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JORDAN DERIVATIONS ON TRIVIAL EXTENSIONS

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Abstract. Let $\mathcal{A}$ be a unital $\mathcal{R}$-algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. It is shown that every Jordan derivation of the trivial extension of $\mathcal{A}$ by $\mathcal{M}$, under some conditions, is the sum of a derivation and an antiderivation.

1. Introduction

Throughout the paper $\mathcal{R}$ will denote a commutative ring with unity. Let $\mathcal{A}$ be an algebra over $\mathcal{R}$. Recall that an $\mathcal{R}$-linear map Δ from $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$ is said to be a *Jordan derivation* if $\Delta(ab+ba) = \Delta(a)b + a\Delta(b) + \Delta(b)a + b\Delta(a)$ for all $a, b \in \mathcal{A}$. It is called a *derivation* if $\Delta(ab) = \Delta(a)b + a\Delta(b)$ for all $a, b \in \mathcal{A}$. Each map of the form $a \to am - ma$, where $m \in \mathcal{M}$, is a derivation which will be called an *inner derivation.* Also Δ is called an *antiderivation* if $\Delta(ab) = \Delta(b)a + b\Delta(a)$ for all $a, b \in \mathcal{A}$. We shall say that an antiderivation Δ is improper if it is a derivation; otherwise we shall say that Δ is proper. Clearly, each derivation or antiderivation is a Jordan derivation. The converse is, in general, not true (see [6]).

It is natural and very interesting to find some conditions under which a Jordan derivation is a derivation. Herstein [14] proved that every Jordan derivation from a 2-torsion free prime ring into itself is a derivation and that there are no nonzero antiderivations on a prime ring. Brešar

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[8] showed that every additive Jordan derivation from a 2-torsion free semiprime ring into itself is a derivation. Sinclair [18] proved that every continuous linear Jordan derivation on semisimple Banach algebras is a derivation. Zhang in [21] proved that every linear Jordan derivation on nest algebras is an inner derivation. Lu [17] proved that every additive Jordan derivation on reflexive algebras is a derivation which generalized the result in [21]. Benkovič [6] determined Jordan derivations on triangular matrices over commutative rings and proved that every Jordan derivation from the algebra of all upper triangular matrices into its arbitrary bimodule is the sum of a derivation and an antiderivation. Zhang and Yu [23] showed that every Jordan derivation of triangular algebras is a derivation, so every Jordan derivation from the algebra of all upper triangular matrices into itself is a derivation.

In this note we study the Jordan derivations on trivial extensions and generalize the Zhang and Yu's result [23].

2. **Preliminaries**

Recall that a *triangular algebra* $Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ is an $\mathcal{R}$-algebra of the form

$$Tri(\mathcal{A},\mathcal{M},\mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mid a \in \mathcal{A},\ m \in \mathcal{M},\ b \in \mathcal{B} \right\}$$

under the usual matrix operations, where $\mathcal{A}$ and $\mathcal{B}$ are unital algebras over $\mathcal{R}$ and $\mathcal{M}$ is a unital $(\mathcal{A},\mathcal{B})$-bimodule which is faithful as a left $\mathcal{A}$-module as well as a right $\mathcal{B}$-module (see [9]). Basic examples of triangular algebras are upper triangular matrix algebras and nest algebras [11], [13]. Recently, there has been a growing interest in the study of special maps on triangular algebras, such as commuting linear maps [9], Lie derivations [10], commuting traces of bilinear maps and commutativity preserving linear maps [7], biderivations [5], functional identities [4], Jordan isomorphisms [19], Jordan derivations [23] and Jordan higher derivations [20].

Let $\mathcal{A}$ be a unital algebra over $\mathcal{R}$ and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. $\mathcal{A} \times \mathcal{M}$ as an $\mathcal{R}$-module together with the algebra product defined by:

$$(a,m).(b,n) = (ab, an + mb) \qquad (a, b \in \mathcal{A}, \quad m, n \in \mathcal{M})$$

is an $\mathcal{R}$-algebra with unity $(1,0)$, which is called the *trivial extension* of $\mathcal{A}$ by $\mathcal{M}$ and denoted by $T(\mathcal{A},\mathcal{M})$. Trivial extensions have been extensively studied in the algebra and analysis (see, for instance, [1], [2], [3], [12], [15], [16] and [22]).

Let $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra over $\mathcal{R}$. Denote by $\mathcal{A} \oplus \mathcal{B}$ the direct sum of $\mathcal{A}$ and $\mathcal{B}$ as $\mathcal{R}$-algebra, and view $\mathcal{M}$ as an $\mathcal{A} \oplus \mathcal{B}$-bimodule with the module actions given by

$$(a, b).m = am, \qquad m.(a, b) = mb, \qquad a \in \mathcal{A}, \quad b \in \mathcal{B}, \quad m \in \mathcal{M}.$$

Then $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is isomorphic to $T(\mathcal{A} \oplus \mathcal{B}, \mathcal{M})$ as an $\mathcal{R}$-algebra. So triangular algebras are examples of trivial extensions.

The following notations will be used in our paper.
Let $\mathcal{A}$ be an $\mathcal{R}$-algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule, define the left annihilator of $\mathcal{M}$ by $l.ann_{\mathcal{A}}\mathcal{M} = \{a \in \mathcal{A} : a\mathcal{M} = \{0\}\}$. Similarly, we define the right annihilator of $\mathcal{M}$ by $r.ann_{\mathcal{A}}\mathcal{M} = \{a \in \mathcal{A} : \mathcal{M}a = \{0\}\}$. Also we denote the unity and zero of $T(\mathcal{A}, \mathcal{M})$ by 1 and 0, respectively.

3. **Main result**

The main result of the paper is the following theorem.

Theorem 3.1. *Let $\mathcal{A}$ be a unital algebra over the 2-torsion free commutative ring $\mathcal{R}$ and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. Suppose that E is a non-trivial idempotent element in $\mathcal{A}$ and E'=1-E such that*

$$E\mathcal{A}E'\mathcal{A}E = \{0\}, \quad E'\mathcal{A}E\mathcal{A}E' = \{0\},$$

$$E(l.ann_{\mathcal{A}}\mathcal{M})E = \{0\}, \quad E'(r.ann_{\mathcal{A}}\mathcal{M})E' = \{0\},$$

and $EME' = M$ for all $M \in \mathcal{M}$. Let $\mathcal{U} = T(\mathcal{A}, \mathcal{M})$ and $\Delta : \mathcal{U} \to \mathcal{U}$ be a Jordan derivation and let $P = (E, 0)$ and $Q = (E', 0)$. Then there exists a derivation $\delta : \mathcal{U} \to \mathcal{U}$ and an antiderivation $J : \mathcal{U} \to \mathcal{U}$ such that $\Delta = \delta + J$, J(PXP)=0 and J(QXQ)=0 for any $X \in \mathcal{U}$. Moreover, δ and J are uniquely determined.

To prove the theorem we need some lemmas. We consider the conditions of this theorem in the lemmas. Note that, P and Q are idempotents of $\mathcal{U}$ such that $P + Q = 1$ and $PQ = 0$.

We will show that the Jordan derivation Δ is a sum of an antiderivation J (see Lemma 3.3), an inner derivation I (see Lemma 3.5) and a derivation D (see Lemma 3.8).

Lemma 3.2. *For every $X, Y \in \mathcal{U}$, we have*

$$PXQYP = 0 \quad \textit{and} \quad QXPYQ = 0.$$

Proof. For all $M \in \mathcal{M}$, since $EME' = M$, we have

$$EME = 0, \quad E'ME = 0, \quad E'ME' = 0,$$
$$EM = M, \quad ME' = M, \quad ME = 0, \quad E'M = 0.$$

Let $X = (A, M)$ and $Y = (B, N)$. So $PXQYP = (EAE'BE, EAE'NE + EME'BE) = 0$ as $ECE = 0$ for all $C \in \mathcal{M}$ and $E\mathcal{A}E'\mathcal{A}E = \{0\}$. Similarly, $QXPYQ = 0$. $\square$

Lemma 3.3. *The mapping $J : \mathcal{U} \to \mathcal{U}$ defined by*

$$J(X) = P\Delta(QXP)Q + Q\Delta(PXQ)P$$

is an antiderivation. Also J(PXP)=0 and J(QXQ)=0 for all $X \in \mathcal{U}$.

Proof. Clearly, J is an $\mathcal{R}$-linear map. Since Δ is a Jordan derivation, for all $X, Y \in \mathcal{U}$ we have

$$\begin{aligned} \Delta(QXPYP) &= \Delta(QXPPYP) \\ &= \Delta(QXPPYP + PYPQXP) \\ &= \Delta(QXP)PYP + QXP\Delta(PYP) \\ &\quad + PYP\Delta(QXP) + \Delta(PYP)QXP. \end{aligned} \tag{3.1}$$

Similarly

$$\begin{aligned} \Delta(QXQYP) &= \Delta(QXQ)QYP + QXQ\Delta(QYP) \\ &\quad + QYP\Delta(QXQ) + \Delta(QYP)QXQ. \end{aligned} \tag{3.2}$$

$$\begin{aligned} \Delta(PXPYQ) &= \Delta(PXP)PYQ + PXP\Delta(PYQ) \\ &\quad + PYQ\Delta(PXP) + \Delta(PYQ)PXP. \end{aligned} \tag{3.3}$$

$$\begin{aligned} \Delta(PXQYQ) &= \Delta(PXQ)QYQ + PXQ\Delta(QYQ) \\ &\quad + QYQ\Delta(PXQ) + \Delta(QYQ)PXQ. \end{aligned} \tag{3.4}$$

Thus,

$$P\Delta(QXPYP)Q = PYP\Delta(QXP)Q;$$
$$P\Delta(QXQYP)Q = P\Delta(QYP)QXQ;$$
$$Q\Delta(PXPYQ)P = Q\Delta(PYQ)PXP;$$
$$Q\Delta(PXQYQ)P = QYQ\Delta(PXQ)P.$$

From these relations and Lemma 3.2 we arrive at

$$\begin{aligned} J(XY) &= P\Delta(QXYP)Q + Q\Delta(PXYQ)P \\ &= P\Delta(QXPYP)Q + P\Delta(QXQYP)Q \\ &\quad + Q\Delta(PXPYQ)P + Q\Delta(PXQYQ)P \\ &= PYP\Delta(QXP)Q + P\Delta(QYP)QXQ \\ &\quad + Q\Delta(PYQ)PXP + QYQ\Delta(PXQ)P \\ &= YP\Delta(QXP)Q + P\Delta(QYP)QX \\ &\quad + Q\Delta(PYQ)PX + YQ\Delta(PXQ)P \\ &= YJ(X) + J(Y)X. \end{aligned}$$

So J is an antiderivation . By the definition of J it is clear that $J(PXP) = 0$ and $J(QXQ) = 0$ for all $X \in \mathcal{U}$. The proof is now complete. □

Lemma 3.4. *If $J : \mathcal{U} \to \mathcal{U}$ is an improper antiderivation, $J(PXP) = 0$ and $J(QXQ) = 0$ for all $X \in \mathcal{U}$, then $J = 0$.*

Proof. First, observe that $J(P) = J(PPP) = 0$. Similarly, we have $J(Q) = 0$. Then, since J is a derivation and an antiderivation, we have

$$\begin{aligned} J(PXQ) &= PJ(XQ) + J(P)XQ = PJ(XQ) \\ &= P(QJ(X) + J(Q)X) = 0. \end{aligned}$$

Similarly, $J(QXP) = 0$. So

$$J(X) = J(PXP) + J(PXQ) + J(QXP) + J(QXQ) = 0$$

for all $X \in \mathcal{U}$. □

Lemma 3.5. *Let $T = P\Delta(P)Q - Q\Delta(P)P$ and the mapping $I : \mathcal{U} \to \mathcal{U}$ be defined by*

$$I(X) = P\Delta(PXP + QXQ)Q + Q\Delta(PXP + QXQ)P.$$

Then for every $X \in \mathcal{U}$ we have

$$I(X) = XT - TX.$$

Proof. All $Y \in \mathcal{U}$ satisfy

$$\begin{aligned} 0 &= \Delta((PYP)(QYQ) + (QYQ)(PYP)) \\ &= PYP\Delta(QYQ) + \Delta(PYP)QYQ \\ &\quad + QYQ\Delta(PYP) + \Delta(QYQ)PYP. \end{aligned} \tag{3.5}$$

From this, for every $Y \in \mathcal{U}$, we obtain

$$PYP\Delta(QYQ)Q + P\Delta(PYP)QYQ = 0 \tag{3.6}$$

and

$$QYQ\Delta(PYP)P + Q\Delta(QYQ)PYP = 0. \tag{3.7}$$

For any $X \in \mathcal{U}$ replace Y by $X + P$ in (3.6). This gives

$$PXP\Delta(QXQ)Q + P\Delta(QXQ)Q + P\Delta(PXP)QXQ + P\Delta(P)QXQ = 0.$$

Hence, replacing X by QXQ in the previous equation, we get that $P\Delta(QXQ)Q + P\Delta(P)QXQ = 0$ for any $X \in \mathcal{U}$. If $X = Q$ in this relation, then $P\Delta(Q)Q + P\Delta(P)Q = 0$.

Now, for any $X \in \mathcal{U}$ replace Y by $PXP + Q$ in (3.6) we obtain

$$PXP\Delta(Q)Q + P\Delta(PXP)Q = 0.$$

According to these relations we have $-PXP\Delta(P)Q + P\Delta(PXP)Q = 0$. Similarly, we can obtain from relation (3.7) that

$$Q\Delta(QXQ)P + QXQ\Delta(P)P = 0 \ \textit{and} \ -Q\Delta(P)PXP + Q\Delta(PXP)P = 0$$

for all $X \in \mathcal{U}$. These relations and Lemma 3.2 imply

$$\begin{aligned} I(X) &= P\Delta(PXP)Q + P\Delta(QXQ)Q + Q\Delta(PXP)P + Q\Delta(QXQ)P \\ &= PXP\Delta(P)Q - P\Delta(P)QXQ + Q\Delta(P)PXP - QXQ\Delta(P)P \\ &= XP\Delta(P)Q - P\Delta(P)QX + Q\Delta(P)PX - XQ\Delta(P)P \\ &= XT - TX. \end{aligned}$$

□

Lemma 3.6. *Let $X \in \mathcal{U}$. Then*

(a) If PXPZQ=0 for all $Z \in \mathcal{U}$, then PXP=0;
(b) If PZQXQ=0 for all $Z \in \mathcal{U}$, then QXQ=0.

Proof. (a) Write $X = (A, N)$. Let $M \in \mathcal{M}$, and set $Z = (0, M)$. We have $EME' = M$ by assumption and $EN = N$ for all $N \in \mathcal{M}$ from the proof of Lemma 3.2. Hence,

$$ENE = 0 \quad \textit{and} \quad 0 = PXPZQ = (0, EAEME') = (0, AM),$$

so $A \in l.ann_{\mathcal{A}}\mathcal{M}$. Hence, by assumptions we obtain $EAE = 0$, therefore

$$PXP = (EAE, ENE) = 0.$$

Similarly, we can show that (b) holds. □

Lemma 3.7. *For every $X \in \mathcal{U}$ we have*

$$P\Delta(QXQ)P = 0, \quad Q\Delta(PXP)Q = 0, \quad P\Delta(PXQ)P = 0,$$
$$Q\Delta(PXQ)Q = 0, \quad P\Delta(QXP)P = 0, \quad Q\Delta(QXP)Q = 0.$$

Proof. Using (3.5) we see that for all $Y \in \mathcal{U}$, we have

$$PYP\Delta(QYQ)P + P\Delta(QYQ)PYP = 0.$$

For any $X \in \mathcal{U}$ replace Y by $QXQ + P$, so $P\Delta(QXQ)P = 0$. Similarly, replacing Y by $PXP+Q$ in (3.5), and multiplying the resulting equation by Q both on the left and on the right, yields $Q\Delta(PXP)Q = 0$, for all $X \in \mathcal{U}$.

If we multiply (3.1) by P and replace Y by P, we obtain $P\Delta(QXP)P = 0$ for all $X \in \mathcal{U}$, since Lemma 3.2 holds. Similarly, multiplying (3.1) by Q and replacing Y by P, we get $Q\Delta(QXP)Q = 0$ for all $X \in \mathcal{U}$.

As above, from (3.4) and Lemma 3.2, we have $P\Delta(PXQ)P = 0$ and $Q\Delta(PXQ)Q = 0$, for all $X \in \mathcal{U}$. □

Lemma 3.8. *The mapping $D : \mathcal{U} \to \mathcal{U}$ defined by $D(X) = P\Delta(PXP)P + P\Delta(PXQ)Q + Q\Delta(QXP)P + Q\Delta(QXQ)Q$ is a derivation.*

Proof. D is an $\mathcal{R}$-linear map. From (3.3) and Lemma 3.7 it follows immediately that

$$P\Delta(PXPYQ)Q = PXP\Delta(PYQ)Q + P\Delta(PXP)PYQ$$

for all $X, Y \in \mathcal{U}$. So for every $X, Y, Z \in \mathcal{U}$ we have

$$P\Delta(PXPYPZQ)Q = PXPYP\Delta(PZQ)Q + P\Delta(PXPYP)PZQ.$$

On the other hand,

$$\begin{aligned} P\Delta(PXPYPZQ)Q \;\; = \;\; & PXPYP\Delta(PZQ)Q \\ & + PXP\Delta(PYP)PZQ + P\Delta(PXP)PYPZQ. \end{aligned}$$

By comparing the two expressions for $P\Delta(PXPYPZQ)Q$, we arrive at

$$P(\Delta(PXPYP) - \Delta(PXP)PY - XP\Delta(PYP))PZQ = 0$$

for any $Z \in \mathcal{U}$. Therefore, by Lemma 3.6, we have

$$P\Delta(PXPYP)P = P\Delta(PXP)PYP + PXP\Delta(PYP))P.$$

Similarly, from (3.4) we get

$$P\Delta(PXQYQ)Q = P\Delta(PXQ)QYQ + PXQ\Delta(QYQ)Q$$

and

$$Q\Delta(QXQYQ)Q = Q\Delta(QXQ)QYQ + QXQ\Delta(QYQ)Q$$

for all $X, Y \in \mathcal{U}$.
Similarly, we can obtain from (3.1), (3.2) and Lemma 3.6 that

$$Q\Delta(QXPYP)P = Q\Delta(QXP)PYP + QXP\Delta(PYP)P$$

and

$$Q\Delta(QXQYP)P = QXQ\Delta(QYP)P + Q\Delta(QXQ)QYP$$

for all $X, Y \in \mathcal{U}$.

These relations with Lemma 3.2 gives us that $D(XY) = XD(Y) + D(X)Y$ for all $X, Y \in \mathcal{U}$. That is, D is a derivation from $\mathcal{U}$ into itself. $\square$

Proof of Theorem 3.1. For any $X \in \mathcal{U}$ we have

$$X = PXP + PXQ + QXP + QXQ$$

so, by Lemmas 3.3, 3.5, 3.7 and 3.8 it follows immediately that $\Delta(X) = J(X) + I(X) + D(X)$ for all $X \in \mathcal{U}$ where $\delta = D + I$ is a derivation and J is an antiderivation from $\mathcal{U}$ into itself such that $J(PXP) = 0$ and $J(QXQ) = 0$ for any $X \in \mathcal{U}$.

Let $\delta' : \mathcal{U} \to \mathcal{U}$ be a derivation and $J' : \mathcal{U} \to \mathcal{U}$ be an antiderivation such that $\Delta = \delta' + J'$, $J'(PXP) = 0$ and $J'(QXQ) = 0$ for any $X \in \mathcal{U}$. So $\delta + J = \delta' + J'$ and hence $\delta - \delta' = J - J'$. Therefore, $J - J'$ is an improper antiderivation such that $(J - J')(PXP) = 0$ and $(J-J')(QXQ) = 0$. Thus, by Lemma 3.4, we have $J = J'$ and hence $\delta = \delta'$. So we have that δ and J are uniquely determined. The proof of Theorem 3.1 is thus complete. $\square$

Note that if $J \neq 0$, then J is a proper antiderivation (by Lemma 3.4).

Remark 3.9. By the above lemmas and the proof of Theorem 3.1, one observes that if $\Delta : \mathcal{U} \to \mathcal{U}$ is a Jordan derivation, then the following are equivalent.

(a) Δ is a derivation.
(b) $P\Delta(QXP)Q = 0$ and $Q\Delta(PXQ)P = 0$ for all $X \in \mathcal{U}$.
(c) $\Delta(P\mathcal{U}Q) \subseteq P\mathcal{U}Q$ and $\Delta(Q\mathcal{U}P) \subseteq Q\mathcal{U}P$.

We have the following corollary, which was proved by a different method in [23].

Corollary 3.10. *Let $\mathcal{A}$, $\mathcal{B}$ be unital algebras over the 2-torsion free commutative ring $\mathcal{R}$, $\mathcal{M}$ be a unital $(\mathcal{A},\mathcal{B})$-bimodule that is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let $\mathcal{T} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ be the triangular algebra. Then every Jordan derivation from $\mathcal{T}$ into itself is a derivation.*

Proof. Let $\mathcal{A} \oplus \mathcal{B}$ be the direct sum of $\mathcal{A}$ and $\mathcal{B}$ as $\mathcal{R}$-algebras and $E = (1, 0)$. Consider $T(\mathcal{A} \oplus \mathcal{B},\mathcal{M})$ as defined in introduction. So this trivial extension satisfies all the requirements in Theorem 3.1 and therefore any Jordan derivation on it satisfies condition (b) of Remark 3.9. Therefore, every Jordan derivation on $T(\mathcal{A} \oplus \mathcal{B},\mathcal{M})$ is a derivation. By the isomorphism given in the introduction we have the result. $\square$

Remark 3.11. Let $\mathcal{T} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular algebra satisfying the conditions of Corollary 3.10, $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the standard idempotent of $\mathcal{T}$ and $Q = 1 - P$. Suppose that $\mathcal{N}$ is a unital $\mathcal{T}$-bimodule such that $Q\mathcal{N}P = \{0\}$ and, let for $N \in \mathcal{N}$, the condition $PNP\mathcal{T}Q = \{0\}$ implies $PNP = 0$ and the condition $P\mathcal{T}QNQ = \{0\}$ implies $QNQ = 0$. Then $(P, 0)$ and $(Q, 0)$ are idempotents of $T(\mathcal{T},\mathcal{N})$ such that

$$(Q, 0)T(\mathcal{T},\mathcal{N})(P, 0) = \{(0, 0)\}.$$

Let $(S,N) \in T(\mathcal{T},\mathcal{N})$ such that

$$(P, 0)(S,N)(P, 0)T(\mathcal{T},\mathcal{N})(Q, 0) = \{(0, 0)\}.$$

So for each $S' \in \mathcal{T}$ we have $(P, 0)(S,N)(P, 0)(S', 0)(Q, 0) = (0, 0)$ and hence $(PSPS'Q, PNPS'Q) = (0, 0)$. Therefore, $PSP\mathcal{T}Q = \{0\}$ and $PNP\mathcal{T}Q = \{0\}$. By assumption, we have $PSP = 0$ and $PNP = 0$. So $(P, 0)(S,N)(P, 0) = 0$. Similarly, if $(P, 0)T(\mathcal{T},\mathcal{N})(Q, 0)(S,N)(Q, 0) = \{(0, 0)\}$, then $(Q, 0)(S,N)(Q, 0) = 0$. Therefore

$$T(\mathcal{T},\mathcal{N}) \cong \begin{pmatrix} (P, 0)T(\mathcal{T},\mathcal{N})(P, 0) & (P, 0)T(\mathcal{T},\mathcal{N})(Q, 0) \\ 0 & (Q, 0)T(\mathcal{T},\mathcal{N})(Q, 0) \end{pmatrix}.$$

Thus, $T(\mathcal{T},\mathcal{N})$ is a triangular algebra. So by Corollary 3.10 every Jordan derivation from $T(\mathcal{T},\mathcal{N})$ into itself is a derivation.

Let $\mathcal{A}$ be a unital algebra over $\mathcal{R}$ and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. An $\mathcal{R}$-linear map δ from $\mathcal{A}$ into $\mathcal{M}$ is a Jordan derivation (derivation) if and only if the $\mathcal{R}$-linear map $\Delta : T(\mathcal{A},\mathcal{M}) \to T(\mathcal{A},\mathcal{M})$, given by $\Delta(A,M) = (0, \delta(A))$, is a Jordan derivation (derivation). From this result and Remark 3.11, we have the next corollary which is a generalization of Corollary 3.10.

Corollary 3.12. *Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra satisfying the conditions of Corollary 3.10 and $\mathcal{N}$ be a unital $\mathcal{T}$-bimodule as in the Remark 3.11. Then every Jordan derivation from $\mathcal{T}$ into $\mathcal{N}$ is a derivation.*

We now provide an example of trivial extension which satisfies conditions of Theorem 3.1, but is not a triangular algebra.

Example 3.13. Let $\mathcal{R}$ be a 2-torsion free commutative ring with unity and $\mathcal{A}$ be the $\mathcal{R}$-algebra of 2×2 lower triangular matrices over $\mathcal{R}$. We make $\mathcal{R}$ into an $\mathcal{A}$-bimodule by defining $RA = RA_{22}$ and $AR = A_{11}R$ for all $R \in \mathcal{R}$, $A \in \mathcal{A}$. Let $E = E_{11}$. Then the conditions of Theorem 3.1 hold for $T(\mathcal{A}, \mathcal{R})$ but this trivial extension is not a triangular algebra because the map $\Delta : T(\mathcal{A}, \mathcal{R}) \to T(\mathcal{A}, \mathcal{R})$ defined by $\Delta(A, R) = (RE_{21}, A_{21})$ is a proper antiderivation, while by the above corollary, triangular algebras have no nonzero proper antiderivations. (We denote E_{ij} for the matrix units, for all i, j.)

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