)$, where $P_{C_i} = (z : z \in C_i)$ and $P_{N_{H_1}(y)} = (z : z \in N_{H_1}(y))$. Then, $N_{H_1}(y) = \emptyset$, since all the minimal generators of $I(H_1)$ are of degree two. Therefore, x, y is a path which is 3-disjoint from the paths of length one in H_2 and disjoint from all paths in H_2. Thus, $a'(H_2)+1 \leq a'(G)$. Since $a'(H_1) \leq a'(G)$, the assertion follows from the inequality $\mathrm{pd}(I(G)^\vee) \leq \max\{a'(H_2) + 1, a'(H_1)\}$.

Hà and Van Tuyl in [6] proved that for any graph G, $\mathrm{reg}(R/I(G)) \leq \alpha'(G)$, where $\alpha'(G)$ is the matching number, the largest number of pairwise disjoint edges in G. It is easy to see that $a'(G) \leq \alpha'(G)$. The following example shows that $a'(G)$ is a smaller upper bound for vertex decomposable graphs. □

Example 2.2. Let G be a graph which is obtained by adding a vertex x to the cycle C_{2n+1} and joining it to one vertex of C_{2n+1}. Let $y \in V(C_{2n+1})$ be a vertex that $xy \in E(G)$. Observe that $H_1 = G \setminus \{y\}$ and $H_2 = G \setminus (\{y\} \cup N_G(y))$ are path graphs and hence they are vertex decomposable. Also, any facet of Δ_{H_1} is a facet of Δ_G. Therefore, G is vertex decomposable. One can see that $\alpha'(G) = n + 1$ and $a'(G) = n$.

The following theorem, was proved in [8].

Theorem 2.3. [8, Lemma 1.5] *Suppose that $I = (u_1, \ldots, u_m)$ is a monomial ideal with linear quotients with the ordering $u_1 < \cdots < u_m$ such that* $\deg(u_1) \leq \deg(u_2) \leq \cdots \leq \deg(u_m)$. *Then, the iterated mapping cone F, derived from the sequence $u_1, \ldots, u_m$, is a minimal graded free resolution of I, and for all $i > 0$, the symbols*

$$f(\sigma; u) \quad \textit{with} \quad u \in G(I), \ \sigma \subseteq \mathrm{set}_I(u), \ |\sigma| = i$$

form a homogeneous basis of the R-module F_i. Moreover, $\deg f(\sigma; u) = |\sigma| + \deg(u)$.

In the following theorem, we show that for a shellable graph there exists a vertex $x \in V(G)$ such that $\mathrm{reg}(R/I(G))$ is bounded by $\mathrm{reg}(R/I(G \setminus (\{x\} \cup N_G(x))) + 1$. For a subset $F \subseteq V(G)$, the monomial $\prod_{x \in F} x$ is denoted by x^F.

Theorem 2.4. *Let G be a shellable graph. There exists a vertex $x \in V(G)$ such that if $H = G \setminus (\{x\} \cup N_G(x))$, then*

$$\mathrm{reg}(R/I(G)) \leq \mathrm{reg}(R/I(H)) + 1.$$

Proof. By Theorem B, we have $\mathrm{reg}(R/I(G)) = \mathrm{pd}(I(G)^\vee)$. Let $J = I(G)^\vee$. From Theorem A, there exists an order of linear quotients $u_1 < \cdots < u_t$ on the minimal generators of J. From [10, Lemma 2.1], one can assume that $\deg(u_1) \leq \cdots \leq \deg(u_t)$. Thus, by Theorem 2.3, we have $\beta_i(J) = \sum_{j=1}^{t} \binom{|\,\mathrm{set}_J(u_j)|}{i}$. Therefore, $\mathrm{pd}(J) = \max\{|\,\mathrm{set}_J(u_i)| : 1 \leq i \leq t\}$. For any i, $1 \leq i \leq t$, we have $u_i = x^{C_i}$, where $C_i \subseteq V(G)$ is a minimal vertex cover of G. Let $\mathrm{pd}(J) = |\,\mathrm{set}_J(x^{C_l})|$, for some $1 \leq l \leq t$, and $\mathrm{set}_J(x^{C_l}) = (x_1, \ldots, x_r)$. Set $x = x_r$, $H = G \setminus (\{x\} \cup N_G(x))$ and $K = I(H)^\vee$. The set of minimal vertex covers of H is $\{C_i \setminus N_G(x) : N_G(x) \subseteq C_i\}$. Let $1 \leq i_1 < \cdots < i_k \leq t$ be all integers such that $N_G(x) \subseteq C_{i_j}$, for $1 \leq j \leq k$. Then, $K = (x^{C_{i_j} \setminus N_G(x)} : 1 \leq j \leq k)$. Also, the ordering $x^{C_{i_1} \setminus N_G(x)} < \cdots < x^{C_{i_k} \setminus N_G(x)}$ is an order of linear quotients for K and is degree increasing. Since $x \in \mathrm{set}_J(x^{C_l})$, we have $x \notin C_l$. Thus, $N_G(x) \subseteq C_l$. Therefore, $l = i_{l'}$, for some $1 \leq l' \leq k$. From the definition of linear quotients we see that for any $1 \leq i \leq r-1$, there exists $\lambda_i < l$ such that $C_{\lambda_i} \setminus C_l = \{x_i\}$. It is easy to see that $x \notin C_{\lambda_i}$ $(1 \leq i \leq r-1)$. This means that $N_G(x) \subseteq C_{\lambda_i}$, and consequently $(x^{C_{\lambda_i} \setminus N_G(x)} : x^{C_l \setminus N_G(x)}) = (x_i)$, for any i, $1 \leq i \leq r-1$. Therefore, $\mathrm{set}_K(x^{C_l \setminus N_G(x)}) = \{x_1, \ldots, x_{r-1}\}$. Thus, $\mathrm{reg}(R/I(H)) = \mathrm{pd}(K) \geq |\,\mathrm{set}_K(x^{C_l \setminus N_G(x)})| = r - 1 = \mathrm{reg}(R/I(G)) - 1$. □

Let G be a graph and $x \in V(G)$. By a whisker we mean adding a new vertex y to G and connecting y to x. This new graph is denoted by $G \cup W(x)$. We denote by $G \cup W(G)$ the graph obtained from G by adding whiskers to all vertices of G. In the following theorem, the set of all induced subgraphs of G is denoted by $\mathcal{S}(G)$.

Theorem 2.5. *Let G be a shellable graph and*

$$n(G) = \max\{|V(H)| : H \in \mathcal{S}(G), H \cup W(H) \in \mathcal{S}(G)\}.$$

Then, $\operatorname{reg}(R/I(G)) \leq n(G)$.

Proof. By Theorem B, it is enough to show that $\operatorname{pd}(I(G)^\vee) \leq n$. With the same notations as in Theorem 2.4, let $x^{C_1} < \cdots < x^{C_t}$ be an order of linear quotients for $I(G)^\vee$ and $\operatorname{pd}(I(G)^\vee) = |\operatorname{set}_{I(G)^\vee}(x^{C_l})| = r$, for some $1 \leq l \leq t$. Let $\operatorname{set}_{I(G)^\vee}(x^{C_l}) = (x_1, \ldots, x_r)$ and $x^{C_{i_1}}, \ldots, x^{C_{i_r}} < x^{C_l}$ be the monomials for which $(x^{C_{i_j}} : x^{C_l}) = (x_j)$, for any $1 \leq j \leq r$. For any $1 \leq j \leq r$, we have $x_j \notin C_l$ and $x_j \in C_{i_j}$. Therefore, $N_G(x_j) \nsubseteq C_{i_j}$, since C_{i_j} is a minimal vertex cover of G. Also, for any $1 \leq j, k \leq r$, where $k \neq j$, we have $x_k \notin C_{i_j}$, since $C_{i_j} \setminus C_l = \{x_j\}$. For any $1 \leq j \leq r$, let $y_j \in N_G(x_j) \setminus C_{i_j}$. Thus, for any $1 \leq j, k \leq r$, where $k \neq j$, we have $x_k y_j \notin E(G)$. Otherwise, for the minimal vertex cover C_{i_j} we have $x_k \in C_{i_j}$ or $y_j \in C_{i_j}$, which is a contradiction. Let H be the induced subgraph of G on the vertex set $\{y_1, \ldots, y_r\}$. We have $x_j y_j \in E(G)$ and $x_j y_k \notin E(G)$, for any $1 \leq j, k \leq r$, where $k \neq j$. This means that $H \cup W(H) \in \mathcal{S}(G)$. Therefore, $\operatorname{pd}(I(G)^\vee) = r = |V(H)| \leq n$. □

Example 2.6. Consider the graph G with vertex set $\{x_1, \ldots, x_4\}$ and edge set $\{x_1x_2, x_1x_3, x_2x_3, x_1x_4\}$. Then, G is shellable with $\operatorname{reg}(R/I(G)) = 1$. We have $\alpha'(G) = 2$ and $n(G) = 1$. This shows that $n(G)$ is a smaller upper bound for shellable graphs.

Example 2.7. Let G be a graph which is obtained by adding a vertex x to the cycle C_{2n+1} and joining it to two adjacent vertices of C_{2n+1}. Then, by [2, Proposition 4.3], G is vertex decomposable and hence it is shellable. One can see that $\alpha'(G) = n + 1$. Obsereve that $n(G) \leq \lfloor \frac{|V(G)|}{2} \rfloor = n + 1$. We show that $n(G) < n + 1$. By contradiction, assume that $n(G) = n + 1$. Let H be an induced subgraph of G such that $n(G) = |V(H)| = n + 1$. Then, $|H \cup W(H)| = 2n + 2$. Hence, $H \cup W(H) = G$. Thus, G has $n + 1$ vertices of degree one, which is a contradiction. Therefore, $n(G) < \alpha'(G)$.

Remark 2.8. There are graphs for which $a'(G) < n(G)$. The path graph of length three is such an example for which $a'(G) = 1$ and $n(G) =$

2. Also, there are graphs for which $n(G) < a'(G)$. Consider the complete graph K_n for $n \geq 6$. We have $n(G) = 1$ and $a'(G) \geq 2$.

Corollary 2.9. *Let G be a vertex decomposable graph. Then,* $\operatorname{reg}(R/I(G)) \leq \min\{a'(G), n(G)\}$.

Proof. This follows from theorems 2.1, 2.5 and the fact that every vertex decomposable graph is shellable, which was proved in [1, Theorem 11.3]. □

Theorem 2.10. *Let G be a graph such that G^c has no triangle. Then,* $\operatorname{reg}(R/I(G)) \leq 2$. *In addition, if G^c is not chordal, then* $\operatorname{reg}(R/I(G)) = 2$.

Proof. From Hochster's formula, we have

$$\beta_{i,j}(R/I(G)) = \sum_{S \subseteq V; |S|=j} \dim \widetilde{H}_{j-i-1}(\Delta(G^c_S), K),$$

where, G_S denotes the induced subgraph of G on the vertex set S. Since G^c has no cycle of length 3, any clique in G^c is of cardinality at most 2. Thus, $\widetilde{H}_i(\Delta(G^c_S), K) = 0$, for any $i > 1$ and any S. Therefore, $\widetilde{H}_{j-i-1}(\Delta(G^c_S), K) = 0$, for any $j - i > 2$. Thus, for any i and j such that $\beta_{i,j}(R/I(G)) \neq 0$, one has $j - i \leq 2$ and the result holds. If G^c is not chordal, then by [4, Theorem 1], $I(G)$ does not have a linear resolution and hence $\operatorname{reg}(R/I(G)) \neq 1$. Thus, $\operatorname{reg}(R/I(G)) = 2$. □

Definition 2.11. A d-tree is a chordal graph defined inductively as follows:
(i) K_{d+1} is a d-tree.
(ii) If H is a d-tree, then so is $G = H \cup_{K_d} K_{d+1}$.

Edge ideals with 2-linear resolution were characterized in [4, Theorem 1] and it was shown that $I(G)$ has linear resolution precisely, when G^c is a chordal graph. Eliahou and Villarreal in [3] conjectured that $\operatorname{pd}(R/I(G))$, where $I(G)$ has 2-linear resolution, is equal to the maximum degree of vertices of G. In the following theorem, we show that for a graph G such that G^c is a d-tree, we have $\operatorname{pd}(R/I(G)) = \max_{v \in V(G)}\{\deg_G(v)\}$. This statement is not true for an arbitrary ideal with 2-linear resolution. Consider the cycle C_4. Clearly C_4^c is chordal, and hence $I(C_4)$ has 2-linear resolution, but $\mathrm{pd}(R/I(C_4)) = 3$, while $\max_{v\in V(C_4)}\{\deg_{C_4}(v)\} = 2$.

To prove Theorem 2.13 we need the following lemma.

Lemma 2.12. *Let G be a d-tree. Then, $\deg_G(v) \geq d$, for any $v \in V(G)$.*

Proof. We proceed inductively in terms of the definition of a d-tree. If $G = K_{d+1}$, then the assertion is clear. Let $G = H \cup_{K_d} K_{d+1}$, where H is a d-tree. Then, by the induction hypothesis, $\deg_H(v) \geq d$, for any $v \in V(H)$. Let $V(G) = V(H) \cup \{x\}$, where $\{x\} = V(K_{d+1}) \setminus V(H)$. Then, $\deg_G(x) = d$, and for any $v \in V(H)$, we have $\deg_G(v) \geq \deg_H(v) \geq d$. □

Theorem 2.13. *Let G be a graph such that G^c is a d-tree. Then,* $\mathrm{pd}(R/I(G)) = \max_{v\in V(G)}\{\deg_G(v)\}$.

Proof. We prove by induction on $|V(G)|$ that $I(G)$ has linear quotients and $\mathrm{pd}(R/I(G)) = \max_{v\in V(G)}\{\deg_G(v)\}$. For $|V(G)| = 2$ the result is clear. Let $|V(G)| > 2$ and $G' = G^c$. Here, we have $G' = H \cup_{K_d} K_{d+1}$, where H is a d-tree. Let $V(G') \setminus V(H) = \{x\}$, $V(H) \cap V(K_{d+1}) = \{x_1, \ldots, x_d\}$ and $V(H) \setminus V(K_d) = \{y_1, \ldots, y_k\}$. Since H is a d-tree, by the induction hypothesis, $I(H^c)$ has linear quotients and $\mathrm{pd}(R/I(H^c)) = \max_{v\in V(H^c)}\{\deg_{H^c}(v)\}$. We have $I(G) = (xy_1, \ldots, xy_k) + I(H^c)$. Let $u_1 < \cdots < u_l$ be an order of linear quotients for the minimal generators of $I(H^c)$. We claim that the ordering $xy_1 < \cdots < xy_k < u_1 < \cdots < u_l$ is an order of linear quotients for $I(G)$. Consider two monomials xy_i and u_j, for some $1 \leq i \leq k$ and $1 \leq j \leq l$, and let $u_j = zw$, for some $z, w \in V(H)$. Since $\{z, w\}$ is not an edge of H, then at least one of z and w is not in $V(K_d)$. Without loss of generality, assume that $w \notin V(K_d)$. Then, $w = y_{j'}$, for some $1 \leq j' \leq k$. We have $x|(xy_i : u_j)$, $xy_{j'} < u_j$ and $(xy_{j'} : u_j) = (x)$. For $xy_i < xy_j$, we have $(xy_i : xy_j) = (y_i)$, and for $u_i < u_j$, since $u_1 < \cdots < u_l$ is an order of linear quotients, the result holds. Now, by Theorem 2.3, we have $\mathrm{pd}(I(G)) = \max\{|\mathrm{set}_{I(G)}(zw)| : \{z, w\} \in E(G)\}$ and $\mathrm{pd}(I(H^c)) = \max\{|\mathrm{set}_{I(H^c)}(zw)| : \{z, w\} \in E(H^c)\}$. For any $1 \leq i \leq k$, we have $\mathrm{set}_{I(G)}(xy_i) = \{y_1, \ldots, y_{i-1}\}$. For any $1 \leq j \leq l$, we know that $u_j = y_{j'}z_j$, for some $1 \leq j' \leq k$ and some $z_j \in V(H)$. Thus, $\mathrm{set}_{I(G)}(u_j) = \{x\} \cup \mathrm{set}_{I(H^c)}(u_j)$. Therefore, $\mathrm{pd}(I(G)) = \max\{\mathrm{pd}(I(H^c)) + 1, k - 1\}$, and hence $\mathrm{pd}(R/I(G)) = \mathrm{pd}(I(G)) + 1 = \max\{\mathrm{pd}(R/I(H^c)) + 1, k\}$. Since $\mathrm{pd}(R/I(H^c)) = \max_{v\in V(H^c)}\{\deg_{H^c}(v)\}$, thus

$$\mathrm{pd}(R/I(G)) = \max_{v\in V(H^c)}\{\deg_{H^c}(v) + 1, k\}.$$

For any i, $1 \leq i \leq k$, we have $\deg_G(y_i) = \deg_{H^c}(y_i) + 1$, because x is adjacent to y_i in G. We claim that for any $1 \leq i \leq d$, $\deg_G(x_i) < k$. Let $1 \leq i \leq d$ be an integer. Since H is a d-tree, by Lemma 2.12 we have $\deg_H(x_i) \geq d$. So, there exists y_j, for some $1 \leq j \leq k$, such that $x_iy_j \in E(H)$. Therefore, $\deg_{H^c}(x_i) < k$. Thus, $\deg_{H^c}(x_i) + 1 \leq k$, for any i, $1 \leq i \leq d$. Since $\deg_{H^c}(x_i) = \deg_G(x_i)$, then $\deg_G(x_i) < k$, for any i, $1 \leq i \leq d$. Since $\deg_G(x) = k$, thus $\max_{v\in V(H^c)}\{\deg_{H^c}(v) + 1, k\} = \max_{v\in V(G)}\{\deg_G(v)\}$ and the proof is complete. □

Acknowledgments

The research of the first and the second author was in part supported by grants from IPM with numbers (No. 89130059) and (No. 89050116). The authors express their deep gratitude to Professor Siamak Yassemi for his helpful suggestions to improve this work. They also thank the referees for their useful comments.

References

[1] A. Björner and M. L. Wachs, Shellable nonpure complexes and posets I, *Trans. Amer. Math. Soc.* **348**(4) (1996) 1299-1327.

[2] A. Dochtermann and A. Engström, Algebraic properties of edge ideals via combinatorial topology, arXiv:0810.4120v1.

[3] S. Eliahou and R. Villarreal, The second Betti number of an edge ideal, *Aportaciones Matematicas, Serie Comunicaciones* **25** (1999) 115-119.

[4] R. Fröberg, On Stanley-Reisner rings, Topics in Algebra *Banach Center Publ.* **26** (2) PWN, Warsaw (1990) 57–70.

[5] H. T. Hà and A. Van Tuyl, Splittable ideals and the resolutions of monomial ideals, *J. Algebra* **309** (2007) 405–425.

[6] H. T. Hà and A. Van Tuyl, Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers, *J. Algebraic Combin.* **27** (2) (2008) 215-245.

[7] J. Herzog, T. Hibi and X. Zheng, Diracs theorem on chordal graphs and Alexander duality, *European J. Combin.* **25** (2004) 949–960.

[8] J. Herzog and Y. Takayama, Resolutions by mapping cones, *Homology, Homotopy Appl.* **4**(2), part 2, (2002) 277-294.

[9] M. Katzmann, Characteristic-independence of Betti numbers of graph ideals, *J. Combin. Theory Ser. A* **113**(3) (2006) 435-454.

[10] A. Soleyman Jahan and X. Zheng, Pretty clean monomial ideals and linear quotients, arXiv:0707.291.4v1.
[11] N. Terai, Alexander duality theorem and Stanley-Reisner rings, *Sūrikaisekikenkyūsho Kōkyūruko* (1999) no. 1078, 174–184, Free resolutions of coordinate rings of projective varieties and related topics (Kyoto 1998).
[12] A. Van Tuyl, Sequentially Cohen-Macaulay bipartite graphs: vertex decomposability and regularity, arXiv:0906.0273v1.
[13] A. Van Tuyl and R. Villarreal, Shellable graphs and sequentially Cohen-Macaulay bipartite graphs, *J. Combin. Theory Ser. A* **115** (2008) 799–814.
[14] R.H. Villarreal, Cohen-Macaulay graphs, *Manuscripta Math* **66**(3) (1990) 277-293.

S. Moradi
Department of Mathematics, Ilam University, P.O. Box 69315-516, Ilam, Iran & School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.
Email: `s_moradi@aut.ac.ir`

D. Kiani
Department of Pure Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Ave., Tehran 15914, Iran & School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.
Email: `dkiani@aut.ac.ir`